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FRACTIONAL HERMITE-HADAMARD INEQUALITIES FOR TWICE DIFFERENTIABLE GEOMETRIC-ARITHMETICALLY s-CONVEX FUNCTIONS

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Abstract. The main objective of this paper is to study the classical Hermite-Hadamard-type inequalities for Riemann-liouville fractional integrals. The Hermite-Hadamard inequalities are derived for k-Riemann-Liouville fractional integrals by using the definitions of different types of convex functions such as s-convex function, m-convex function, (s, m)- convex function and twice differentiable geometric-arithmetically s-convex function. Our results generalize the results of [8] and [18].

1. Introduction

In literature inequalities are very important for convex functions defined by C. Hermite and J. Hadamard (see, e.g.,[12, p.137]). These inequalities define that if $\Psi : I \to \mathbb{R}$ is a convex on the interval I of real numbers and $\theta, \zeta \in I$ with $\theta < \zeta$, then

$$\Psi\left(\frac{\theta+\zeta}{2}\right) \leq \frac{1}{\zeta-\theta}\int_{\theta}^{\zeta}\Psi(\xi)d\xi \leq \frac{\Psi(\theta)+\Psi(\zeta)}{2}.$$

In last few years the theory of Hermite-Hadamard inequalities progress very fast (see [1], [9], [10], [15], [16], [17]). We note that Hadamard's inequality has attracted a lot number of researchers in last few decades (see, for example [7], [13], [18]). In continuation of the development of Hadamard's inequality we produce Hadamard's inequalities for k-Riemann-Liouville fractional integral by using the definition of different convex functions of fractional calculus.

Definition 1.1. $\Psi : [\theta, \zeta] \to \mathbb{R}$ *is said to be convex if the inequality holds on an interval* $\Psi \subseteq \mathbb{R}$ *if*

$$\Psi(\tau\xi + (1-\tau)\eta) \leq \tau\Psi(\xi) + (1-\tau)\Psi(\eta) \tag{1.1}$$

holds for $\xi, \eta \in [\theta, \zeta]$ *and* $\tau \in [0, 1]$.

Following definition is given in [18].

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Definition 1.2. *The function* $\Psi : [0, \zeta^*] \to \mathbb{R}$ *is* m*-convex, where* $m \in [0,1]$ *and* $\zeta^* > 0$*, if for every* $\xi, \eta \in [0, \zeta^*]$ *and* $\mu \in [0,1]$*, we have*

$$\Psi(\mu\xi + m(1-\mu)\eta) \leq \mu\Psi(\xi) + m(1-\mu)\Psi(\eta). \tag{1.2}$$

Following definition is given in [18].

Definition 1.3. *The function* $\Psi : [0, \zeta^*] \to \mathbb{R}$ *is* (s,m)*-convex, where* $(s,m) \in [0,1]^2$ *and* $\zeta^* > 0$*, if for every* $\xi, \eta \in [0, \zeta^*]$ *and* $\mu \in [0,1]$*, we have*

$$\Psi(\mu\xi + m(1-\mu)\eta) \leq \mu^s\Psi(\xi) + m(1-\mu^s)\Psi(\eta). \tag{1.3}$$

Following definition is given in [6]

Definition 1.4. *Let* $\Psi \in L_1[\theta, \zeta]$*. The Riemann-Liouville integrals* $I^{\lambda}_{\theta+}\Psi$ *and* $I^{\lambda}_{\zeta-}\Psi$ *of order* $\lambda > 0$ *with* $\theta \geq 0$ *are stated by*

$$I^{\lambda}_{\theta+}\Psi(\xi) = \frac{1}{\Gamma(\lambda)}\int_{\theta}^{\xi}(\xi-\mu)^{\lambda-1}\Psi(\mu)d\mu, \qquad \xi > \theta$$

and

$$I^{\lambda}_{\zeta-}\Psi(\xi) = \frac{1}{\Gamma(\lambda)}\int_{\xi}^{\zeta}(\mu-\xi)^{\lambda-1}\Psi(\mu)d\mu, \qquad \xi < \zeta$$

respectively. At this spot $\Gamma(\lambda)$ *is the Gamma function and* $I^{0}_{\theta+}\Psi = I^{0}_{\zeta-}\Psi = \Psi(\xi)$*. For some current outcomes, (see* [14]*).*

Definition 1.5. [8] *Let* $\Psi : I \subset \mathbb{R}^+ \to \mathbb{R}^+$ *and* $s \in (0,1]$*. A function* $\Psi(\xi)$ *is geometric-arithmetically* s*-convex function on* I *if for every* $\xi, \eta \in I$ *and* $\mu \in [0,1]$*, we have*

$$\Psi(\xi^{\mu}\eta^{1-\mu}) \leq \mu^s(\Psi(\xi)) + (1-\mu)^s\Psi(\eta). \tag{1.4}$$

Lemma 1.6. [8] *For* $\mu \in [0,1]$*,* $\xi, \eta > 0$*, we have*

$$\mu\xi + (1-\mu)\eta \geq \eta^{1-\mu}\xi^{\mu}.$$

Lemma 1.7. [3] *For* $\mu \in [0,1]$*, we have*

$$(1-\mu)^{\chi} \leq 2^{1-\chi} - \mu^{\chi} \qquad \chi \in [0,1],$$

$$(1-\mu)^{\chi} \geq 2^{1-\chi} - \mu^{\chi} \qquad \chi \in [1,\infty],$$

Definition 1.8. [4] *The definition of incomplete beta function is given as:*

$$\beta_{\xi}(\theta,\zeta) = \int_0^{\xi}\mu^{\theta-1}(1-\mu)^{\zeta-1}d\mu, \tag{1.5}$$

where $\xi \in [0,1]$*,* $\theta, \zeta > 0$*.*

Theorem 1.9. [8] *Let* $\Psi : [\theta, \zeta] \to \mathbb{R}$ *be differentiable. If* $|\Psi''|$ *is measurable and* $|\Psi''|$ *is decreasing and geometric-arithmetically* s*-convex on* $[\theta, \zeta]$ *for some fixed* $\lambda \in (0,\infty)$*,* $s \in (0,1]$*,* $0 \leq \theta < \zeta$*, then*

$$\begin{aligned}&\left|\frac{\Psi(\theta)+\Psi(\zeta)}{2} - \frac{\Gamma(\lambda+1)}{2(\zeta-\theta)^{\lambda}}\left[I^{\lambda}_{\theta+}\Psi(\zeta) + I^{\lambda}_{\zeta-}\Psi(\theta)\right]\right| \\ \leq\ & \frac{(\zeta-\theta)^2(|\Psi''(\theta)|+|\Psi''(\zeta)|)}{2(\lambda+1)}\left(\frac{1}{s+1} - \frac{1}{\lambda+s+2}\right).\end{aligned} \tag{1.6}$$

Theorem 1.10. [8] *Consider $\Psi : [\theta, \zeta] \to \mathbb{R}$ a differentiable. If $|\Psi''|^h$ is measurable and $|\Psi''|^h$ is decreasing and geometric-arithmetically s-convex on $[\theta, \zeta]$ for some fixed $\lambda \in (0, \infty)$, $s \in (0, 1]$, $0 \leq \theta < \zeta$, then*

$$\begin{aligned} &\left| \frac{\Psi(\theta) + \Psi(\zeta)}{2} - \frac{\Gamma(\lambda+1)}{2(\zeta-\theta)^\lambda} \left[I^\lambda_{\theta+} \Psi(\zeta) + I^\lambda_{\zeta-} \Psi(\theta) \right] \right| \\ \leq \quad & \frac{(\zeta-\theta)^2 \max\{1 - 2^{1-\lambda}, 2^{1-\lambda} - 1\}}{2(\lambda+1)} \left(\frac{|\Psi''(\theta)|^h + |\Psi''(\zeta)|^h}{s+1} \right)^{\frac{1}{h}}. \end{aligned} \tag{1.7}$$

The following lemma is given in [19].

Lemma 1.11. *Assume that $\Psi : [\theta, \zeta] \to \mathbb{R}$ is twice differentiable on (θ, ζ) with $\theta < \zeta$. If $\Psi'' \in L[\theta, \zeta]$, then*

$$\begin{aligned} &\frac{\Gamma(\lambda+1)}{2(\zeta-\theta)^\lambda} \left[I^\lambda_{\theta+} \Psi(\zeta) + I^\lambda_{\zeta-} \Psi(\theta) \right] - \Psi\left(\frac{\theta+\zeta}{2} \right) \\ = \quad & \frac{(\zeta-\theta)^2}{2} \int_0^1 m(\mu) \Psi''(\mu\theta + (1-\mu)\zeta) d\mu, \end{aligned} \tag{1.8}$$

where

$$m(\mu) = \begin{cases} \mu - \frac{1-(1-\mu)^{\lambda+1} - \mu^{\lambda+1}}{(\lambda+1)}, & \mu \in [0, \frac{1}{2}) \\ (1-\mu) - \frac{1-(1-\mu)^{\lambda+1} - \mu^{\lambda+1}}{(\lambda+1)}, & \mu \in [\frac{1}{2}, 1). \end{cases}$$

The following theorem is given in [8].

Theorem 1.12. *Let $\Psi : [0, \zeta] \to \mathbb{R}$ be differentiable. $|\Psi''|^h$ is measurable and $1 < h < \infty$. If $|\Psi''|^h$ is measurable and $|\Psi''|^h$ is decreasing and geometric-arithmetically s-convex on $[0, \zeta]$ for some fixed $\lambda \in (0, \infty)$, $s \in (0, 1]$, $1 \leq \theta < \zeta$, then*

$$\begin{aligned} &\frac{\Gamma(\lambda+1)}{2(\zeta-\theta)^\lambda} \left[I^\lambda_{\theta+} \Psi(\zeta) + I^\lambda_{\zeta-} \Psi(\theta) \right] - \Psi\left(\frac{\theta+\zeta}{2} \right) \\ \leq \quad & \frac{(\zeta-\theta)^2}{2(\lambda+1)} \left(\frac{|\Psi''(\theta)|^h + |\Psi''(\zeta)|^h}{s+1} \right)^{\frac{1}{h}} \\ \times \quad & \left(\frac{(\lambda+1)2^{-g-1} + (\lambda+0.5)^{g+1} - \lambda^{g+1}}{g+1} \right)^{\frac{1}{g}}, \end{aligned} \tag{1.9}$$

where $\frac{1}{g} + \frac{1}{h} = 1$

The following theorem is given in [18].

Theorem 1.13. *Consider $\Psi : [0, \zeta^*] \to \mathbb{R}$ a twice differentiable with $\zeta^* > 0$. If $|\Psi''|^h$ is measurable and m-convex on $[\theta, \frac{\zeta}{m}]$ for some fixed $h \geq 1$, $0 \leq \theta < \zeta$ and $m \in (0, 1]$ with $\frac{\zeta}{m} \leq \zeta^*$, with then*

$$\begin{aligned} &\left| \frac{\Psi(\theta) + \Psi(\zeta)}{2} - \frac{\Gamma(\lambda+1)}{2(\zeta-\theta)^\lambda} \left[I^\lambda_{\theta+} \Psi(\zeta) + I^\lambda_{\zeta-} \Psi(\theta) \right] \right| \\ &\leq \frac{\lambda(\zeta-\theta)^2}{2(\lambda+1)(\lambda+2)} \left[\frac{|\Psi''(\theta)|^h + m|\Psi''(\frac{\zeta}{m})|^h}{2} \right]^{\frac{1}{h}} \end{aligned} \tag{1.10}$$

The following theorem is given in [18].

Theorem 1.14. *Let the condition of Theorem* 1.13, *be satisfied, then following inequality holds*

$$\begin{aligned}&\left|\frac{\Psi(\theta)+\Psi(\zeta)}{2}-\frac{\Gamma(\lambda+1)}{2(\zeta-\theta)^{\lambda}}\left[I_{\theta^+}^{\lambda}\Psi(\zeta)+I_{\zeta^-}^{\lambda}\Psi(\theta)\right]\right|\\ \leq\;& \frac{(\zeta-\theta)^2}{2(\lambda+1)}\left(1-\frac{2}{g(\lambda+1)+1}\right)^{\frac{1}{g}}\left[\frac{|\Psi''(\theta)|^h+m|\Psi''(\frac{\zeta}{m})|^h}{2}\right]^{\frac{1}{h}}, \qquad (1.11)\end{aligned}$$

where $\frac{1}{g}+\frac{1}{h}=1$

Theorem 1.15. [18] *Assume that* $\Psi:[\theta,\zeta]\to\mathbb{R}$ *is a twice differentiable mapping with* $\theta<\zeta$. *If* $|\Psi''|^h$ *is measurable and* (s,m)*-convex on* $[\theta,\zeta]$ *for some fixed* $h\geq 1$ *and* $(s,m)\in(0,1]^2$, *then*

$$\begin{aligned}&\left|\frac{\Psi(\theta)+\Psi(m\zeta)}{2}-\frac{\Gamma(\lambda+1)}{2(\zeta-\theta)^{\lambda}}\left[I_{\theta^+}^{\lambda}\Psi(m\zeta)+I_{m\zeta^-}^{\lambda}\Psi(\theta)\right]\right|\\ \leq\;& \frac{(m\zeta-\theta)^2}{2(\lambda+1)}\left(1-\frac{2}{g\,(\lambda+1)+1}\right)^{\frac{1}{g}}\left(|\Psi''(\theta)|^h\frac{1}{s+1}+|\Psi''(\zeta)|^h\frac{s}{s+1}\right)^{\frac{1}{h}} (1{,}12)\end{aligned}$$

where $\frac{1}{g}+\frac{1}{h}=1$.

Theorem 1.16. [18] *All is same as in Theorem* 1.15, *then the inequality holds*

$$\begin{aligned}&\left|\frac{\Psi(\theta)+\Psi(m\zeta)}{2}-\frac{\Gamma(\lambda+1)}{2(m\zeta-\theta)^{\lambda}}\left[I_{\theta^+}^{\lambda}\Psi(m\zeta)+I_{m\zeta^-}^{\lambda}\Psi(\theta)\right]\right|\\ \leq\;& \frac{(m\zeta-\theta)^2}{2(\lambda+1)}\left[|\Psi''(\theta)|^h\left(\frac{1}{s+1}-\beta(s+1,h(\lambda+1)+1)-\frac{1}{h(\lambda+1)+s+1}\right)\right.\\ +\;& m|\Psi''(\zeta)|^h\left(1-\frac{1}{s+1}-\frac{2}{(h(\lambda+1)+1)}+\beta(s+1,h(\lambda+1)+1)\right.\\ +\;& \left.\left.\frac{1}{h(\lambda+1)+s+1}\right)\right]^{\frac{1}{h}}. \qquad (1.13)\end{aligned}$$

The following lemma is given in [5].

Lemma 1.17. *Let* $k>0$, $I_{\theta^+,k}^{\lambda}\Psi$ *and* $I_{\zeta^-,k}^{\lambda}\Psi$ *be defined in Definition* 2.1. *Let* $\Psi:[\theta,\zeta]\to\mathbb{R}$ *be a differentiable mapping on* (θ,ζ) *with* $\theta<\zeta$. *If* $\Psi'\in L[\theta,\zeta]$, *then*

$$\begin{aligned}&\frac{\Psi(\theta)+\Psi(\zeta)}{2}-\frac{\Gamma_k(\lambda+k)}{2(\zeta-\theta)^{\frac{\lambda}{k}}}[I_{\theta^+,k}^{\lambda}\Psi(\zeta)+I_{\zeta^-,k}^{\lambda}\Psi(\theta)]\\ =\;& \frac{(\zeta-\theta)}{2}\int_0^1\left[(1-\mu)^{\frac{\lambda}{k}}-\mu^{\frac{\lambda}{k}}\right]\Psi'(\mu\theta+(1-\mu)\zeta)d\mu. \qquad (1.14)\end{aligned}$$

2. Hermite-Hadamard inequalities for twice differentiable geometric-arithmetically s-convex function

Definition 2.1. *Let* $\Psi\in L_1[\theta,\zeta]$. *The* k*-Riemann-Liouville integrals* $I_{\theta^+,k}^{\lambda}\Psi$ *and* $I_{\zeta^-,k}^{\lambda}\Psi$ *of order* $\lambda>0$ *and* $k>0$ *with* $\theta\geq 0$ *are defined by*

$$I_{\theta^+,k}^{\lambda}\Psi(\xi)=\frac{1}{\Gamma_k(\lambda)}\int_\theta^\xi(\xi-\mu)^{\frac{\lambda}{k}-1}\Psi(\mu)d\mu, \qquad \xi>\theta$$

and

$$I^{\lambda}_{\zeta^-,k}\Psi(\xi)=\frac{1}{\Gamma_k(\lambda)}\int\limits_{\xi}^{\zeta}(\mu-\xi)^{\frac{\lambda}{k}-1}\Psi(\mu)d\mu, \qquad \xi<\zeta,$$

respectively. Here $\Gamma_k(\lambda)$ *is the Gamma function and* $I^0_{\theta^+,k}\Psi=I^0_{\zeta^-,k}\Psi=\Psi(\xi)$.

Lemma 2.2. *Let* $k>0$, $I^{\lambda}_{\theta^+,k}\Psi$ *and* $I^{\lambda}_{\zeta^-,k}\Psi$ *be defined in Definition* 2.1. *Let* $\Psi:[\theta,\zeta]\to\mathbb{R}$ *be twice differentiable on* (θ,ζ) *with* $\theta<\zeta$. *If* $\Psi''\in L[\theta,\zeta]$, *then*

$$\frac{\Psi(\theta)+\Psi(\zeta)}{2}-\frac{\Gamma_k(\lambda+k)}{2(\zeta-\theta)^{\frac{\alpha}{k}}}\left[I^{\lambda}_{\theta^+,k}\Psi(\zeta)+I^{\lambda}_{\zeta^-,k}\Psi(\theta)\right]$$
$$=\frac{(\zeta-\theta)^2}{2}\int\limits_0^1\frac{1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}}{\frac{\lambda}{k}+1}\Psi''(\mu\theta+(1-\mu)\zeta)d\mu. \tag{2.1}$$

Proof. We take Lemma 1.17, i.e.

$$\frac{\Psi(\theta)+\Psi(\zeta)}{2}-\frac{\Gamma_k(\lambda+k)}{2(\zeta-\theta)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\theta^+,k}\Psi(\zeta)+I^{\lambda}_{\zeta^-,k}\Psi(\theta)\right]$$
$$=\frac{\zeta-\theta}{2}\int\limits_0^1[(1-\mu)^{\frac{\lambda}{k}}-\mu^{\frac{\lambda}{k}}]\Psi'(\mu\theta+(1-\mu)\zeta)d\mu. \tag{2.2}$$

Now let us call

$$I = \int\limits_0^1[(1-\mu)^{\frac{\lambda}{k}}-\mu^{\frac{\lambda}{k}}]\Psi'(\mu\theta+(1-\mu)\zeta)d\mu \tag{2.3}$$
$$= \int\limits_0^1(1-\mu)^{\frac{\lambda}{k}}\Psi'(\mu\theta+(1-\mu)\zeta)d\mu-\int\limits_0^1\mu^{\frac{\lambda}{k}}\Psi'(\mu\theta+(1-\mu)\zeta)d\mu.$$

Integrating the integral I by parts, then we obtain

$$\begin{aligned} I &= -\frac{(1-\mu)^{\frac{\lambda}{k}+1}}{\frac{\lambda}{k}+1}\Psi'(\mu\theta+(1-\mu)\zeta)|_0^1+\int\limits_0^1\frac{\mu^{\frac{\lambda}{k}+1}}{\frac{\lambda}{k}+1}\Psi''(\mu\theta+(1-\mu)\zeta)(\theta-\zeta)d\mu \\ &\quad -\frac{\mu^{\frac{\lambda}{k}+1}}{\frac{\lambda}{k}+1}\Psi'(\mu\theta+(1-\mu)\zeta)|_0^1+\int\limits_0^1\frac{\mu^{\frac{\lambda}{k}+1}}{\frac{\lambda}{k}+1}\Psi''(\mu\theta+(1-\mu)\zeta)(\theta-\zeta)d\mu \\ &= \frac{\Psi'(\zeta)}{\frac{\lambda}{k}+1}-\frac{\zeta-\theta}{\frac{\lambda}{k}+1}\int\limits_0^1(1-\mu)^{\frac{\lambda}{k}+1}\Psi''(\mu\theta+(1-\mu)\zeta)d\mu \\ &\quad -\frac{\Psi'(\theta)}{\frac{\lambda}{k}+1}-\frac{\zeta-\theta}{\frac{\lambda}{k}+1}\int\limits_0^1\mu^{\frac{\lambda}{k}+1}\Psi''(\mu\theta+(1-\mu)\zeta)d\mu \\ &= \frac{\Psi'(\zeta)-\Psi'(\theta)}{\frac{\lambda}{k}+1}-\frac{\zeta-\theta}{\frac{\lambda}{k}+1}\int\limits_0^1[(1-\mu)^{\frac{\lambda}{k}}-\mu^{\frac{\lambda}{k}}]\Psi''(\mu\theta+(1-\mu)\zeta)d\mu, \end{aligned} \tag{2.4}$$

and

$$\Psi'(\zeta)-\Psi'(\theta)=\int_{\theta}^{\zeta}\Psi''(\xi)d\xi=(\zeta-\theta)\int_{0}^{1}\Psi''(\mu\theta+(1-\mu)\zeta)d\mu. \quad (2.5)$$

Submitting (2.5) to (2.4), we obtain

$$I = \frac{(\zeta-\theta)\int_0^1\Psi''(\mu\theta+(1-\mu)\zeta)d\mu}{\frac{\lambda}{k}+1}-\frac{\zeta-\theta}{\frac{\lambda}{k}+1}\int_0^1[(1-\mu)^{\frac{\lambda}{k}+1}+\mu^{\frac{\lambda}{k}+1}]\Psi''(\mu\theta+(1-\mu)\zeta)d\mu.$$

This implies that

$$\begin{aligned}&\int_0^1[(1-\mu)^{\frac{\lambda}{k}+1}+\mu^{\frac{\lambda}{k}+1}]\Psi'(\mu\theta+(1-\mu)\zeta)d\mu\\ &= (\zeta-\theta)\int_0^1\frac{1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}}{\frac{\lambda}{k}+1}\Psi''(\mu\theta+(1-\mu)\zeta)d\mu \quad (2.6)\end{aligned}$$

Multiplying $\frac{\zeta-\theta}{2}$ on both sides of equality (2.6), we get

$$\frac{\zeta-\theta}{2}\int_0^1[(1-\mu)^{\frac{\lambda}{k}+1}+\mu^{\frac{\lambda}{k}+1}]\Psi'(\mu\theta+(1-\mu)\zeta)d\mu \quad (2.7)$$

$$= \frac{(\zeta-\theta)^2}{2}\int_0^1\frac{1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}}{\frac{\lambda}{k}+1}\Psi''(\mu\theta+(1-\mu)\zeta)d\mu \quad (2.8)$$

Comparing (2.8) with (2.2), we get

$$\begin{aligned}&\frac{\Psi(\theta)+\Psi(\zeta)}{2}-\frac{\Gamma_k(\lambda+k)}{2(\zeta-\theta)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\theta^+,k}\Psi(\zeta)+I^{\lambda}_{\zeta^-,k}\Psi(\theta)\right]\\ &=\frac{(\zeta-\theta)^2}{2}\int_0^1\frac{1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}}{\frac{\lambda}{k}+1}\Psi''(\mu\theta+(1-\mu)\zeta)d\mu.\end{aligned}$$

Hence the required result is proved. □

Theorem 2.3. *Let $k>0$, $I^{\lambda}_{\theta^+,k}\Psi$ and $I^{\lambda}_{\zeta^-,k}\Psi$ be defined in Definition 2.1. Let $\Psi:[\theta,\zeta]\to\mathbb{R}$ be a differentiable function. If $|\Psi''|$ is measurable and $|\Psi''|$ is decreasing and geometric-arithmetically s-convex on $[\theta,\zeta]$ for some fixed $\lambda\in(0,\infty)$, $s\in(0,1]$ $0\le\theta<\zeta$, then*

$$\begin{aligned}&\left|\frac{\Psi(\theta)+\Psi(\zeta)}{2}-\frac{\Gamma_k(\lambda+k)}{2(\zeta-\theta)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\theta^+,k}\Psi(\zeta)+I^{\lambda}_{\zeta^-,k}\Psi(\theta)\right]\right|\\ &\le \frac{(\zeta-\theta)^2(|\Psi''(\theta)|+|\Psi''(\zeta)|)}{2(\frac{\lambda}{k}+1)}\left(\frac{1}{s+1}-\frac{1}{\frac{\lambda}{k}+s+2}\right).\end{aligned}$$

Proof. By using Definition 1.5, Lemma 1.6 and Lemma 2.2, we have

$$\begin{aligned}
&\left|\frac{\Psi(\theta)+\Psi(\zeta)}{2}-\frac{\Gamma_k(\lambda+k)}{2(\zeta-\theta)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\theta+,k}\Psi(\zeta)+I^{\lambda}_{\zeta-,k}\Psi(\theta)\right]\right|\\
\leq\;& \frac{(\zeta-\theta)^2}{2}\int_0^1\left|\frac{1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}}{\frac{\lambda}{k}+1}\right||\Psi''(\mu\theta+(1-\mu)\zeta)|d\mu\\
=\;& \frac{(\zeta-\theta)^2}{2(\frac{\lambda}{k}+1)}\int_0^1\left(1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}\right)|\Psi''(\mu\theta+(1-\mu)\zeta)|\,d\mu\\
\leq\;& \frac{(\zeta-\theta)^2}{2(\frac{\lambda}{k}+1)}\int_0^1\left(1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}\right)\left|\Psi''(\theta^{\mu}\zeta^{1-\mu})\right|d\mu\\
\leq\;& \frac{(\zeta-\theta)^2}{2(\frac{\lambda}{k}+1)}\int_0^1\left(1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}\right)\left[\mu^s|\Psi''(\theta)|+(1-\mu)^s|\Psi''(\zeta)|\right]d\mu\\
=\;& \frac{(\zeta-\theta)^2}{2(\frac{\lambda}{k}+1)}\Bigg[\int_0^1\left[\mu^s|\Psi''(\theta)|+(1-\mu)^s|\Psi''(\zeta)|\right]d\mu-\int_0^1\Big[\mu^s(1-\mu)^{\frac{\lambda}{k}+1}|\Psi''(\theta)|\\
+\;& (1-\mu)^{\frac{\lambda}{k}+s+1}|\Psi''(\zeta)|\Big]d\mu-\int_0^1\left[\mu^{\frac{\lambda}{k}+s+1}|\Psi''(\theta)|+\mu^{\frac{\lambda}{k}+1}(1-\mu)^s|\Psi''(\zeta)|\right]d\mu\Bigg].
\end{aligned}$$

By using the definition of beta function, then we have

$$\begin{aligned}
&\left|\frac{\Psi(\theta)+\Psi(\zeta)}{2}-\frac{\Gamma_k(\lambda+k)}{2(\zeta-\theta)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\theta+,k}\Psi(\zeta)+I^{\lambda}_{\zeta-,k}\Psi(\theta)\right]\right|\\
\leq\;& \frac{(\zeta-\theta)^2}{2}\int_0^1\left|\frac{1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}}{\frac{\lambda}{k}+1}\right||\Psi''(\mu\theta+(1-\mu)\zeta)|d\mu\\
\leq\;& \frac{(\zeta-\theta)^2|\Psi''(\theta)|}{2(\frac{\lambda}{k}+1)(s+1)}+\frac{(\zeta-\theta)^2|\Psi''(\zeta)|}{2(\frac{\lambda}{k}+1)(s+1)}-\frac{(\zeta-\theta)^2|\Psi''(\theta)|}{2(\frac{\lambda}{k}+1)(\frac{\lambda}{k}+s+2)}-\frac{(\zeta-\theta)^2|\Psi''(\zeta)|}{2(\frac{\lambda}{k}+1)(s+1)}\\
&\beta(s+1,\frac{\lambda}{k}+2)-\frac{(\zeta-\theta)^2|\Psi''(\theta)|}{2(\frac{\lambda}{k}+1)}\beta(s+1,\frac{\lambda}{k}+2)-\frac{(\zeta-\theta)^2|\Psi''(\zeta)|}{2(\frac{\lambda}{k}+1)(\frac{\lambda}{k}+s+2)}\\
\leq\;& \frac{(\zeta-\theta)^2(|\Psi''(\theta)|+|\Psi''(\zeta)|)}{2(\frac{\lambda}{k}+1)}\left(\frac{1}{s+1}-\frac{1}{\frac{\lambda}{k}+s+2}\right).
\end{aligned}$$

It implies that

$$\begin{aligned}
&\left|\frac{\Psi(\theta)+\Psi(\zeta)}{2}-\frac{\Gamma_k(\lambda+k)}{2(\zeta-\theta)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\theta+,k}\Psi(\zeta)+I^{\lambda}_{\zeta-,k}\Psi(\theta)\right]\right|\\
\leq\;& \frac{(\zeta-\theta)^2(|\Psi''(\theta)|+|\Psi''(\zeta)|)}{2(\frac{\lambda}{k}+1)}\left(\frac{1}{s+1}-\frac{1}{\frac{\lambda}{k}+s+2}\right).
\end{aligned}$$

So the required proof is completed. □

Theorem 2.4. *Let $k>0$, $I^{\lambda}_{\theta+,k}\Psi$ and $I^{\lambda}_{\zeta-,k}$ be defined in Definition 2.1. Let $\Psi:[\theta,\zeta]\to\mathbb{R}$ be differentiable. If $|\Psi''|^h$ is measurable and $|\Psi''|^h$ is decreasing and geometric-arithmetically s-convex on $[\theta,\zeta]$ for some fixed $\lambda\in(0,\infty)$, $s\in(0,1]$ $0\le\theta<\zeta$, then*

$$\left|\frac{\Psi(\theta)+\Psi(\zeta)}{2}-\frac{\Gamma_k(\lambda+k)}{2(\zeta-\theta)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\theta+,k}\Psi(\zeta)+I^{\lambda}_{\zeta-,k}\Psi(\theta)\right]\right|$$
$$\le\quad \frac{(\zeta-\theta)^2\max\{1-2^{1-\frac{\lambda}{k}},2^{1-\frac{\lambda}{k}}-1\}}{2(\frac{\lambda}{k}+1)}\left(\frac{|\Psi''(\theta)|^h+|\Psi''(\zeta)|^h}{s+1}\right)^{\frac{1}{h}}.$$

Proof. We shall solve it in two cases for our aim.
Case 1: $\lambda\in(0,1)$, using Holder's inequality, Definition 1.5, Lemma 1.6, Lemma 1.7 and Lemma 2.2, we have

$$\begin{aligned}
&\left|\frac{\Psi(\theta)+\Psi(\zeta)}{2}-\frac{\Gamma_k(\lambda+k)}{2(\zeta-\theta)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\theta+,k}\Psi(\zeta)+I^{\lambda}_{\zeta-,k}\Psi(\theta)\right]\right|\\
\le\quad&\frac{(\zeta-\theta)^2}{2}\int_0^1\left|\frac{1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}}{\frac{\lambda}{k}+1}\right||\Psi''(\mu\theta+(1-\mu)\zeta)|d\mu\\
\le\quad&\frac{(\zeta-\theta)^2}{2(\frac{\lambda}{k}+1)}\left(\int_0^1\left|1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}\right|^g d\mu\right)^{\frac{1}{g}}\left(\int_0^1|\Psi''(\mu\theta+(1-\mu)\zeta)|^h d\mu\right)^{\frac{1}{h}}\\
\le\quad&\frac{(\zeta-\theta)^2}{2(\frac{\lambda}{k})}\left(\int_0^1\left|1-(1-\mu)^{\frac{\lambda}{k}}-\mu^{\frac{\lambda}{k}}\right|^g d\mu\right)^{\frac{1}{g}}\left(\int_0^1|\Psi''\theta^{\mu}\zeta^{1-\mu}|^h d\mu\right)^{\frac{1}{h}}\\
\le\quad&\frac{(\zeta-\theta)^2}{2(\frac{\lambda}{k}+1)}\left(\int_0^1\left|1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}}\right|^g d\mu\right)^{\frac{1}{g}}\left(\int_0^1[\mu^s|\Psi''(\theta)|^h+(1-\mu)^s|\Psi''(\zeta)|^h]d\mu\right)^{\frac{1}{h}}\\
=\quad&\frac{(\zeta-\theta)^2}{2(\frac{\lambda}{k}+1)}\left(\frac{|\Psi''(\theta)|^h}{s+1}+\frac{|\Psi''(\zeta)|^h}{s+1}\right)^{\frac{1}{h}}\left(\int_0^1[(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}}-1]^g d\mu\right)^{\frac{1}{g}}\\
\le\quad&\frac{(\zeta-\theta)^2}{2(\frac{\lambda}{k}+1)}\left(\frac{|\Psi''(\theta)|^h+|\Psi''(\zeta)|^h}{s+1}\right)^{\frac{1}{h}}\left(\int_0^1(2^{1-\frac{\lambda}{k}}-1)^g d\mu\right)^{\frac{1}{g}}\\
=\quad&\frac{(\zeta-\theta)^2}{2(\frac{\lambda}{k}+1)}\left(\frac{|\Psi''(\theta)|^h+|\Psi''(\zeta)|^h}{s+1}\right)^{\frac{1}{h}}\left(2^{1-\frac{\lambda}{k}}-1\right),
\end{aligned}\tag{2.9}$$

where $\frac{1}{g}+\frac{1}{h}=1$

Case 2: $\lambda\in[1,\infty)$. By using the Definition 1.5, Lemma 1.6, Lemma 1.7, Holder's inequality and Lemma 2.2, we have

$$\begin{aligned} & \left| \frac{\Psi(\theta)+\Psi(\zeta)}{2} - \frac{\Gamma_k(\lambda+k)}{2(\zeta-\theta)^{\frac{\lambda}{k}}} \left[I^{\lambda}_{\theta^+,k}\Psi(\zeta) + I^{\lambda}_{\zeta^-,k}\Psi(\theta) \right] \right| \\ \leq \quad & \frac{(\zeta-\theta)^2}{2(\frac{\lambda}{k}+1)} \left(\frac{|\Psi''(\theta)|^h + |\Psi''(\zeta)|^h}{s+1} \right)^{\frac{1}{h}} \left(\int_0^1 (1-2^{1-\frac{\lambda}{k}})^g d\mu \right)^{\frac{1}{g}} \\ = \quad & \frac{(\zeta-\theta)^2}{2(\frac{\lambda}{k}+1)} \left(\frac{|\Psi''(\theta)|^h + |\Psi''(\zeta)|^h}{s+1} \right)^{\frac{1}{h}} \left(1-2^{1-\frac{\lambda}{k}} \right). \end{aligned} \tag{2.10}$$

Now from (2.9) and (2.10), we obtain the required result. □

Lemma 2.5. *Let $k > 0$, $I^{\lambda}_{\theta^+,k}\Psi$ and $I^{\lambda}_{\zeta^-,k}\Psi$ be defined in Definition 2.1. Let $\Psi : [\theta, \zeta] \to \mathbb{R}$ be twice differentiable on (θ, ζ) with $\theta < \zeta$. If $\Psi'' \in L[\theta, \zeta]$, then*

$$\begin{aligned} & \frac{\Gamma_k(\lambda+k)}{2(\zeta-\theta)^{\frac{\lambda}{k}}} \left[I^{\lambda}_{\theta^+,k}\Psi(\zeta) + I^{\lambda}_{\zeta^-,k}\Psi(\theta) \right] - \Psi\left(\frac{\theta+\zeta}{2}\right) \\ = \quad & \frac{(\zeta-\theta)^2}{2} \int_0^1 m(\mu)\Psi''(\mu\theta+(1-\mu)\zeta)d\mu, \end{aligned} \tag{2.11}$$

where

$$m(\mu) = \begin{cases} \mu - \frac{1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}}{(\frac{\lambda}{k}+1)}, & \mu \in [0, \frac{1}{2}), \\ (1-\mu) - \frac{1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}}{(\frac{\lambda}{k}+1)}, & \mu \in [\frac{1}{2}, 1). \end{cases}$$

Proof. we have

$$\begin{aligned} & \frac{\Gamma_k(\lambda+k)}{2(\zeta-\theta)^{\frac{\lambda}{k}}} \left[I^{\lambda}_{\theta^+,k}\Psi(\zeta) + I^{\lambda}_{\zeta^-,k}\Psi(\theta) \right] - \Psi\left(\frac{\theta+\zeta}{2}\right) \\ = \quad & \frac{(\zeta-\theta)^2}{2} \int_0^1 m(\mu)\Psi''(\mu\theta+(1-\mu)\zeta)d\mu \qquad (2.12) \\ = \quad & \int_0^{\frac{1}{2}} \left(\mu - \frac{1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}}{\frac{\lambda}{k}+1} \right) \Psi''(\mu\theta+(1-\mu)\zeta)d\mu \\ & + \int_{\frac{1}{2}}^1 \left(1-\mu - \frac{1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}}{\frac{\lambda}{k}+1} \right) \Psi''(\mu\theta+(1-\mu)\zeta)d\mu \\ = \quad & \int_0^{\frac{1}{2}} \mu\Psi''(\mu\theta+(1-\mu)\zeta)d\mu - \int_0^{\frac{1}{2}} \left(\frac{1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}}{\frac{\lambda}{k}+1} \right) \Psi''(\mu\theta+(1-\mu)\zeta)d\mu \\ & + \int_{\frac{1}{2}}^1 (1-\mu)\Psi''(\mu\theta+(1-\mu)\zeta)d\mu + \int_{\frac{1}{2}}^1 \left(\frac{1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}}{\frac{\lambda}{k}+1} \right) \Psi''(\mu\theta+(1-\mu)\zeta)d\mu \end{aligned}$$

$$= \int_0^{\frac{1}{2}} \mu\Psi''(\mu\theta+(1-\mu)\zeta)d\mu + \int_{\frac{1}{2}}^1 (1-\mu)\Psi''(\mu\theta+(1-\mu)\zeta)d\mu$$
$$- \int_0^1 \left(\frac{1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}}{\frac{\lambda}{k}+1}\right)\Psi''(\mu\theta+(1-\mu)\zeta)d\mu. \tag{2.13}$$

Let us say that

$$I = \int_0^{\frac{1}{2}} \mu\Psi''(\mu\theta+(1-\mu)\zeta)d\mu + \int_{\frac{1}{2}}^1 (1-\mu)\Psi''(\mu\theta+(1-\mu)\zeta)d\mu. \tag{2.14}$$

Now we take

$$I_1 = \int_0^{\frac{1}{2}} \mu\Psi''(\mu\theta+(1-\mu)\zeta)d\mu.$$

Integrating the integral I_1 by parts, we get

$$I_1 = \frac{\Psi'(\frac{\theta+\zeta}{2})}{2(\theta-\zeta)} - \frac{\left[\Psi(\frac{\theta+\zeta}{2})-\Psi(\zeta)\right]}{(\theta-\zeta)^2}. \tag{2.15}$$

Similarly, we say that

$$I_2 = \int_{\frac{1}{2}}^1 (1-\mu)\Psi''(\mu\theta+(1-\mu)\zeta)d\mu,$$

again integrating the integral I_2 by parts, we get

$$I_2 = -\frac{\Psi'(\frac{\theta+\zeta}{2})}{2(\theta-\zeta)} + \frac{\left[-\Psi(\frac{\theta+\zeta}{2})+\Psi(\theta)\right]}{(\theta-\zeta)^2}. \tag{2.16}$$

Submitting (2.15) and (2.16) to (2.14), it holds

$$I = \frac{\Psi(\theta)+\Psi(\zeta)}{(\zeta-\theta)^2} - \frac{2\Psi\left(\frac{\theta+\zeta}{2}\right)}{(\zeta-\theta)^2}. \tag{2.17}$$

Thus by multiplying both sides of (2.12) by $\frac{(\zeta-\theta)^2}{2}$, we have

$$\frac{(\zeta-\theta)^2}{2}\int_0^1 m(\mu)\Psi''(\mu\theta+(1-\mu)\zeta)d\mu$$
$$= \frac{\Psi(\theta)+\Psi(\zeta)}{2} - \Psi\left(\frac{\theta+\zeta}{2}\right)$$
$$-\frac{(\zeta-\theta)^2}{2}\int_0^1 \left(\frac{1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}}{\frac{\lambda}{k}+1}\right)\Psi''(\mu\theta+(1-\mu)\zeta)d\mu. \tag{2.18}$$

On the other hand, we have

$$\frac{(\zeta-\theta)^2}{2}\int_0^1 \left(\frac{1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}}{\frac{\lambda}{k}+1}\right)\Psi''(\mu\theta+(1-\mu)\zeta)d\mu$$
$$=\frac{\Psi(\theta)+\Psi(\zeta)}{2}-\frac{\Gamma_k(\lambda+k)}{2(\zeta-\theta)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\theta+,k}\Psi(\zeta)+I^{\lambda}_{\zeta-,k}\Psi(\theta)\right]. \tag{2.19}$$

So combining (2.18) and (2.19), we obtain (2.11)

$$\frac{\Gamma_k(\lambda+k)}{2(\zeta-\theta)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\theta+,k}\Psi(\zeta)+I^{\lambda}_{\zeta-,k}\Psi(\theta)\right]-\Psi\left(\frac{\theta+\zeta}{2}\right)$$
$$=\frac{(\zeta-\theta)^2}{2}\int_0^1 m(\mu)\Psi''(\mu\theta+(1-\mu)\zeta)d\mu.$$

Hence proved. □

Theorem 2.6. *Let $k>0$, $I^{\lambda}_{\theta+,k}\Psi$ and $I^{\lambda}_{\zeta-,k}\Psi$ be defined in the Definition 2.1. Let $\Psi:[0,\zeta]\to\mathbb{R}$ be differentiable function. $|\Psi''|$ is measurable and $1<h<\infty$. If $|\Psi''|^h$ is measurable and $|\Psi''|$ is decreasing and geometric-arithmetically s-convex on $[0,\zeta]$ for some fixed $\lambda\in(0,\infty)$, $s\in(0,1]$ $1\le\theta<\zeta$ then*

$$\frac{\Gamma_k(\lambda+k)}{2(\zeta-\theta)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\theta+,k}\Psi(\zeta)+I^{\lambda}_{\zeta-,k}\Psi(\theta)\right]-\Psi\left(\frac{\theta+\zeta}{2}\right)$$
$$\le\frac{(\zeta-\theta)^2}{2(\frac{\lambda}{k}+1)}\left(\frac{|\Psi''(\theta)|^h+|\Psi''(\zeta)|^h}{s+1}\right)\left(\frac{(\frac{\lambda}{k}+1)2^{-g-1}+(\frac{\lambda}{k}+0.5)^{g+1}-(\frac{\lambda}{k})^{g+1}}{g+1}\right)^{\frac{1}{g}},$$

where $\frac{1}{g}+\frac{1}{h}=1$.

Proof. By using Holder's inequality, Lemma 1.6, Definition 1.5 and Lemma 2.5, we have

$$\frac{\Gamma_k(\lambda+k)}{2(\zeta-\theta)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\theta+,k}\Psi(\zeta)+I^{\lambda}_{\zeta-,k}\Psi(\theta)\right]-\Psi\left(\frac{\theta+\zeta}{2}\right)$$
$$\le\frac{(\zeta-\theta)^2}{2}\int_0^1 m(\mu)\Psi''(\mu\theta+(1-\mu)\zeta)d\mu$$
$$\le\frac{(\zeta-\theta)^2}{2}\left(\int_0^1|m(\mu)|^g d\mu\right)^{\frac{1}{g}}\left(\int_0^1|\Psi''(\mu\theta+(1-\mu)\zeta)|^h d\mu\right)^{\frac{1}{h}}$$
$$\le\frac{(\zeta-\theta)^2}{2}\left(\int_0^1|m(\mu)|^g d\mu\right)^{\frac{1}{g}}\left(\int_0^1|\Psi''(\theta^{\mu}\zeta^{1-\mu})|^h d\mu\right)^{\frac{1}{h}}$$

$$\begin{aligned}
&\leq \frac{(\zeta-\theta)^2}{2}\left(\int_0^1 |m(\mu)|^g d\mu\right)^{\frac{1}{g}}\left(\int [\mu^s|\Psi''(\theta)|^h+(1-\mu)^s|\Psi''(\zeta)|^h]d\mu\right)^{\frac{1}{h}}\\
&= \frac{(\zeta-\theta)^2}{2}\left(\int_0^1 |m(\mu)|^g d\mu\right)^{\frac{1}{g}}\left(\frac{|\Psi''(\theta)|^h+|\Psi''(\zeta)|^h}{s+1}\right)^{\frac{1}{h}}\\
&= \frac{(\zeta-\theta)^2}{2}\left(\frac{|\Psi''(\theta)|^h+|\Psi''(\zeta)|^h}{s+1}\right)^{\frac{1}{h}}\left[\left(\int_0^{\frac{1}{2}}\left|\left(\mu-\frac{1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}}{\frac{\lambda}{k}+1}\right)\right|^g d\mu\right)\right.\\
&\quad\left.+\left(\int_{\frac{1}{2}}^1\left|\left(1-\mu-\frac{1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}}{\frac{\lambda}{k}+1}\right)\right|^g d\mu\right)\right]^{\frac{1}{g}}\\
&= \frac{(\zeta-\theta)^2}{2(\frac{\lambda}{k}+1)}\left(\frac{|\Psi''(\theta)|^h+|\Psi''(\zeta)|^h}{s+1}\right)^{\frac{1}{h}}\left[\left(\int_0^{\frac{1}{2}}\left|\left(\left(\frac{\lambda}{k}+1\right)\mu-1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}\right)\right|^g d\mu\right)\right.\\
&\quad\left.+\left(\int_{\frac{1}{2}}^1\left|\left(\frac{\lambda}{k}+1-\left(\frac{\lambda}{k}+1\right)\mu-1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}\right)\right|^g d\mu\right)\right]^{\frac{1}{g}}\\
&= \frac{(\zeta-\theta)^2}{2(\frac{\lambda}{k}+1)}\left(\frac{|\Psi''(\theta)|^h+|\Psi''(\zeta)|^h}{s+1}\right)^{\frac{1}{h}}\left(\int_0^{\frac{1}{2}}\left(\frac{\lambda}{k}+1\right)\mu^g d\mu+\int_{\frac{1}{2}}^1\left(\frac{\lambda}{k}-\mu+1\right)^g d\mu\right)^{\frac{1}{g}}\\
&= \frac{(\zeta-\theta)^2}{2(\frac{\lambda}{k}+1)}\left(\frac{|\Psi''(\theta)|^h+|\Psi''(\zeta)|^h}{s+1}\right)^{\frac{1}{h}}\left(\left(\frac{\lambda}{k}+1\right)\frac{\mu^{g+1}}{g+1}\Big|_0^{\frac{1}{2}}-\frac{\left(\frac{\lambda}{k}-\mu+1\right)^{g+1}}{g+1}\Big|_{\frac{1}{2}}^1\right)^{\frac{1}{g}}\\
&= \frac{(\zeta-\theta)^2}{2(\frac{\lambda}{k}+1)}\left(\frac{|\Psi''(\theta)|^h+|\Psi''(\zeta)|^h}{s+1}\right)^{\frac{1}{h}}\left(\frac{\left(\frac{\lambda}{k}+1\right)(0.5)^{g+1}}{g+1}-\frac{(\frac{\lambda}{k})^{g+1}}{g+1}+\frac{(\frac{\lambda}{k}+0.5)^{g+1}}{g+1}\right)^{\frac{1}{g}}\\
&= \frac{(\zeta-\theta)^2}{2(\frac{\lambda}{k}+1)}\left(\frac{|\Psi''(\theta)|^h+|\Psi''(\zeta)|^h}{s+1}\right)^{\frac{1}{h}}\left(\frac{\left(\frac{\lambda}{k}+1\right)2^{-g-1}-(\frac{\lambda}{k})^{g+1}+\left(\frac{\lambda}{k}+0.5\right)^{g+1}}{g+1}\right)^{\frac{1}{g}}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\frac{\Gamma_k(\lambda+k)}{2(\zeta-\theta)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\theta^+,k}\Psi(\zeta)+I^{\lambda}_{\zeta^-,k}\Psi(\theta)\right]-\Psi\left(\frac{\theta+\zeta}{2}\right)\\
&\leq \frac{(\zeta-\theta)^2}{2(\frac{\lambda}{k}+1)}\left(\frac{|\Psi''(\theta)|^h+|\Psi''(\zeta)|^h}{s+1}\right)\left(\frac{(\frac{\lambda}{k}+1)2^{-g-1}+(\frac{\lambda}{k}+0.5)^{g+1}-(\frac{\lambda}{k})^{g+1}}{g+1}\right)^{\frac{1}{g}}.
\end{aligned}$$

This is required result. □

3. Hermite-Hadamard-type inequalities for m convex functions

Theorem 3.1. *Let $k>0$, $I^{\lambda}_{\theta+,k}\Psi$ and $I^{\lambda}_{\zeta-,k}\Psi$ be defined in Definition 2.1. Let $\Psi:[0,\zeta^*]\to\mathbb{R}$ be a twice differentiable function with $\zeta^*>0$. If $|\Psi''|^h$ is measurable and m-convex on $[\theta,\frac{\zeta}{m}]$ for some fixed $h\geq 1$, $0\leq\theta<\zeta$ and $m\in(0,1]$ with $\frac{\zeta}{m}\leq\zeta^*$, then*

$$\left|\frac{\Psi(\theta)+\Psi(\zeta)}{2}-\frac{\Gamma_k(\lambda+k)}{2(\zeta-\theta)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\theta+,k}\Psi(\zeta)+I^{\lambda}_{\zeta-,k}\Psi(\theta)\right]\right|$$
$$\leq\frac{\frac{\lambda}{k}(\zeta-\theta)^2}{2\left(\frac{\lambda}{k}+1\right)\left(\frac{\lambda}{k}+2\right)}\left[\frac{|\Psi''(\theta)|^h+m|\Psi''(\frac{\zeta}{m})|^h}{2}\right]^{\frac{1}{h}}.$$

Proof. Firstly, we suppose that $h=1$. From Lemma 2.2, we have

$$\left|\frac{\Psi(\theta)+\Psi(\zeta)}{2}-\frac{\Gamma_k(\lambda+k)}{2(\zeta-\theta)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\theta+,k}\Psi(\zeta)+I^{\lambda}_{\zeta-,k}\Psi(\theta)\right]\right|$$
$$\leq\frac{(\zeta-\theta)^2}{2}\int_0^1\frac{1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}}{\frac{\lambda}{k}+1}|\Psi''(\mu\theta+(1-\mu)\zeta)|d\mu, \tag{3.1}$$

because $(1-\mu)^{\frac{\lambda}{k}+1}+\mu^{\frac{\lambda}{k}+1}\leq 1$ for any $\mu\in[\theta,\zeta]$. Since $|\Psi''|$ is m-convex on $[\theta,\frac{\zeta}{m}]$, we know that for any $\mu\in[0,1]$,

$$|\Psi''(\mu\theta+(1-\mu)\zeta)|\leq\mu|\Psi''(\theta)|+m(1-\mu)|\Psi''(\frac{\zeta}{m})|.$$

Therefore,

$$\left|\frac{\Psi(\theta)+\Psi(\zeta)}{2}-\frac{\Gamma_k(\lambda+k)}{2(\zeta-\theta)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\theta+,k}\Psi(\zeta)+I^{\lambda}_{\zeta-,k}\Psi(\theta)\right]\right|$$
$$\leq\frac{(\zeta-\theta)^2}{2}\int_0^1\frac{1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}}{\frac{\lambda}{k}+1}\left(\mu|\Psi''(\theta)|+m(1-\mu)|\Psi''(\frac{\zeta}{m})|\right)d\mu$$
$$\leq\frac{(\zeta-\theta)^2|\Psi''(\theta)|}{2\left(\frac{\lambda}{k}+1\right)}\int_0^1\mu\left[1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}\right]d\mu$$
$$+\frac{m(\zeta-\theta)^2|\Psi''(\frac{\zeta}{m})|}{2\left(\frac{\lambda}{k}+1\right)}\int_0^1(1-\mu)\left[1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}\right]d\mu$$

$$
\begin{aligned}
= \quad & \frac{(\zeta-\theta)^2|\Psi''(\theta)|}{2\left(\frac{\lambda}{k}+1\right)}\left[\int_0^1 \mu d\mu - \int_0^1 \mu(1-\mu)^{\frac{\lambda}{k}+1} - \int_0^1 \mu^{\frac{\lambda}{k}+2}\right] \\
& + \frac{m(\zeta-\theta)^2|\Psi''(\frac{\zeta}{m})|}{2\left(\frac{\lambda}{k}+1\right)}\left[\int_0^1 (1-\mu)d\mu - \int_0^1 (1-\mu)^{\frac{\lambda}{k}+2}d\mu \int_0^1 \mu^{\frac{\lambda}{k}+1}d\mu + \int_0^1 \mu^{\frac{\lambda}{k}+2}d\mu\right] \\
= \quad & \frac{(\zeta-\theta)^2|\Psi''(\theta)|}{2\left(\frac{\lambda}{k}+1\right)}\left[\int_0^1 \mu d\mu - \int_0^1 (1-1+\mu)(1-\mu)^{\frac{\lambda}{k}+1} - \int_0^1 \mu^{\frac{\lambda}{k}+2}\right] \\
& + \frac{m(\zeta-\theta)^2|\Psi''(\frac{\zeta}{m})|}{2\left(\frac{\lambda}{k}+1\right)}\left[\int_0^1 (1-\mu)d\mu - \int_0^1 (1-\mu)^{\frac{\lambda}{k}+2}d\mu \int_0^1 \mu^{\frac{\lambda}{k}+1}d\mu + \int_0^1 \mu^{\frac{\lambda}{k}+2}d\mu\right] \\
= \quad & \frac{(\zeta-\theta)^2|\Psi''(\theta)|}{2\left(\frac{\lambda}{k}+1\right)}\left[\int_0^1 \mu d\mu + \int_0^1 (1-\mu)^{\frac{\lambda}{k}+2} - \int_0^1 (1-\mu)^{\frac{\lambda}{k}+1} \int_0^1 \mu^{\frac{\lambda}{k}+2}\right] \\
& + \frac{m(\zeta-\theta)^2|\Psi''(\frac{\zeta}{m})|}{2\left(\frac{\lambda}{k}+1\right)}\left[\int_0^1 (1-\mu)d\mu - \int_0^1 (1-\mu)^{\frac{\lambda}{k}+2}d\mu \int_0^1 \mu^{\frac{\lambda}{k}+1}d\mu + \int_0^1 \mu^{\frac{\lambda}{k}+2}d\mu\right] \\
= \quad & \frac{(\zeta-\theta)^2|\Psi''(\theta)|}{2\left(\frac{\lambda}{k}+1\right)}\left[\frac{1}{2} - \frac{1}{\frac{\lambda}{k}+3} - \frac{1}{\frac{\lambda}{k}+2} + \frac{1}{\frac{\lambda}{k}+3}\right] \\
& + \frac{m(\zeta-\theta)^2|\Psi''(\frac{\zeta}{m})|}{2\left(\frac{\lambda}{k}+1\right)}\left[\frac{1}{2} - \frac{1}{\frac{\lambda}{k}+3} - \frac{1}{\frac{\lambda}{k}+2} + \frac{1}{\frac{\lambda}{k}+3}\right] \\
= \quad & \frac{(\zeta-\theta)^2|\Psi''(\theta)|}{2\left(\frac{\lambda}{k}+1\right)}\left[\frac{\frac{\lambda}{k}}{2\left(\frac{\lambda}{k}+2\right)}\right] + \frac{m(\zeta-\theta)^2|\Psi''(\frac{\zeta}{m})|}{2\left(\frac{\lambda}{k}+1\right)}\left[\frac{\frac{\lambda}{k}}{2\left(\frac{\lambda}{k}+2\right)}\right] \\
= \quad & \frac{\frac{\lambda}{k}(\zeta-\theta)^2}{2\left(\frac{\lambda}{k}+1\right)\left(\frac{\lambda}{k}+2\right)}\left[\frac{|\Psi''(\theta)| + m|\Psi''(\frac{\zeta}{m})|}{2}\right],
\end{aligned}
$$

which complete the proof for this case.
Secondly, we suppose that $h > 1$. Using Lemma 2.2 and power mean inequality for h, we obtain

$$
\begin{aligned}
& \int_0^1 \left(1-(1-\mu)^{\frac{\lambda}{k}+1} - \mu^{\frac{\lambda}{k}+1}\right)|\Psi''(\mu\theta+(1-\mu)\zeta)|d\mu \\
\leq \quad & \left(\int_0^1 \left(1-(1-\mu)^{\frac{\lambda}{k}+1} - \mu^{\frac{\lambda}{k}+1}\right)d\mu\right)^{1-\frac{1}{h}} \\
& \times \left(\int_0^1 \left(1-(1-\mu)^{\frac{\lambda}{k}+1} - \mu^{\frac{\lambda}{k}+1}\right)|\Psi''(\mu\theta+(1-\mu)\zeta)|^h d\mu\right)^{\frac{1}{h}}. \qquad (3.2)
\end{aligned}
$$

Since $|\Psi''|$ is m-convex on $[\theta, \frac{\zeta}{m}]$, for any $\mu \in [0,1]$

$$|\Psi''(\mu\theta+(1-\mu)\zeta)|^h \le \mu|\Psi''(\theta)|^h + m(1-\mu)|\Psi''(\frac{\zeta}{m})|^h. \tag{3.3}$$

Hence from (3.1), (3.2) and (3.3), we obtain

$$\begin{aligned}
&\left|\frac{\Psi(\theta)+\Psi(\zeta)}{2} - \frac{\Gamma_k(\lambda+k)}{2(\zeta-\theta)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\theta^+,k}\Psi(\zeta)+I^{\lambda}_{\zeta^-,k}\Psi(\theta)\right]\right| \\
\le\ & \frac{(\zeta-\theta)^2}{2\left(\frac{\lambda}{k}+1\right)}\left(\int_0^1\left(1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}\right)\right)^{1-\frac{1}{h}} \\
&\times\left(\int_0^1\left(1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}\right)|\Psi''(\mu\theta+(1-\mu)\zeta)|^h\right)^{\frac{1}{h}},
\end{aligned} \tag{3.4}$$

this implies that

$$\begin{aligned}
&\left|\frac{\Psi(\theta)+\Psi(\zeta)}{2} - \frac{\Gamma_k(\lambda+k)}{2(\zeta-\theta)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\theta^+,k}\Psi(\zeta)+I^{\lambda}_{\zeta^-,k}\Psi(\theta)\right]\right| \\
\le\ & \frac{(\zeta-\theta)^2}{2\left(\frac{\lambda}{k}+1\right)}\left(\int_0^1\left(1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}\right)\right)^{1-\frac{1}{h}} \\
&\times\left(\int_0^1\left(1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}\right)\left[\mu|\Psi''(\theta)|^h+m(1-\mu)|\Psi''(\frac{\zeta}{m})|^h\right]\right)^{\frac{1}{h}} \\
\le\ & \frac{(\zeta-\theta)^2}{2\left(\frac{\lambda}{k}+1\right)}\left(\frac{\frac{\lambda}{k}}{\frac{\lambda}{k}+2}\right)^{1-\frac{1}{h}}\left(|\Psi''(\theta)|^h\left(\int_0^1 \mu-(1-1+\mu)(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+2}\right)d\mu\right. \\
&\left.\times m|\Psi''(\frac{\zeta}{m})|^h\left(\int_0^1(1-\mu)-(1-\mu)^{\frac{\lambda}{k}+2}-(1-\mu)\mu^{\frac{\lambda}{k}+1}\right)d\mu\right)^{\frac{1}{h}} \\
\le\ & \frac{(\zeta-\theta)^2}{2\left(\frac{\lambda}{k}+1\right)}\left(\frac{\frac{\lambda}{k}}{\frac{\lambda}{k}+2}\right)^{1-\frac{1}{h}}\left(|\Psi''(\theta)|^h\left(\frac{1}{2}+\frac{1}{\frac{\lambda}{k}+3}-\frac{1}{\frac{\lambda}{k}+2}-\frac{1}{\frac{\lambda}{k}+3}\right)\right. \\
&\left.\times m|\Psi''(\frac{\zeta}{m})|^h\left(\frac{1}{2}+\frac{1}{\frac{\lambda}{k}+3}-\frac{1}{\frac{\lambda}{k}+2}-\frac{1}{\frac{\lambda}{k}+3}\right)\right)^{\frac{1}{h}} \\
=\ & \frac{\frac{\lambda}{k}(\zeta-\theta)^2}{2\left(\frac{\lambda}{k}+1\right)\left(\frac{\lambda}{k}+2\right)}\left(\frac{|\Psi''(\theta)|^h+m|\Psi''(\frac{\zeta}{m})|^h}{2}\right)^{\frac{1}{h}},
\end{aligned}$$

which completes the proof for this case. □

Theorem 3.2. *With the same conditions of Theorem* 3.1, *then*

$$\left|\frac{\Psi(\theta)+\Psi(\zeta)}{2}-\frac{\Gamma_k(\lambda+k)}{2(\zeta-\theta)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\theta^+,k}\Psi(\zeta)+I^{\lambda}_{\zeta^-,k}\Psi(\theta)\right]\right|$$
$$\leq \frac{(\zeta-\theta)^2}{2(\frac{\lambda}{k}+1)}\left(1-\frac{2}{g(\frac{\lambda}{k}+1)+1}\right)^{\frac{1}{g}}\left[\frac{|\Psi''(\theta)|^h+|\Psi''(\zeta)|^h}{2}\right]^{\frac{1}{h}},$$

where $\frac{1}{g}+\frac{1}{h}=1$

Proof. By using the Holder's inequality and from Lemma 2.2, we obtain

$$\left|\frac{\Psi(\theta)+\Psi(\zeta)}{2}-\frac{\Gamma_k(\lambda+k)}{2(\zeta-\theta)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\theta^+,k}\Psi(\zeta)+I^{\lambda}_{\zeta^-,k}\Psi(\theta)\right]\right|$$
$$\leq \frac{(\zeta-\theta)^2}{2}\int_0^1 \frac{1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}}{\frac{\lambda}{k}+1}|\Psi''(\mu\theta+(1-\mu)\zeta)|d\mu$$
$$\leq \frac{(\zeta-\theta)^2}{2\left(\frac{\lambda}{k}+1\right)}\left(\int_0^1\left(1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}\right)^g d\mu\right)^{\frac{1}{g}}\left(|\Psi''(\mu\theta+(1-\mu)\zeta)|^h d\mu\right)^{\frac{1}{h}}$$
$$= \frac{(\zeta-\theta)^2}{2\left(\frac{\lambda}{k}+1\right)}\left(\int_0^1\left(1-(1-\mu)^{g(\frac{\lambda}{k}+1)}-\mu^{g(\frac{\lambda}{k}+1)}\right)d\mu\right)^{\frac{1}{g}}$$
$$\times\left(|\Psi''(\theta)|^h\int_0^1\mu d\mu+m|\Psi''(\frac{\zeta}{m})|^h\int_0^1(1-\mu)d\mu\right)^{\frac{1}{h}}$$
$$= \frac{(\zeta-\theta)^2}{2\left(\frac{\lambda}{k}+1\right)}\left(\mu|_0^1+\frac{(1-\mu)^{g(\frac{\lambda}{k}+1)+1}}{g(\frac{\lambda}{k}+1)+1}-\frac{\mu^{g(\frac{\lambda}{k}+1)+1}}{g(\frac{\lambda}{k}+1)+1}\right)^{\frac{1}{g}}$$
$$\times\left(|\Psi''(\theta)|^h\frac{\mu^2}{2}|_0^1-m|\Psi''(\frac{\zeta}{m})|^h\frac{(1-\mu)^2}{2}\right)^{\frac{1}{h}}$$
$$= \frac{(\zeta-\theta)^2}{2\left(\frac{\lambda}{k}+1\right)}\left(1-\frac{1}{g(\frac{\lambda}{k}+1)+1}-\frac{1}{g(\frac{\lambda}{k}+1)+1}\right)^{\frac{1}{g}}\left(\frac{|\Psi''(\theta)|^h+m|\Psi''(\frac{\zeta}{m})|^h}{2}\right)^{\frac{1}{h}}$$
$$= \frac{(\zeta-\theta)^2}{2\left(\frac{\lambda}{k}+1\right)}\left(1-\frac{2}{g(\frac{\lambda}{k}+1)+1}\right)^{\frac{1}{g}}\left(\frac{|\Psi''(\theta)|^h+|m\Psi''(\frac{\zeta}{m})|^h}{2}\right)^{\frac{1}{h}}.$$

Here we use

$$\left(1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}\right)^h\leq 1-(1-\mu)^{h(\frac{\lambda}{k}+1)}-\mu^{h(\frac{\lambda}{k}+1)},$$

for any $\mu\in[0,1]$, which follows from $(C-D)^h\leq C^h-D^h$, for any $C>D\geq 0$ and $h\geq 1$. Hence proved. □

4. Hermite-Hadamard-type inequalities for (s,m)-convex functions

Lemma 4.1. *Let* $k>0$, $I^{\lambda}_{\theta^+,k}\Psi$ *and* $I^{\lambda}_{\zeta^-,k}\Psi$ *be defined in Definition* 2.1. *Let* $\Psi:[\theta,\zeta]\to\mathbb{R}$ *be twice differentiable function on* (θ,ζ) *with* $\theta<m\zeta\leq\zeta$. *If*

$\Psi'' \in L^1[\theta, \zeta]$, *then*

$$\frac{\Psi(\theta)+\Psi(m\zeta)}{2}-\frac{\Gamma_k(\lambda+k)}{2(m\zeta-\theta)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\theta^+,k}\Psi(m\zeta)+I^{\lambda}_{m\zeta^-,k}\Psi(\theta)\right]$$
$$=\frac{(m\zeta-\theta)^2}{2}\int_0^1\frac{1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}}{\frac{\lambda}{k}+1}\Psi''(\mu\theta+m(1-\mu)\zeta)d\mu. \quad (4.1)$$

This is just Lemma 2.2 *on the interval* $[\theta, m\zeta] \subset [\theta, \zeta]$.

Theorem 4.2. *Let* $k > 0$, $I^{\lambda}_{\theta^+,k}\Psi$ *and* $I^{\lambda}_{\zeta^-,k}\Psi$ *be defined in the Definition* 2.1. *Let* $\Psi : [\theta, \zeta] \to \mathbb{R}$ *be a twice differentiable function with* $\theta < \zeta$. *If* $|\Psi''|^h$ *is measurable and* (s, m)*-convex on* $[\theta, \zeta]$ *for some fixed* $h \geq 1$ *and* $(s, m) \in (0, 1]^2$, *then*

$$\left|\frac{\Psi(\theta)+\Psi(m\zeta)}{2}-\frac{\Gamma_k(\lambda+k)}{2(m\zeta-\theta)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\theta^+,k}\Psi(m\zeta)+I^{\lambda}_{m\zeta^-,k}\Psi(\theta)\right]\right|$$
$$\leq\frac{(m\zeta-\theta)^2}{2(\frac{\lambda}{k}+1)}\left(1-\frac{2}{g\left(\frac{\lambda}{k}+1\right)+1}\right)^{\frac{1}{g}}\left(|\Psi''(\theta)|^h\frac{1}{s+1}+|\Psi''(\zeta)|^h\frac{s}{s+1}\right)^{\frac{1}{h}},$$

where $\frac{1}{g}+\frac{1}{h}=1$.

Proof. By using the Holder's inequality and Lemma 4.1, we get

$$\left|\frac{\Psi(\theta)+\Psi(m\zeta)}{2}-\frac{\Gamma_k(\lambda+k)}{2(m\zeta-\theta)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\theta^+,k}\Psi(m\zeta)+I^{\lambda}_{m\zeta^-,k}\Psi(\theta)\right]\right|$$
$$\leq \frac{(m\zeta-\theta)^2}{2}\int_0^1\frac{1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}}{\frac{\lambda}{k}+1}|\Psi''(\mu\theta+m(1-\mu)\zeta)|d\mu$$
$$\leq \frac{(m\zeta-\theta)^2}{2\left(\frac{\lambda}{k}+1\right)}\left(\int_0^1\left(1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}\right)^g d\mu\right)^{\frac{1}{g}}\left(\int_0^1|\Psi''(\mu\theta+m(1-\mu)\zeta)|^h d\mu\right)^{\frac{1}{h}}$$
$$\leq \frac{(m\zeta-\theta)^2}{2\left(\frac{\lambda}{k}+1\right)}\left(\int_0^1\left(1-(1-\mu)^{g(\frac{\lambda}{k}+1)}-\mu^{g(\frac{\lambda}{k}+1)}\right)d\mu\right)^{\frac{1}{g}}$$
$$\times\left(|\Psi''(\theta)|^h\int_0^1\mu^s d\mu+m|\Psi''(\zeta)|^h\int_0^1(1-\mu^s)d\mu\right)^{\frac{1}{h}}$$
$$\leq \frac{(m\zeta-\theta)^2}{2\left(\frac{\lambda}{k}+1\right)}\left(\mu|_0^1+\frac{(1-\mu)^{g(\frac{\lambda}{k}+1)+1}}{g(\frac{\lambda}{k}+1)+1}|_0^1-\frac{(1-\mu)^{g(\frac{\lambda}{k}+1)+1}}{g(\frac{\lambda}{k}+1)+1}|_0^1\right)^{\frac{1}{g}}$$
$$\times\left(|\Psi''(\theta)|^h\frac{\mu^s}{s+1}|_0^1+m|\Psi''(\zeta)|^h\left(\mu|_0^1-\frac{\mu^s}{s+1}|_0^1\right)\right)^{\frac{1}{h}}$$
$$\leq \frac{(m\zeta-\theta)^2}{2\left(\frac{\lambda}{k}+1\right)}\left(1-\frac{2}{g(\frac{\lambda}{k}+1)+1}\right)^{\frac{1}{g}}\left(|\Psi''(\theta)|^h\frac{1}{s+1}+m|\Psi''(\zeta)|^h\frac{s}{s+1}\right)^{\frac{1}{h}}.$$

The proof is completed. □

Theorem 4.3. *With the same condition of Theorem* 4.2, *then*

$$\left|\frac{\Psi(\theta)+\Psi(m\zeta)}{2}-\frac{\Gamma_k(\lambda+k)}{2(m\zeta-\theta)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\theta^+,k}\Psi(m\zeta)+I^{\lambda}_{m\zeta^-,k}\Psi(\theta)\right]\right|$$
$$\leq\frac{(m\zeta-\theta)^2}{2(\frac{\lambda}{k}+1)}\left[|\Psi''(\theta)|^h\left(\frac{1}{s+1}-\beta(s+1,h(\frac{\lambda}{k}+1)+1)-\frac{1}{h(\frac{\lambda}{k}+1)+s+1}\right)\right.$$
$$\left.+m|\Psi''(\zeta)|^h\left(1-\frac{1}{s+1}-\frac{2}{h(\frac{\lambda}{k}+1)+1}+\beta(s+1,h(\frac{\lambda}{k}+1)+1)+\frac{1}{h(\frac{\lambda}{k}+1)+s+1}\right)\right]^{\frac{1}{h}}.$$

Proof. Using the Holder's inequality and from Lemma 4.1, we obtain

$$\left|\frac{\Psi(\theta)+\Psi(m\zeta)}{2}-\frac{\Gamma_k(\lambda+k)}{2(m\zeta-\theta)^{\frac{\lambda}{k}}}\left[I^{\lambda}_{\theta^+,k}\Psi(m\zeta)+I^{\lambda}_{m\zeta^-,k}\Psi(\theta)\right]\right|$$
$$\leq\frac{(m\zeta-\theta)^2}{2}\int_0^1\frac{1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}}{\frac{\lambda}{k}+1}|\Psi''(\mu\theta+m(1-\mu)\zeta)|d\mu$$
$$\leq\frac{(m\zeta-\theta)^2}{2\left(\frac{\lambda}{k}+1\right)}\left(\int_0^1 1d\mu\right)^{\frac{1}{g}}\left(\int_0^1\left(1-(1-\mu)^{\frac{\lambda}{k}+1}-\mu^{\frac{\lambda}{k}+1}\right)^h|\Psi''(\mu\theta+m(1-\mu)\zeta)|^h\right)^{\frac{1}{h}}$$
$$\leq\frac{(m\zeta-\theta)^2}{2\left(\frac{\lambda}{k}+1\right)}\left(|\Psi''(\theta)|^h\int_0^1\left[\mu^s-(1-\mu)^{h(\frac{\lambda}{k}+1)}\mu^s-^{\frac{\lambda}{k}+1+s}\right]d\mu\right.$$
$$\left.+m|\Psi''(\zeta)|^h\int_0^1\left[(1-\mu^s)-(1-\mu)^{h(\frac{\lambda}{k}+1)}(1-\mu^s)-\mu^{h(\frac{\lambda}{k}+1)}(1-\mu^s)\right]d\mu\right)^{\frac{1}{h}}$$
$$=\frac{(m\zeta-\theta)^2}{2(\frac{\lambda}{k}+1)}\left[|\Psi''(\theta)|^h\left(\frac{1}{s+1}-\beta(s+1,h(\frac{\lambda}{k}+1)+1)-\frac{1}{h(\frac{\lambda}{k}+1)+s+1}\right)\right.$$
$$\left.+m|\Psi''(\zeta)|^h\left(1-\frac{1}{s+1}-\frac{2}{h(\frac{\lambda}{k}+1)+1}+\beta(s+1,h(\frac{\lambda}{k}+1)+1)+\frac{1}{h(\frac{\lambda}{k}+1)+s+1}\right)\right]^{\frac{1}{h}},$$

which is required result. □

Corollary 4.4. *In Theorem* 4.3, *by taking* $s=m=1$ *and* $k=1$, *we get the result proved in* [13].

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