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AN INVERSE BOUNDARY VALUE PROBLEM FOR THE EQUATION OF FLEXURAL VIBRATIONS OF A BAR WITH AN INTEGRAL CONDITIONS OF THE FIRST KIND

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Abstract. In this work, we study one inverse boundary value problem for the equations of flexural vibrations of a bar with an integral condition of the first kind. Using the Fourier method, the problem reduces to solving a system of integral equations, and using method of contracting mappings, the existence and uniqueness of a solution to a system of integral equations is proved. The existence and uniqueness of the classical solution of the original problem are proved.

1. Introduction

In modern technology, it is necessary to regulate vibration processes in one-dimensional distributed systems, and the relevance of these problems is increasing. For shafts, which are the basic principles of mechanical transmission, dangerous transverse vibrations are not allowed [1]. In aircraft such elements are constructed simultaneously by bending and torsional vibrations. One of the objectives of the project is to prevent the use of shaft vibrations with an adjustable speed [2,3]. For such problems, mathematical models of transverse vibrations of rods are built on the basis of a refined theory [4]. Solutions of unknown parameters in accordance with the known data of its solutions [5,6]. Such problems are called inverse problems of mathematical physics, which in many works [6] - [10], [12] , [14], [15],[17,18] were studied for partial differential equations. In problems associated with initial and boundary conditions, additional information is required. The necessary additional information is due to the presence of unknown coefficients or the right-hand sides of the equations [13].

2. Problem statement and its reduction to an equivalent task.

In this paper, we study the inverse boundary value problem with integral conditions for the equation of transverse vibrations of a bar for the case of rigidly fixed ends. Consider for the equation

$$u_{tt}(x,t) + u_{xxxx}(x,t) = a(t)u(x,t) + b(t)u_t(x,t) + f(x,t) \tag{2.1}$$

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in the domain $D_T = \{(x,t) : 0 \le x \le 1, 0 \le t \le T\}$ an inverse boundary problem with boundary conditions

$$u(x,0) = \varphi(x), u_t(x,0) = \psi(x)(0 \le x \le 1) \tag{2.2}$$

with boundary conditions

$$u_x(0,t) = 0, u_x(1,t) = 0, u_{xxx}(0,t) = 0(0 \le t \le T) \tag{2.3}$$

with non-local integral conditions

$$\int_0^1 u(x,t)dx = 0(0 \le t \le T) \tag{2.4}$$

and with the additional condition

$$u(0,t) = h_1(t), u(1,t) = h_2(t)(0 \le t \le T) \tag{2.5}$$

where $f(x,t)$, $\varphi(x)$,$\psi(x)$, $h_i(t), (i = 1,2)$ -given functions,$u(x,t)$, $a(t)$ and $b(t)$ -desired functions.

Definition We call the trio $\{u(x,t), a(t), b(t)\}$ the classic solution of inverse boundary value problem problem (2.1)-(2.5), if the following conditions are satisfied:

1) the function $u(x,t)$ is continuous in D_T together with all its derivatives, $u_x(x,t), u_{xx}(x,t), u_{xxx}(x,t), u_{xxxx}(x,t), u_t(x,t), u_{tt}(x,t)$;
2) the functions $a(t), b(t)$ are continuous on $[0,T]$;
3) the problem (2.1)-(2.5) is satisfied in the ordinary sense.

The following lemma holds:

Lemma 2.1. *Let* $\varphi(x), \psi(x) \in C[0,1], h_i(t) \in C^2[0,T](i = 1,2), h(t) \equiv h_1(t)h_2'(t) - h_2(t)h_1'(t) \ne 0, (0 \le t \le T)$,

$$f(x,t) \in C(D_T), \int_0^1 f(x,t)dx = 0, (0 \le t \le T)$$

and the consistency conditions

$$\int_0^1 \varphi(x)dx = 0, \int_0^1 \psi(x)dx = 0, \tag{2.6}$$

$$\varphi(0) = h_1(0), \varphi(1) = h_2(0),$$

$$\psi(0) = h_1'(0), \psi(1) = h_2'(0). \tag{2.7}$$

be satisfied. Then the problem of finding a classical solution to problem (2.1)-(2.5)*is equivalent to the problem of determining functions* $u(x,t)$ *and* $a(t)$ *having properties 1) and 2), determining a solution to problem* (2.1)-(2.5), *from* (2.1)-(2.3),

$$u_{xxx}(1,t) = 0(0 \le t \le T), \tag{2.8}$$

$$h_1''(t) + u_{xxxx}(0,t) = a(t)h_1'(t) + b(t)h_1'(t) + f(0,t)(0 \le t \le T), \tag{2.9}$$

$$h_2''(t) + u_{xxxx}(1,t) = a(t)h_2(t) + b(t)h_2'(t) + f(1,t)(0 \le t \le T). \tag{2.10}$$

Proof. Let $\{u(x,t), a(t), b(t)\}$ be a solution of problem (2.1)- (2.5). Integrating equation (2.1) over x from 0 to 1, we have:

$$\frac{d^2}{dt^2}\int\limits_0^1 u(x,t)dx + u_{xxx}(1,t) - u_{xxx}(0,t) =$$

$$= a(t)\int\limits_0^1 u(x,t)dx + b(t)\int\limits_0^1 u_t(x,t)dx + \int\limits_0^1 f(x,t)dx \ \ (0 \le t \le T). \tag{2.11}$$

Assuming that $\int\limits_0^1 f(x,t)dx = 0 (0 \le t \le T)$, in view of (2.3), (2.4), we arrive at fulfillment (2.8).

Under the assumption $h_i(t) \in C^2[0,T]$ $(i = 1,2)$ and differentiating two times (2.5) we have:

$$u_{tt}(0,t) = h_1''(t), \tag{2.12}$$

$$u_{tt}(1,t) = h_2''(t) \ \ (0 \le t \le T), \tag{2.13}$$

Substituting $x = 0$ in equation (2.1), respectively, we find:

$$u_{tt}(0,t) + u_{xxxx}(0,t) = a(t)u(0,t) + b(t)u_t(0,t) + f(0,t) \ \ (0 \le t \le T), \tag{2.14}$$

Considering these from (2.1) follows,

$$u_{tt}(1,t) + u_{xxxx}(1,t) = a(t)u(1,t) + b(t)u_t(1,t) + f(1,t) \ \ (0 \le t \le T). \tag{2.15}$$

From (2.14), by virtue of (2.5) and (2.12), it follows that (2.9) holds. Similarly, from (2.15), taking into account (2.5) and (2.13), we arrive (2.10).

Now, suppose that $u(x,t), b(t)$ and $a(t)$ are a solution to problem (2.1)-(2.3), (2.8)-(2.10). Then from (2.11), with regard to (2.3)-(2.8) we have:

$$y''(t) - b(t)y'(t) - a(t)y(t) = 0, \ \ (0 \le t \le T), \tag{2.16}$$

where

$$y(t) = \int\limits_0^1 u(x,t)dx \tag{2.17}$$

By virtue of (2.2), (2.6) it is obvious that

$$y(0) = \int\limits_0^1 u(x,0)dx = \int\limits_0^1 \varphi(x)dx = 0, y'(0) = \int\limits_0^1 u_t(x,0)dx = \int\limits_0^1 \psi(x)dx = 0. \tag{2.18}$$

From (2.16), taking into account (2.18), it is obvious that $y(t) \equiv 0, (0 \le t \le T)$. Hence, by virtue of (2.17), we easily come to the fulfillment of (2.4). Further, from (2.9), (2.14) and (2.10), (2.15) we find:

$$\frac{d^2}{dt^2}(u(0,t) - h_1(t)) - b(t)\frac{d}{dt}(u(0,t) - h_1(t)) - a(t)(u(0,t) - h_1(t)) = 0, \ \ (0 \le t \le T), \tag{2.19}$$

$$\frac{d^2}{dt^2}(u(1,t) - h_2(t)) - b(t)\frac{d}{dt}(u(1,t) - h_2(t)) - a(t)(u(1,t) - h_2(t)) = 0, \ \ (0 \le t \le T), \tag{2.20}$$

From (2.2) and (2.7), we have:

$$u(0,0)-h_1(0)=\varphi(0)-h_1(0)=0, u_t(0,0)-h_1'(0)=\psi(0)-h_1'(0)=0, \quad (2.21)$$

$$u(1,0)-h_2(0)=\varphi(1)-h_2(0)=0, u_t(1,0)-h_2'(0)=\psi(0)-h_{12}'(0)=0. \quad (2.22)$$

From (2.19) -(2.22) we conclude that condition (2.5) are satisfied, respectively. The Lemma is proved.

3. Solvability of the inverse boundary value problem

The first komponent $u(x,t)$ of the solution $\{u(x,t),a(t),b(t)\}$ of the problem (2.1) -(2.3), (2.8)- (2.11) will be sought in the form:

$$u(x,t)=\sum_{k=0}^{\infty}u_k(t)\cos\lambda_k x(\lambda_k=\pi k), \quad (3.1)$$

where

$$u_k(t)=m_k\int_0^1 u(x,t)\cos\lambda_k x dx(k=0,1,2,...),$$

$$m_k=\begin{cases}1, k=0,\\ 0, k=1,2,...\end{cases}$$

Then applying the formal scheme of the Fourier method, from (2.1) and (2.2))we obtained:

$$u_k''(t)+\lambda_k^4 u_k(t)=F_k(t;u,a,b)(0\le t\le T;k=0,1,\dots), \quad (3.2)$$

$$u_k(0)=\varphi_k, u_k'(0)=\psi_k \quad (3.3)$$

where

$$F_k(t;u,a,b)=f_k(t)+a(t)u_k(t)+b(t)u_k'(t)(k=0,1,...)$$

$$f_k(t)=m_k\int_0^1 f(x,t)\cos\lambda_k x dx,$$

$$\varphi_k=m_k\int_0^1\varphi(x)\cos\lambda_k x dx, \psi_k=m_k\int_0^1\psi(x)\cos\lambda_k x dx, \quad (k=0,1,\dots).$$

Further, from (3.2),(3.3) we find:

$$u_0(t)=\varphi_0+\psi_0 t+\int_0^t(t-\tau)F_0(\tau;u,a,b)d\tau \quad (3.4)$$

$$u_k(t)=\varphi_k\cos\lambda_k^2 t+\frac{1}{\lambda_k^2}\psi_k\sin\lambda_k^2 t+\frac{1}{\lambda_k^2}\int_0^t F_k(\tau;u,a,b)\sin\lambda_k^2(t-\tau)d\tau, (k=1,2,...). \quad (3.5)$$

It's obvious that

$$u_0'(t)=\psi_0+\int_0^t F_0(\tau;u,a,b)d\tau, \quad (3.6)$$

$$u_k'(t) = -\lambda_k^2 \varphi_k \sin \lambda_k^2 t + \psi_k \cos \lambda_k^2 t + \int\limits_0^t F_k(\tau; u, a, b) \cos \lambda_k^2 (t-\tau) d\tau, (k = 1, 2, ...). \tag{3.7}$$

After expression substitution $u_k(t) (k = 0, 1, ...)$ in (3.1), to determine the components $u(x, t)$ of the solution to problem $(2.1) - (2.3), (2.8), (2.10)$ we obtain:

$$u(x,t) = \varphi_0 + t\psi_0 + \int\limits_0^t (t-\tau) F_0(\tau; u, a) d\tau +$$

$$+ \sum_{k=1}^{\infty} \left\{ \varphi_k \cos \lambda_k^2 t + \frac{1}{\lambda_k^2} \psi_k \sin \lambda_k^2 t + \frac{1}{\lambda_k^2} \int\limits_0^t F_k(\tau; u, a) \sin \lambda_k^2 (t-\tau) d\tau \right\} \cos \lambda_k x. \tag{3.8}$$

Now, from (2.9) and (2.10) , considering (3.1), we have:

$$a(t) = [h(t)]^{-1} \{ (h_1''(t) - f(0,t)) h_2'(t) - (h_2''(t) - f(1,t)) h_1'(t) + \\ + \sum_{k=1}^{\infty} \lambda_k^4 u_k(t) \left(h_2'(t) - h_1'(t)(-1)^k \right). \Big\} \tag{3.9}$$

$$b(t) = [h(t)]^{-1} \{ (h_2''(t) - f(1,t)) h_1(t) - (h_1''(t) - f(0,t)) h_2(t) + \\ + \sum_{k=1}^{\infty} \lambda_k^4 u_k(t) \left(-h_1(t)(-1)^k - h_2(t) \right). \Big\} \tag{3.10}$$

where

$$h(t) \equiv h_1(t) h_2'(t) - h_2(t) h_1'(t) \neq 0, (0 \le t \le T). \tag{3.11}$$

Substitute the expression (3.5) in (3.9) and (3.10) obtained:

$$a(t) = [h(t)]^{-1} \{ (h_1''(t) - f(0,t)) h_2'(t) - (h_2''(t) - f(1,t)) h_1'(t) + \\ + \sum_{k=1}^{\infty} \lambda_k^4 \left(h_2'(t) - h_1'(t)(-1)^k \right) \Big[\varphi_k \cos \lambda_k^2 t + \frac{1}{\lambda_k^2} \psi_k \sin \lambda_k^2 t + \\ + \frac{1}{\lambda_k^2} \int\limits_0^t F_k(\tau; u, a, b) \sin \lambda_k^2 (t-\tau) d\tau \Big] \Big\} \tag{3.12}$$

$$b(t) = [h(t)]^{-1} \{ (h_2''(t) - f(1,t)) h_1(t) - (h_1''(t) - f(0,t)) h_2(t) + \\ + \sum_{k=1}^{\infty} \lambda_k^4 \left(-h_1(t)(-1)^k - h_2(t) \right) \Big[\varphi_k \cos \lambda_k^2 t + \frac{1}{\lambda_k^2} \psi_k \sin \lambda_k^2 t + \\ + \frac{1}{\lambda_k^2} \int\limits_0^t F_k(\tau; u, a, b) \sin \lambda_k^2 (t-\tau) d\tau \Big] \Big\} \tag{3.13}$$

Thus, the solution of the problem (2.1)-(2.3), (2.8)-(2.10) was reduced to the solution of the problem (3.8),(3.12), (3.14) for the unknown functions $u(x,t), a(t)$ and $b(t)$.

Lemma 3.1. *If* $\{u(x,t),a(t),b(t)\}$ *any classical solution of the problem* (2.1)-(2.3),(2.8)-(2.10) *then functions* $u_k(t)=m_k\int\limits_0^1 u(x,t)\cos\lambda_k x dx$, $(k=0,1,2,\dots)$ *satisfy the system* (3.4)-(3.5).

From Lemma 3.1 it follows that:

Remark. *Let system* (3.8), (3.12), (3.13) *have a unique solution. Then problem* (2.1)-(2.3),(2.8)-(2.10) *cannot have more than one solution.*

Now, we consider the following spaces:

We denote by $B_{2,T}^{\alpha,\alpha-1}$ [12] ,a consisting of all functions $u(x,t)$ of the form

$$u(x,t)=\sum_{k=0}^{\infty}u_k(t)\cos\lambda_k x dx \;\; (\lambda_k=\pi k),$$

considered in D_T , where $u_k(t)(k=0,1,...)$ is continuous on $[0,T]$ and

$$J(u)\equiv\|u_0(t)\|_{C[0,T]}+\left(\sum_{k=1}^{\infty}\left(\lambda_k^{\alpha}\|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+$$

$$+\|u_0'(t)\|_{C[0,T]}+\left(\sum_{k=1}^{\infty}\left(\lambda_k^{\alpha-1}\|u_k'(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}<+\infty.$$

The norm in this set is defined as follows:

$$\|u(x,t)\|_{B_{2,T}^{\alpha}-1}=J(u)$$

The spaces $E_T^{\alpha,\alpha-1}$ denote the space consisting of a topological product

$$B_{2,T}^{\alpha,\alpha-1}\times C[0,T]\times C[0,T].$$

The norm of element $z(x,t)=\{u,a(t),b(t)\}$ is determined by the formula

$$\|z\|_{E_T^{\alpha,\alpha-1}}=\|u(x,t)\|_{B_{2,T}^{\alpha,\alpha-1}}+\|a(t)\|_{C[0,T]}+\|b(t)\|_{C[0,T]}.$$

It is obvious that $B_{2,T}^{\alpha,\alpha-1}$ and $E_T^{\alpha,\alpha-1}$ are Banach spaces.

Now in the space $E_T^{3,2}$ consider the operator

$$\Phi(u,a)=\{\Phi_1(u,a,b),\Phi_2(u,a,b),\Phi_3(u,a,b)\}$$

where

$$\Phi_1(u,a)=\tilde{u}(x,t)\equiv\sum_{k=0}^{\infty}\tilde{u}_k(t)\cos\lambda_k x,$$

$$\Phi_2(u,a)=\tilde{a}(t),\Phi_3(u,a)=\tilde{b}(t),$$

$\tilde{u}_0(t),\tilde{u}_k(t)(k=1,2,...)$ and $\tilde{a}(t)$,$\tilde{b}(t)$ are equal to the right hand sides of (3.4), (3.5), (3.12),(3.13).

Now with the help of easy transformations we get:

$$\|\tilde{u}_0(t)\|_{C[0,T]}\le|\varphi_0|+T|\psi_0|+T\sqrt{T}\left(\int\limits_0^T|f_0(\tau)|^2d\tau\right)^{\frac{1}{2}}+$$

$$+T^2\|a(t)\|_{C[0,T]}\|u_0(t)\|_{C[0,T]}+T^2\|b(t)\|_{C[0,T]}\|u_0'(t)\|_{C[0,T]}, \tag{3.14}$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\left\|\tilde{u}_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \le \sqrt{5}\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\left|\varphi_k\right|\right)^2\right)^{\frac{1}{2}} + \sqrt{5}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left|\psi_k\right|\right)^2\right)^{\frac{1}{2}} +$$

$$+\sqrt{5T}\left(\int\limits_0^T\sum_{k=1}^{\infty}\left(\lambda_k^3\left|f_k(\tau)\right|\right)^2 d\tau\right)^{\frac{1}{2}} + \sqrt{5}T\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^4\left\|u_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}$$

$$+\sqrt{5}T\left\|b(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\left\|u_k'(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}, \tag{3.15}$$

$$\|\tilde{u'}_0(t)\|_{C[0,T]} \le |\psi_0| + \sqrt{T}\left(\int\limits_0^T |f_0(\tau)|^2 d\tau\right)^{\frac{1}{2}} +$$

$$+T\|a(t)\|_{C[0,T]}\|u_0(t)\|_{C[0,T]} + T\|b(t)\|_{C[0,T]}\|u_0'(t)\|_{C[0,T]}, \tag{3.16}$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^4\left\|\tilde{u}_k'(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \le \sqrt{5}\left(\sum_{k=1}^{\infty}\left(\lambda_k^6\left|\varphi_k\right|\right)^2\right)^{\frac{1}{2}} + \sqrt{5}\left(\sum_{k=1}^{\infty}\left(\lambda_k^4\left|\psi_k\right|\right)^2\right)^{\frac{1}{2}} +$$

$$+\sqrt{6T}\left(\int\limits_0^T\sum_{k=1}^{\infty}\left(\lambda_k^4\left|f_k(\tau)d\tau\right|\right)^2\right)^{\frac{1}{2}} + \sqrt{6}T\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\left\|u_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} +$$

$$+\sqrt{5}T\left\|b(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^4\left\|u_k'(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}, \tag{3.17}$$

$$\left\|\tilde{a}(t)\right\|_{C[0,T]} \le$$

$$\le \left\|[h(t)]^{-1}\right\|_{C[0,T]}\Big\{\|(h_1''(t)-f(0,t))\,h_2'(t)-(h_2''(t)-f(1,t))\,h_1'(t)\|_{C[0,T]} +$$

$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\||h_1'(t)|+|h_2'(t)|\|_{C[0,T]}\left[\left(\sum_{k=1}^{\infty}(\lambda_k^5\left|\varphi_k\right|)^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty}(\lambda_k^3\left|\psi_k\right|)^2\right)^{\frac{1}{2}}\right.$$

$$+\sqrt{T}\left(\int\limits_0^T\sum_{k=1}^{\infty}(\lambda_k^3\left|f_k(\tau)\right|)^2 d\tau\right)^{\frac{1}{2}} + T\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\left\|u_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}} +$$

$$\left.+T\left\|b(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^4\left\|u_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right]\Bigg\} \tag{3.18}$$

$$\left\|\tilde{b}(t)\right\|_{C[0,T]} \le$$

$$\le \left\|[h(t)]^{-1}\right\|_{C[0,T]}\Big\{\|(h_2''(t)-f(1,t))\,h_1(t)-(h_1''(t)-f(0,t))\,h_2(t)\|_{C[0,T]} +$$

$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\||h_1(t)|+|h_2(t)|\|_{C[0,T]}\left[\left(\sum_{k=1}^{\infty}(\lambda_k^5\left|\varphi_k\right|)^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty}(\lambda_k^3\left|\psi_k\right|)^2\right)^{\frac{1}{2}} +\right.$$

$$+\sqrt{T}\left(\int_0^T \sum_{k=1}^{\infty}(\lambda_k^3\,|f_k(\tau)|)^2 d\tau\right)^{\frac{1}{2}} + T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} +$$
$$+T\,\|b(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^4\,\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\Bigg]\Bigg\} \tag{3.19}$$

Suppose that the data of the problem (2.1)- (2.3),(2.8)-(2.10)satisfy the following conditions:

(1) $\varphi(x) \in C^5[0,1], \varphi^{(6)}(x) \in L_2(0,1), \varphi'(0) = \varphi'(1) = \varphi^{(3)}(0) = \varphi^{(3)}(1) = \varphi^{(5)}(0) = \varphi^{(5)}(1) = 0;$
(2) $\psi(x) \in C^3[0,1], \psi^{(4)}(x) \in L_2(0,1), \psi'(0) = \psi'(1) = \psi^{(3)}(0) = \psi^{(3)}(1) = 0;$
(3) $f(x,t), f_x(x,t), f_{xx}(x,t) \in C(D_T), f_{xxx}(x,t) \in C(D_T), f_{xxxx}(x,t) \in L_2(D_T),$ $f_x(0,t) = f_x(1,t) = f_{xxx}(0,t) = f_{xxx}(1,t) = 0 (0 \le t \le T);$
(4) $h_i(t) \in C^2[0,T], (i = 1,2), h(t) \equiv h_1(t)h_2'(t) - h_2(t)h_1'(t) \ne 0. (0 \le t \le T).$

Further, from (3.14)- (3.19) we have:

$$\|\tilde{u}_0(t)\|_{C[0,T]} \le \|\tilde{\varphi}(x)\|_{L_2(0,1)} + T\,\|\psi(x)\|_{L_2(0,1)} + T\sqrt{T}\,\|f(x,t)\|_{L_2(D_T)} + \tag{3.20}$$
$$+T^2\,\|a(t)\|_{C[0,T]}\,\|u_0(t)\|_{C[0,T]} + T^2\,\|b(t)\|_{C[0,T]}\,\|u_0'(t)\|_{C[0,T]}$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\,\|\tilde{u}_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \le \sqrt{5}\,\left\|\varphi^5(x)\right\|_{L_2(0,1)} + \sqrt{5}\,\left\|\psi^3(x)\right\|_{L_2(0,1)} + \sqrt{5T}\,\|f_{xxx}(x,t)\|_{L_2(D_T)} +$$
$$+\sqrt{5}T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\,\|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} + \tag{3.21}$$
$$+\sqrt{5}T\,\|b(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^4\,\|u_k'(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}},$$

$$\|\tilde{u}_0'(t)\|_{C[0,T]} \le \|\psi(x)\|_{L_2(0,1)} + \sqrt{T}\,\|f(x,t)\|_{L_2(D_T)} + \tag{3.22}$$
$$+T\,\|a(t)\|_{C[0,T]}\,\|u_0(t)\|_{C[0,T]} + T\,\|b(t)\|_{C[0,T]}\,\|u_0'(t)\|_{C[0,T]}$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\,\|\tilde{u}_k'(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \le \sqrt{5}\,\left\|\varphi^6(x)\right\|_{L_2(0,1)} + \sqrt{5}\,\left\|\psi^4(x)\right\|_{L_2(0,1)} + \sqrt{5T}\,\|f_{xxxx}(x,t)\|_{L_2(D_T)} +$$
$$+\sqrt{5}T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\,\|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} + \tag{3.23}$$
$$+\sqrt{5}T\,\|b(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^4\,\|u_k'(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}},$$

$$\|\tilde{a}(t)\|_{C[0,T]} \le$$
$$\le \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\|(h_1''(t) - f(0,t))\,h_2'(t) - (h_2''(t) - f(1,t))\,h_1'(t)\|_{C[0,T]} +\right.$$

$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\||h_1'(t)|+|h_2'(t)|\|_{C[0,T]}\left[\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}+\left\|\psi^{(3)}(x)\right\|_{L_2(0,1)}+\right.$$

$$+\sqrt{T}\left\|f_{xxx}(x,t)\right\|_{L_2(D_T)}+T\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\left\|u_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+$$

$$\left.\left.+T\left\|b(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^4\left\|u_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\right]\right\} \tag{3.24}$$

$$\left\|\tilde{b}(t)\right\|_{C[0,T]}\leq$$

$$\leq\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|\left(h_2''(t)-f(1,t)\right)h_1(t)-\left(h_1''(t)-f(0,t)\right)h_2(t)\right\|_{C[0,T]}+\right.$$

$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\||h_1(t)|+|h_2(t)|\|_{C[0,T]}\left[\left[\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}+\left\|\psi^{(3)}(x)\right\|_{L_2(0,1)}+\right.\right.$$

$$+\sqrt{T}\left\|f_{xxx}(x,t)\right\|_{L_2(D_T)}+T\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\left\|u_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+$$

$$\left.\left.+T\left\|b(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^4\left\|u_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\right]\right\}. \tag{3.25}$$

Then, from (3.20)-(3.23) we get:

$$\|\tilde{u}(x,t)\|_{B_{2,T}^{5,4}}\leq A_1(T)+B_1(T)\left(\|a(t)\|_{C[0,T]}+\|b(t)\|_{C[0,T]}\right)\|u(x,t)\|_{B_{2,T}^{5,4}} \tag{3.26}$$

where

$$A_1(T)=\|\varphi(x)\|_{L_2(0,1)}+(T+1)\|\psi(x)\|_{L_2(0,1)}+T\sqrt{T}\|f(x,t)\|_{L_2(D_T)}+$$

$$+\sqrt{5}\left(\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}+\left\|(\varphi^{(6)}(x)\right\|_{L_2(0,1)}\right)+\sqrt{5}\left(\left\|\psi^{(3)}(x)\right\|_{L_2(0,1)}+\left\|\psi^{(4)}(x)\right\|_{L_2(0,1)}\right)+$$

$$+\sqrt{5T}\left(\|f_{xxx}(x,t)\|_{L_2(D_T)}+\|f_{xxxx}(x,t)\|_{L_2(D_T)}\right), \tag{3.27}$$

$$B_1(T)=T(T+1+2\sqrt{5}).$$

From (3.24)-(3.25) accordingly, we have:

$$\|\tilde{a}(t)\|_{C[0,T]}\leq A_2(T)+B_2(T)\left(\|a(t)\|_{C[0,T]}+\|b(t)\|_{C[0,T]}\right)\|u(x,t)\|_{B_{2,T}^{5,4}}, \tag{3.28}$$

$$\left\|\tilde{b}(t)\right\|_{C[0,T]}\leq A_3(T)+B_3(T)\left(\|a(t)\|_{C[0,T]}+\|b(t)\|_{C[0,T]}\right)\|u(x,t)\|_{B_{2,T}^{5,4}}, \tag{3.29}$$

where

$$A_2(T)=$$

$$\leq\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|\left(h_1''(t)-f(0,t)\right)h_2'(t)-\left(h_2''(t)-f(1,t)\right)h_1'(t)\right\|_{C[0,T]}+\right.$$

$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\||h_1'(t)|+|h_2'(t)|\|_{C[0,T]}\left[\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}+\left\|\psi^{(3)}(x)\right\|_{L_2(0,1)}+\right.$$

$$+\sqrt{T}\,\|f_{xxx}(x,t)\|_{L_2(D_T)}\Big]\Big\},$$

$$A_3(T)=$$

$$+\left\|[h(t)]^{-1}\right\|_{C[0,T]}\Big\{\|(h_2''(t)-f(1,t))\,h_1(t)-(h_1''(t)-f(0,t))\,h_2(t)\|_{C[0,T]}+$$

$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\||h_1(t)|+|h_2(t)|\|_{C[0,T]}\Big[\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}+\left\|\psi^{(3)}(x)\right\|_{L_2(0,1)}+$$

$$+\sqrt{T}\,\|f_{xxx}(x,t)\|_{L_2(D_T)}\Big]\Big\},$$

$$B_2(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\||h_1'(t)|+|h_2'(t)|\|_{C[0,T]}\,T,$$

$$B_3(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\||h_1(t)|+|h_2(t)|\|_{C[0,T]}\,T,$$

From inequalities (3.26)-(3.28) we conclude:

$$\|\widetilde{u}(x,t)\|_{B_{2,T}^{5,4}}+\|\widetilde{a}(t)\|_{C[0,T]}+\|\widetilde{b}(t)\|_{C[0,T]}\leq$$

$$\leq A(T)+B(T)\left(\|a(t)\|_{C[0,T]}+\|b(t)\|_{C[0,T]}\right)\|u(x,t)\|_{B_{2,T}^{5,4}} \tag{3.30}$$

where

$$A(T)=A_1(T)+A_2(T)+A_3(T),$$
$$B(T)=B_1(T)+B_2(T)+B_3(T).$$

So, we can prove the following theorem:

Theorem 3.2. *Let conditions* 1.-4. *be satisfied. and*

$$B(T)((A(T)+2)^2<1. \tag{3.31}$$

Then problem (2.1)-(2.3), (2.8)-(2.10) *has a unique solution in the sphere* $K=K_R(\|z\|_{E_T^{5,4}}\leq R=A(T)+2)$ *of the space* $E_T^{5,4}$.

Proof. In the space $E_T^{5,4}$ consider the equation

$$z=\Phi z, \tag{3.32}$$

where $z=\{u,a,b\}$ the components $\Phi_i(u,a,b)(i=1,2,3)$ of the operator $\Phi(u,a,b)$ are determined by the right-hand sides of equations (3.8), (3.12), (3.13). Consider the operator $\Phi(u,a,b)$ in the sphere $K_R=K$ from $E_T^{5,4}$. Similiar to (3.29), we get that for any $z,z_1,z_2\in K_R$ the following estimates are valid:

$$\|\Phi z\|_{E_T^{5,4}}\leq A(T)+B(T)\left(\|a(t)\|_{C[0,T]}+\|b(t)\|_{C[0,T]}\right)\|u(x,t)\|_{B_{2,T}^{5,4}}\leq$$

$$\leq A(T)+B(T)\left((A(T)+2)\right)^2\leq A(t)+2, \tag{3.33}$$

$$\|\Phi z_1-\Phi z_2\|_{E_T^{5,4}}\leq B(T)R\left(\|a_1(t)-a_2(t)\|_{C[0,T]}+\|b_1(t)-b_2(t)\|_{C[0,T]}+\right.$$

$$\left.+\|u_1(x,t)-u_2(x,t)\|_{B_{2,T}^{5,4}}\right) \tag{3.34}$$

Then, using (3.32) and (3.33), it follows from estimations (3.30) that the operator Φ acts in the sphere $K_R=K$ and it is contraction mapping. Therefore, in the sphere $K_R=K$, the operator has a unique fixed point $\{u,a,b\}$ that is a solution of equation (3.8),(3.12), (3.13).

The function $u(x,t)$, as the element of the space $B^{5,4}_{2,T}$, has continuous derivatives $u_x(x,t), u_{xx}(x,t), u_{xxx}(x,t), u_{xxxx}(x,t), u_t(x,t), u_{tx}(x,t)$ in D_T. Now from (3.2) we get:

$$\left(\sum_{k=1}^{\infty}(\lambda_k\|u_k''(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \le \sqrt{2}a^2\left(\sum_{k=1}^{\infty}(\lambda_k^5\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} +$$

$$+\sqrt{3}\left\|\|a(t)u_x(x,t)x + b(t)u_{tx}(x,t) + f_x(x,t)x\|_{C[0,T]}\right\|_{L_2(0,1)}.$$

Hence it follows that $u_{tt}(x,t)$is continuous in D_T.

It is easy to verify that equation (2.1) and conditions (2.2), (2.3), (2.8) - (2.10) are satisfied in the ordinary sense. Consequently, $\{u(x,t), a(t), b(t)\}$ is a solution of problem (2.1), (2.3), (2.8) - (2.10), and by Lemma 3.1, it is unique in the ball $K_R = K$. The theorem is proved.

The following theorem is proved by means of Lemma 2.1.

Theorem 3.3. *Let all the conditions of Theorem 3.2 be satisfied:*

$$\int_0^1 f(x,t)dx = 0 (0 \le t \le T), \ \int_0^1 \varphi(x)dx = 0, \ \int_0^1 \psi(x)dx = 0,$$

$$\varphi(0) = h_1(0), \psi(0) = h_1'(0), \ \varphi(1) = h_2(0), \psi(1) = h_2'(0),$$

Then in the sphere $K = K_R(\|z\|_{E_T^{5,4}} \le R = A(T) + 2)$ *of the space* E_T^5*, problem* (2.1)*,* (2.5) *has a unique classical solution.*

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