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AN APPROXIMATE SOLUTION OF LINEAR SINGULARLY PERTURBED PROBLEM WITH NONLOCAL BOUNDARY CONDITION

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Abstract. This study is about the finite difference method for the solution of the singularly perturbed problem with the boundary layer and integral boundary condition. Firstly, we determine the behavior of the exact solution and its derivative. Then we construct the finite difference scheme on Shishkin mesh. Finally, we prove the uniform convergence of the proposed difference scheme and give a numerical example, which shows the efficiency of the proposed method.

1. Introduction

An ordinary differential equation with a very small parameter ε multiplied by the highest order derivative term is known as the singular perturbation. Because of this epsilon parameter, the solution to these problems could result in several difficulties and so it becomes necessary to choose appropriate numerical methods. The mentioned problem occurs in various areas of applied mathematics and physics, for instance, in chemistry and physics, reaction-diffusion processes, describing exothermic and isothermal chemical reactions, the steady-state temperature distributions, heat transfer problems, heat conduction, chemical engineering, underground water flow, oceanography, meteorology, etc., [15, 18, 19, 20, 21, 25]. In recent years, there exists an improving interest in the numerical behavior of the singularly perturbed differential equations. The literature review indicates that there are existence and uniqueness studies on the solution of singularly perturbed problems [16, 20, 21, 24] and the references therein. For the numerical methods concerning singularly perturbed differential equations, one can see, e.g., [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 17, 22, 23]. In [8, 17] authors developed a finite difference scheme for a problem with integral boundary conditions.

The present study takes the following linear singularly perturbed problem with integral boundary condition into consideration as follows:

$$\varepsilon u''(x) + a(x)\, u'(x) = g(x)\,,\ 0 < x < 1, \tag{1.1}$$

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$$u'(0) = \frac{A}{\varepsilon}, \tag{1.2}$$

$$\int_0^1 b(x)u(x)dx = B, \tag{1.3}$$

where $0 < \varepsilon << 1$ is a small perturbation parameter, A and B are constants; $a^* > a(x) \geq \alpha > 0$; $g(x)$ and $b(x)$ are assumed to be sufficiently smooth functions in $[0,1]$.

The solution of the problem (1.1)-(1.3) has boundary layer near $x = 0$, where the behavior of the solution is observed to be highly rapid and irregular. Therefore, the finite difference method, which is an appropriate numerical method, is used to overcome this disadvantage.

Numerical solution and error estimation of the problem (1.1)-(1.3) were obtained on uniform mesh [17]. This present paper provides error estimation on Shishkin (nonuniform) mesh differently from [17]. In particular, P^N convergence rates are closer to one. The present study is structured as follows: Initially, several asymptotic properties of the solution (1.1)-(1.3) are investigated in Section 2. In Section 3, the finite difference scheme is constructed through interpolation quadrature rules. The error estimation of the difference scheme is evaluated in Section 4. A numerical experiment is presented to show how the method worked in Section 5.

In the following sections, C and C_0 are used to refer to the mean positive constants independent of ε and the mesh parameter.

2. Certain Properties of the Continuous Problem

In this section, highly important asymptotic properties of the exact solution (1.1)-(1.3) are provided.

Lemma 2.1. *Let $a(x)$, $g(x)$, $b(x) \in C^1[0,1]$ and*

$$b_1 = \int_0^1 b(x)dx \neq 0. \tag{2.1}$$

Then, the solution of the problem (1.1)-(1.3) fulfills the following inequalities:

$$|u(x)| \leq C_0, \tag{2.2}$$

where,

$$C_0 = |u(0)| + \alpha^{-1}(|A| + \int_0^x |g(s)|ds),$$

and

$$|u'(x)| \leq C\left\{1 + \frac{1}{\varepsilon}e^{\frac{-\alpha x}{\varepsilon}}\right\}. \tag{2.3}$$

Proof. Taking in account $u'(x) = v(x)$ in the Eq. (1.1), we obtain,

$$\varepsilon v' + a(x)v(x) = g(x), \tag{2.4}$$

$$v(0) = \frac{A}{\varepsilon}. \tag{2.5}$$

Once the problem (2.4)-(2.5) is solved, the following solution is obtained:

$$v(x) = \frac{A}{\varepsilon} e^{\frac{-1}{\varepsilon}\int_0^x a(\xi)d\xi} + \int_0^x g(\tau) e^{\frac{1}{\varepsilon}\int_\tau^x a(\eta)d\eta} d\tau. \tag{2.6}$$

Subsequently, the Eq. (2.6) is integrated over $(0, x)$ and we have,

$$u(x) = u(0) + \frac{A}{\varepsilon}\int_0^x e^{\frac{-1}{\varepsilon}\int_0^x a(\xi)d\xi} ds + \frac{-1}{\varepsilon}\int_0^s (\int_0^x g(\tau) e^{\frac{1}{\varepsilon}\int_\tau^s a(\eta)d\eta} d\tau) ds. \tag{2.7}$$

Herein, Eq. (2.7) is written in boundary condition (1.3) as follows:

$$\begin{aligned} u(0) &= \frac{1}{\int_0^1 b(x)dx} B - \frac{A}{\varepsilon}\int_0^1 b(x)\int_0^s e^{\frac{-1}{\varepsilon}\int_0^s a(\xi)d\xi} ds dx \\ &+ \frac{1}{\int_0^1 b(x)dx}(\frac{-1}{\varepsilon}\int_0^1 b(x)\int_0^s\int_0^x g(\tau) e^{\frac{1}{\varepsilon}\int_\tau^s a(\eta)d\eta} d\tau ds dx). \end{aligned} \tag{2.8}$$

We deduce that,

$$|u(0)| \leq \frac{|B - C|}{|b_1|} \leq C. \tag{2.9}$$

Finally, from (2.7) and (2.9), we obtain,

$$|u(x)| \leq C_0,$$

which proved (2.2). Here, the inequality (2.3) is examined,

$$|u'(x)| \leq |u'(0)||e^{\frac{-1}{\varepsilon}\int_0^x a(\xi)d\xi} + \int_0^x g(\tau) e^{\frac{1}{\varepsilon}\int_\tau^x a(\eta)d\eta} d\tau|. \tag{2.10}$$

Due to several calculations in Eq. (2.10), we obtain,

$$|u'(x)| \leq C + \frac{C}{\varepsilon}\left(e^{-\frac{\alpha x}{\varepsilon}}\right) \leq C\left\{1 + \frac{1}{\varepsilon}\left(e^{-\frac{\alpha x}{\varepsilon}}\right)\right\}.$$

Eventually, inequality (2.3) is obtained and the proof of Lemma 2.1 is proven. □

3. The Establishment of Difference Scheme

In this section, the problem (1.1)-(1.3) is discretized using a finite difference method on a piecewise uniform mesh of Shishkin type, which is introduced for the present study.

3.1. **Shishkin Mesh.** A Shishkin mesh is a piecewise uniform mesh, described for problem (1.1)-(1.3). For an even number N, the piecewise uniform mesh takes $N/2$ points in the interval $[0, \sigma]$ and also $N/2$ points in the interval $[\sigma, 1]$, where the transition point σ, which separates the fine and coarse portions of the mesh, is obtained by taking

$$\sigma = \min\left\{\frac{1}{2}, \alpha^{-1}\varepsilon \ln N\right\}. \tag{3.1}$$

In practice, $\sigma << 1$ is usually employed, so the mesh is fine on $[0, \sigma]$ and coarse on $[\sigma, 1]$. Hence, if we denote by hand $h^{(1)}$ and $h^{(2)}$ the step sizes in $[0, \sigma]$ and $[\sigma, 1]$, respectively, we have

$$h^{(1)} < N^{-1}, \quad N^{-1} < h^{(2)} < 2N^{-1}.$$

A set of the mesh points $\bar{\omega}_N = \{x_i\}_{i=0}^N$, are introduced

$$\bar{\omega}_N = \begin{cases} x_i = ih^{(1)}, \quad 0 \le i \le \frac{N}{2}; \\ x_i = \sigma + \left(i - \frac{N}{2}\right) h^{(2)}, \quad \frac{N}{2} + 1 \le i \le N; \\ h^{(1)} = \frac{2\sigma}{N}, \quad h^{(2)} = \frac{2(1-\sigma)}{N}. \end{cases}$$

For each $i \ge 1$ the step-size is set as $h_i = x_i - x_{i-1}$, $1 \le i \le N$.

3.2. **Construction of the Finite Difference Scheme.** A non-uniform mesh on the interval $[0, 1]$ is introduced

$$\omega_N = \{0 < x_1 < x_2 < ... < x_{N-1} < 1\},$$

and

$$\bar{\omega}_N = \omega_N \cup \{x_0 = 0, x_N = 1\}.$$

Prior to the description of the numerical method employ in the present study, it is essential to introduce several notations regarding the mesh functions. The following finite difference for any mesh function $u_i = u(x_i)$ given on $\bar{\omega}_N$ is defined:

$$u_{\bar{x},i} = \frac{u_i - u_{i-1}}{h_i}, \quad u_{x,i} = \frac{u_{i+1} - u_i}{h_{i+1}}, \quad u_{\overset{0}{x},i} = \frac{u_{x,i} + u_{\bar{x},i}}{2},$$

$$u_{\hat{x},i} = \frac{u_{i+1} - u_i}{\hbar_i}, \quad u_{\bar{x}\hat{x},i} = \frac{u_{x,i} - u_{\bar{x},i}}{\hbar_i}, \quad \hbar_i = \frac{h_i + h_{i+1}}{2},$$

$$\|u\|_\infty \equiv \|u\|_{\infty,\bar{\omega}_N} := \max_{0 \le i \le N} |u_i|.$$

The difference scheme for Eq. (1.1) is constructed. Initially, Eq. (1.1) is integrated over (x_{i-1}, x_{i+1}),

$$\hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} Lu(x)\varphi_i(x)dx = \hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} g(x)\varphi_i(x)dx, \ 1 \le i \le N-1, \tag{3.2}$$

where $\{\varphi_i(x)\}_{i=1}^{N-1}$ are the functions in the form

$$\varphi_i(x) = \begin{cases} \varphi_i^{(1)}(x) = \frac{e^{\frac{a_i(x-x_{i-1})}{\varepsilon}} - 1}{e^{\frac{a_i h_i}{\varepsilon}} - 1}, & x_{i-1} < x < x_i, \\ \varphi_i^{(2)}(x) = \frac{1 - e^{\frac{a_i(x-x_{i+1})}{\varepsilon}}}{1 - e^{\frac{-a_i h_{i+1}}{\varepsilon}}}, & x_i < x < x_{i+1}, \\ 0, & x \notin (x_{i-1}, x_{i+1}). \end{cases}$$

Due to several arrangements in Eq. (3.2), we obtain,

$$-\varepsilon\hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} u'(x)\varphi'_i(x)dx + a_i\hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} u'(x)\varphi_i(x)dx = g_i + R_{a,i} + R_{g,i}, \tag{3.3}$$

where

$$\begin{aligned} R_i = R_{a,i} + R_{g,i} \quad &= \quad \hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} [a(x_i) - a(x)] u'(x)\varphi_i(x)dx \\ &\quad + \hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} [g(x) - g(x_i)] \varphi_i(x)dx. \end{aligned} \tag{3.4}$$

The interpolating quadrature rules from [4] with weight functions $\varphi_i(x)$ in Eq. (3.3) and the following precise relation is obtained:

$$\varepsilon\theta_i u_{\widehat{x}\widehat{x},i} + \eta_i u_{\widehat{x},i} = g_i + R_i, 1 \leq i \leq N-1, \tag{3.5}$$

where,

$$\theta_i = \frac{a_i h_i}{e^{\frac{a_i h_i}{\varepsilon}} - 1}, \tag{3.6}$$

and

$$\eta_i = \frac{a_i}{1 - e^{\frac{-a_i h_{i+1}}{\varepsilon}}} - \frac{a_i h_i}{h_{i+1}(e^{\frac{a_i h_i}{\varepsilon}} - 1)}. \tag{3.7}$$

Herein, the difference scheme for the boundary condition (1.2) is obtained.
As a first step, Eq. (1.1) is integrated over (x_0, x_1),

$$\int_{x_0}^{x_1} Lu(x)\varphi_0(x)dx = \int_{x_0}^{x_1} g(x)\varphi_0(x)dx, \tag{3.8}$$

here $\varphi_0(x)$ is the basis function and it take the following form:

$$\varphi_0(x) = \begin{cases} \frac{1-e^{\frac{a_0(x-x_1)}{\varepsilon}}}{1-e^{\frac{-a_0 h_1}{\varepsilon}}}, & x_0 < x < x_1, \\ 0, & x \notin (x_0, x_1). \end{cases}$$

Consequent to using interpolation quadrature rules in Eq.(3.8), we obtain,

$$\varepsilon\phi_1 u_{x,0} - g_0\phi_0 = A + r_0,$$

where,

$$\begin{aligned} r_0 &= \int_{x_0}^{x_1} [a(x) - a(0)]\, u'(x)\varphi_0(x)dx \\ &\quad + \int_{x_0}^{x_1} [g(x) - g(0)]\, \varphi_0(x)dx, \end{aligned} \tag{3.9}$$

and

$$\phi_0 = \frac{-\varepsilon}{a_0} + \frac{h_1}{1 - e^{\frac{-a_0 h_1}{\varepsilon}}},$$

$$\phi_1 = \frac{a_0 h_1}{\varepsilon(1 - e^{\frac{-a_0 h_1}{\varepsilon}})}.$$

Subsequently, the difference scheme for the boundary condition (1.3) is acquired as:

$$\int_0^1 b(x)u(x)dx = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} b(x)u(x)dx. \tag{3.10}$$

Using rectangular rule for Eq. (3.10), we have,

$$\sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} b(x)u(x)dx = \sum_{i=1}^{N} b_i u_i h_i + r_1, \tag{3.11}$$

where,

$$r_1 = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (x - x_{i-1}) \frac{d}{dx} \left(b(x)u(x) \right) dx. \tag{3.12}$$

Therefore, the difference scheme for the boundary condition (1.3) is suggested as:

$$\sum_{i=1}^{N} b_i u_i h_i = B - r_1. \tag{3.13}$$

Once R_i, r_0 and r_1 are neglected in equations (3.4), (3.9) and (3.12), respectively, we propose the following difference scheme for approximation (1.1)-(1.3):

$$\varepsilon\theta_i y_{\bar{x}\hat{x},i} + \eta_i y_{\hat{x},i} = g_i, \quad 1 \le i \le N-1, \tag{3.14}$$

$$\varepsilon\phi_1 y_{x,0} - g_0\phi_0 = A, \tag{3.15}$$

$$\sum_{i=1}^{N} b_i y_i h_i = B. \tag{3.16}$$

4. Uniform Error Estimate

In this section, the convergence of the method proposed for the problem (1.1)-(1.3) is examined. The error function is provided as $z_i = y_i - u_i$, $0 \leqslant i \leqslant N$, where z_i is considered as the solution of the following discrete problem:

$$\varepsilon\theta_i z_{\bar{x}\hat{x},i} + \eta_i z_{\hat{x},i} = R_i, \quad 1 \le i \le N-1, \tag{4.1}$$

$$\varepsilon\phi_1 z_{x,0} = r_0, \tag{4.2}$$

$$\sum_{i=1}^{N} b_i z_i h_i = r_1. \tag{4.3}$$

Lemma 4.1. *Given that z_i is the solution (4.1)-(4.3), then the estimate holds,*

$$\|z\|_{\infty,\bar{\omega}_N} \le C \left\{ \|R\|_{\infty,\omega_N} + |r_0| + |r_1| \right\}. \tag{4.4}$$

Proof. Let $z_{\hat{x},i} = v_i$ in the Eq. (4.1). Then the following equation is obtained

$$\varepsilon\theta_i v_{\bar{x},i} + \eta_i v_{x,i} = R_i, \quad 1 \le i \le N-1,$$

and we have first-order difference equation with respect to v_i as,

$$v_i = \frac{\varepsilon\theta_i h_i^{-1}}{\varepsilon\theta_i h_i^{-1} + \eta_i} v_{i-1} + \frac{R_i}{\varepsilon\theta_i h_i^{-1} + \eta_i}.$$

Solving this first-order difference equation with respect to v_i, we get

$$v_i = v_0\phi_i + \sum_{k=1}^{i} \varphi_k \phi_{i-k}, \tag{4.5}$$

where

$$\phi_{i-k} = \begin{cases} 1, & k = i, \\ \prod_{j=k+1}^{i} \frac{\varepsilon\theta_j h_j^{-1}}{\varepsilon\theta_j h_j^{-1} + \eta_j}, & 0 \le k \le i-1, \end{cases}$$

and

$$\varphi_k = \frac{R_i}{\varepsilon\theta_i h_i^{-1} + \eta_i}.$$

Now, we obtain v_0 from Eq.(4.2) and it is written in the Eq.(4.5) as following:

$$v_0 = \frac{2r_0h_1}{\varepsilon\theta_1(h_0+h_1}),$$

and

$$z_i = z_{i-1} + (\frac{2\hbar_{i-1}r_0h_1}{\varepsilon\theta_1(h_0+h_1)})\frac{R_i}{\varepsilon\theta_i h_i^{-1}+\eta_i} + \hbar_{i-1}\sum_{k=1}^{i-1}\varphi_k\phi_{i-1-k}. \tag{4.6}$$

If the Eq.(4.6) is written in the second boundary condition Eq.(4.3), we have the following inequality

$$\|z_i\| \leq C\left\{|r_0| + |r_1| + \|R\|_{\infty,\omega_N}\right\}.$$

Thus, Lemma 4.1 is proven. □

Lemma 4.2. *Based on the assumptions from Section 1 and Lemma* 1 *the solution of the problem (1.1)-(1.3) fulfill the following estimates for the remainder terms* R_i, r_0 *and* r_1*:*

$$\|R\|_{\infty,\omega_N} \leq CN^{-1}\ln N, \tag{4.7}$$

$$|r_0| \leq CN^{-1}\ln N, \tag{4.8}$$

$$|r_1| \leq CN^{-1}\ln N. \tag{4.9}$$

Proof. Reminder term R_i is arranged as:

$$\begin{aligned} |R_i| \quad \leq \quad & |\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}}[a(x_i)-a(x)]\,u'(x)\varphi_i(x)dx \\ & +\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}}[g(x)-g(x_i)]\,\varphi_i(x)dx|, \end{aligned} \tag{4.10}$$

where,

$$|a(x_i)-a(x)| \leq Ch_i, \tag{4.11}$$

$$|g(x)-g(x_i)| \leq Ch_i, \tag{4.12}$$

and

$$\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}}\varphi_i(x)dx \leq Ch_i. \tag{4.13}$$

Through applying equations (4.11)-(4.13) with in Eq. (4.10), we obtain,

$$|R_i| \leq Ch_i + \frac{1}{\varepsilon}Ch_{i+1}\int_{x_{i-1}}^{x_{i+1}}|e^{\frac{-\alpha x}{\varepsilon}}|dx. \tag{4.14}$$

The remainder terms R_i, r_0 and r_1 are evaluated for the intervals $[0,\sigma]$ and $[\sigma,1]$, respectively:

In the first case for $\sigma = \frac{1}{2}$, $\frac{1}{2} < \alpha^{-1}\varepsilon\ln N$, $h^1 = h^2 = h = N^{-1}$ and $1 < i < N$:

$|R_i| \leq C\left\{h + \varepsilon^{-1}h\right\} \leq C\left\{N^{-1} + \varepsilon^{-1}(2\alpha^{-1}\varepsilon\ln N)N^{-1}\right\} \leq CN^{-1}\ln N.$

In the second case:

1) For $\sigma < \frac{1}{2}$, $\frac{1}{2} > \alpha^{-1}\varepsilon \ln N$ and $x_i \in [0, \sigma]$,

$$\begin{aligned} |R_i| &\leq C\left\{h^{(1)} + \varepsilon^{-1}h^{(1)}\right\} \leq C\left\{N^{-1} + \varepsilon^{-1}(2\alpha^{-1}\varepsilon \ln N)N^{-1}\right\} \\ &\leq CN^{-1}\ln N, \quad 1 \leq i \leq \frac{N}{2} - 1. \end{aligned}$$

2) For $\sigma < \frac{1}{2}$, $\frac{1}{2} > \alpha^{-1}\varepsilon \ln N$ and $x_i \in [\sigma, 1]$,

$$\begin{aligned} |R_i| &\leq C\left\{h^{(2)} + \alpha^{-1}(e^{\frac{-\alpha x_{i+1}}{\varepsilon}} - e^{\frac{-\alpha x_{i-1}}{\varepsilon}})\right\} \\ &\leq C\left\{2(1 - \alpha^{-1}\varepsilon \ln N)N^{-1} + \alpha^{-1}N\right\} \\ &\leq CN^{-1}\ln N, \quad \frac{N}{2} + 1 \leq i \leq N, \end{aligned}$$

where,

$$x_i = \sigma + (i - \frac{N}{2})h^{(2)} = \alpha^{-1}\varepsilon \ln N + (i - \frac{N}{2})h^{(2)},$$

$$e^{\frac{-\alpha x_{i+1}}{\varepsilon}} - e^{\frac{-\alpha x_{i-1}}{\varepsilon}} = e^{\frac{-\alpha x_{i+1}}{\varepsilon}}(1 - e^{\frac{-\alpha(x_{i+1}+x_{i-1})}{\varepsilon}}) \leq N^{-1}.$$

In the third case: R_i is evaluated for $i = \frac{N}{2}$,

$$\begin{aligned} \left|R_{\frac{N}{2}}\right| &\leq C\left\{(1 + \varepsilon^{-1})h^{(1)} + h^{(2)} + \frac{1}{\varepsilon}\int_{x_{i-1}}^{x_{i+1}} e^{\frac{-\alpha x}{\varepsilon}}dx\right\} \\ &\leq C\left\{(1 + \varepsilon^{-1})(2\alpha^{-1}\varepsilon \ln N)N^{-1} + 2(1 - \alpha^{-1}\varepsilon \ln N)N^{-1}\right\} \\ &\leq CN^{-1}\ln N. \end{aligned}$$

Based on all above mentioned situations,

$$|R_i| \leq CN^{-1}\ln N.$$

Herein, error function r_0 will be evaluated,

$$\begin{aligned} |r_0| &\leq |\int_{x_0}^{x_1} [a(x) - a(0)]\, u'(x)\varphi_0(x)dx \\ &\quad + \int_{x_0}^{x_1} [g(x) - g(0)]\, \varphi_0(x)dx|. \end{aligned} \tag{4.15}$$

Once the mean value theorem is applied for Eq. (4.15), it is possible to deduce that,

$$\begin{aligned} |r_0| &\leq C\left\{h^{(1)} + h^{(1)}\int_{x_0}^{x_1} |u'(x)|dx\right\} \\ &\leq C\left\{2N^{-1}\alpha^{-1}\varepsilon \ln N(1 + e^{\frac{-\alpha x_1}{\varepsilon}})\right\} \\ &\leq CN^{-1}\ln N, \end{aligned}$$

where,

$$e^{\frac{-\alpha x_1}{\varepsilon}} - e^{\frac{-\alpha x_0}{\varepsilon}} = e^{\frac{-\alpha x_1}{\varepsilon}}(1 - e^{\frac{-\alpha(x_0+x_1)}{\varepsilon}}) \leq N^{-1}.$$

Subsequently, error function r_1 will be evaluated as,

$$|r_1| \leq \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \mid (x - x_{i-1}) \frac{d}{dx} [b(x)u(x)] \, |dx, \tag{4.16}$$

where,

$$(x - x_{i-1}) \leq h_i, b^*(x) \leq C, u(x) \leq C_0. \tag{4.17}$$

It is possible to observe from Equations (4.16) and (4.17) that

$$|r_1| \leq CN^{-1} \ln N. \tag{4.18}$$

Thus, the proof of Lemma 4.2 is achieved. □

The convergence result of the present study could be indicated as the following Theorem 4.3.

Theorem 4.3. *Given $u(x)$ as the solution of the problem (1.1)-(1.3) and y_i as the solution of the difference scheme (3.14)-(3.16), the following uniform error estimate fulfills,*

$$\|y - u\|_{\infty, \bar{\omega}_N} \leq CN^{-1} \ln N.$$

5. Results for the Numerical Algorithm

a) Here, a new algorithm for the difference scheme (3.14)-(3.16) is introduced and the numerical results are presented as follows:

$$\left(\frac{\varepsilon\theta_i}{\hbar_i h_i}\right) y_{i-1}^{(n)} - \left(\frac{2\varepsilon\theta_i}{h_i h_{i+1}} + \frac{\eta_i}{\hbar_i}\right) y_i^{(n)} + \left(\frac{\varepsilon\theta_i}{\hbar_i h_{i+1}} + \frac{\eta_i}{\hbar_i}\right) y_{i+1}^{(n)} = -g_i, \quad 1 \leq i \leq N-1, \tag{5.1}$$

$$y_0^{(n)} = y_1^{(n)} - \frac{g_0\phi_0 + A}{\varepsilon\phi_1 h_1^{-1}}, \tag{5.2}$$

$$y_N^{(n)} = \left[B - \sum_{i=1}^{N-1} h_i b_i y_i^{(n-1)}\right] h_N^{-1} b_N^{-1}, \; n = 1, 2, 3, ..., \; y_i^{(0)} = 0.5. \tag{5.3}$$

The iteration (5.1)-(5.3) is easily calculated using the Thomas algorithm.

$$A_i = \frac{\varepsilon\theta_i}{\hbar_i h_i}, \quad B_i = \frac{\varepsilon\theta_i}{\hbar_i h_{i+1}} + \frac{\eta_i}{\hbar_i}, \quad C_i = \frac{2\varepsilon\theta_i}{h_i h_{i+1}} + \frac{\eta_i}{\hbar_i},$$

$$\alpha_1 = 1, \; \beta_1 = -\frac{g_0\phi_0 + A}{\varepsilon\phi_1 h_1^{-1}},$$

$$\begin{aligned} \alpha_{i+1} &= \frac{B_i}{C_i - A_i\alpha_i}, \; \beta_{i+1} = \frac{F_i + A_i\beta_i}{C_i - A_i\alpha_i}, \quad 1 \leq i \leq N-1, \\ y_i^{(n)} &= \alpha_{i+1} y_{i+1}^{(n)} + \beta_{i+1}, \quad i = N-1, ..., 2, 1. \end{aligned}$$

b) The following problem is examined in order to determine how the method worked:

$$\varepsilon u''(x) + u'(x) = 1, \quad 0 < x < 1,$$

$$u'(0) = \frac{1}{\varepsilon}, \qquad \int_0^1 u(x)dx = \frac{1}{2}.$$

The exact solution of this problem is provided via,

$$u(x) = x - \varepsilon(\varepsilon - 1)(1 - e^{\frac{-1}{\varepsilon}}) - (\varepsilon - 1)(1 - e^{\frac{-x}{\varepsilon}}) + \varepsilon - 1.$$

The corresponding $\varepsilon-$ uniform convergence rates are computed using the formula

$$P^N = \frac{\ln\left(e^N/e^{2N}\right)}{\ln 2}.$$

The error estimates are denoted by

$$e^N = \max_{\varepsilon} e_{\varepsilon}^N, \quad e_{\varepsilon}^N = \|y - u\|_{\infty,\bar{\omega}_N}.$$

TABLE 1. The computed maximum pointwise errors e^N and rates of convergence p^N

ε	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^{-1}	0.0089361	0.0044891	0.0022453	0.0011308	0.0006008	0.0003175
	0.99	0.99	0.98	0.99	0.99	
2^{-2}	0.0299224	0.0149851	0.0074905	0.0037376	0.0018802	0.0009082
	0.99	1.00	1.00	0.99	1.00	
2^{-3}	0.0692956	0.0360931	0.0183037	0.0091322	0.0045525	0.0021890
	0.94	0.97	1.00	1.00	1.04	
2^{-4}	0.1301416	0.0626738	0.0315452	0.0162414	0.0081723	0.0040476
	1.05	0.99	0.95	0.99	1.01	
2^{-5}	0.2531037	0.130746	0.0637075	0.0320654	0.0167370	0.0083105
	0.95	1.03	0.99	0.99	1.01	
e^N	0.2531037	0.130746	0.0637075	0.0320654	0.0167370	0.0083105
p^N	0.95	1.03	0.99	0.99	1.01	

The approximate errors and rates of convergence are tabulated in Table 1 for the given test problem in support of the theoretical results. The graph of the approximate solution of the example for different values of perturbation parameter is plotted in Figure 1. Furthermore, error distributions are demonstrated for $N = 64$ and different values of ε in Figure 2.

6. Conclusion

The present paper was successful in solving the problem (1.1)-(1.3) by using a robust numerical method. It was indicated that the method demonstrated uniform convergence for perturbation parameter ε on Shishkin mesh. Based on the results of the error estimation, it was proved that the convergence was first-order. A numerical experiment was conducted to demonstrate the effectiveness and accuracy of the present method. The results were demonstrated through tables and figures. We confidently recommend that finite difference methods can be used to solve different types of differential equations i.e. [1, 2].

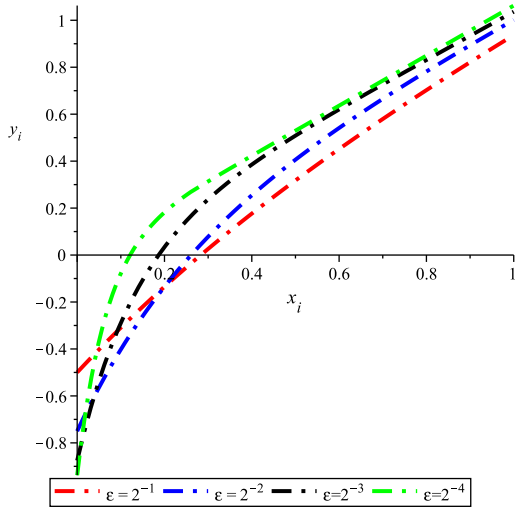

FIGURE 1. Approximate solution of test problem for $N = 64$.

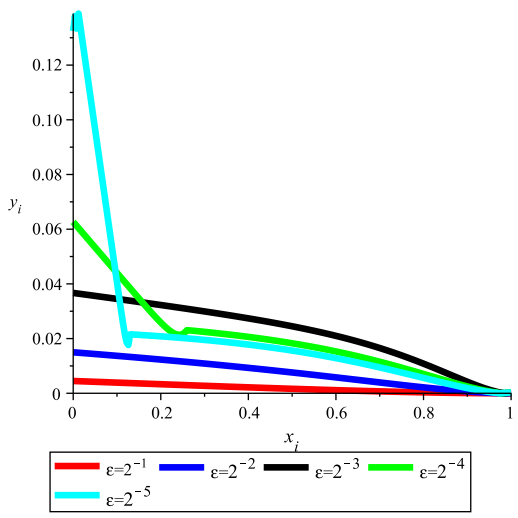

FIGURE 2. Error distribution of test problem for $N = 64$.

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