# DYNAMICAL STUDY OF A CLASS OF SYSTEMS OF DIFFERENCE EQUATIONS 

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#### Abstract

This paper presents a study of the system of nonlinear difference equations $x_{n+1}=\frac{x_{n-2 q+1}}{A+B x_{n-q+1} y_{n-2 q+1}}, y_{n+1}=\frac{y_{n-2 q+1}}{A+B y_{n-q+1} x_{n-2 q+1}}$ with nonzero arbitrary initial conditions, where $A$ and $B$ are arbitrary parameters and $q$ is an arbitrary non-negative integer. We obtain closed forms for the solutions of this system and give a complete investigation of their convergence. Local and global stability of the equilibrium points are discussed. Numerical examples are given to confirm the correctness of the analytical results.


## 1. Introduction

Difference equations have been extensively studied in recent years. They play a vital role in describing dynamical systems and presenting numerical schemes. Various applications of discrete dynamical systems and difference equations have widely grown in many fields such as ecology, population dynamics, engineering, mathematical biology, physics, and game theory. For instance, systems of difference equations are used in mathematical biology to model competitive interaction between two species or to describe predator-prey models (see [17, 13, 32, 41] and references therein). Khan and Qureshi [20] investigated a modified Nicholson-Bailey model which describes a host-parasitoid phenomena. For more applications of difference equations we refer the reader to $[1,2,3,8,10,11,22,26,27,38]$. In addition to their applications in other fields, difference equations play an important role in mathematics as a whole. For example, Mazzia and Trigiante [28] described how difference equations are used in numerical analysis to approximate solutions of differential equations. Studying the qualitative behavior of nonlinear rational difference equations has attracted many researchers due to their paramount importance.

Papaschinopoulos and Schinas [30] considered the system

$$
x_{n+1}=A+\frac{y_{n}}{x_{n-p}}, y_{n+1}=B+\frac{x_{n}}{y_{n-q}}
$$

[^0]where $p, q$ are positive integers. They studied some of its properties such as oscillatory behavior, boundedness of the solutions, and the global asymptotic stability of the equilibrium points.

Clark and Kulenovic [5] and Clark et al., [6] discussed the asymptotic behavior and the global stability of the system

$$
x_{n+1}=\frac{x_{n}}{a+c y_{n}}, y_{n+1}=\frac{y_{n}}{b+d x_{n}} .
$$

Kurbanli et al. [24] studied the positive solutions of the system

$$
x_{n+1}=\frac{x_{n-1}}{y_{n} x_{n-1}+1}, y_{n+1}=\frac{y_{n-1}}{x_{n} y_{n-1}+1} .
$$

As an extension, Elsayed [16] obtained the solutions of the system

$$
x_{n+1}=\frac{x_{n-1}}{ \pm 1+x_{n-1} y_{n}}, y_{n+1}=\frac{y_{n-1}}{ \pm 1+y_{n-1} x_{n}}
$$

with nonzero real number initial conditions.
Touafek and Elsayed [35] investigated the periodicity and the solutions of the system

$$
x_{n+1}=\frac{x_{n-3}}{ \pm 1 \pm x_{n-3} y_{n-1}}, y_{n+1}=\frac{y_{n-3}}{ \pm 1 \pm y_{n-3} x_{n-1}} .
$$

In [37], Wang et al. considered a more general class where they studied the the asymptotic behavior and determined the solution expression of the system

$$
x_{n+1}=\frac{x_{n-3}}{A+x_{n-3} y_{n-1}}, y_{n+1}=\frac{y_{n-3}}{B+y_{n-3} x_{n-1}}, A, B \in[0, \infty) .
$$

Khan et. al. [21] investigated the qualitative behavior of the following two systems

$$
\begin{aligned}
x_{n+1} & =\frac{\alpha x_{n-1}}{\beta+\gamma y_{n} y_{n-1}}, y_{n+1}=\frac{\alpha_{1} y_{n-1}}{\beta_{1}+\gamma_{1} x_{n} x_{n-1}} \\
x_{n+1} & =\frac{a y_{n-1}}{b+c x_{n} x_{n-1}}, y_{n+1}=\frac{a_{1} x_{n-1}}{b_{1}+c_{1} y_{n} y_{n-1}}
\end{aligned}
$$

where all of the parameters are positive real numbers.
In 2017, Wang et al. [36] used variational iteration techniques to describe the asymptotic behavior of the equilibrium points of the systems of difference equations

$$
x_{n+1}=\frac{x_{n-1} x_{n-2}}{A+B y_{n-3}}, y_{n+1}=\frac{y_{n-1} y_{n-2}}{C+D x_{n-3}} .
$$

In 2019, Liu et al. [25] used the variational iteration techniques to study the system

$$
x_{n+1}=\frac{x_{n-3}-y_{n-1}}{A+x_{n-3} y_{n-1}}, y_{n+1}=\frac{y_{n-3}-x_{n-1}}{A+y_{n-3} x_{n-1}}
$$

Recently, Gelisken and Kara [18] investigated four general systems of rational difference equations of order $3 k$, where $k$ is a positive integer. For more studies of difference equations and systems of difference equations see also $[4,7,9,15,29,33$, $34,36,40]$.

Motivated by the above studies, our main goal in this paper is to study the class of nonlinear rational difference equations given by

$$
\begin{equation*}
x_{n+1}=\frac{x_{n-2 q+1}}{A+B x_{n-2 q+1} y_{n-q+1}}, y_{n+1}=\frac{y_{n-2 q+1}}{A+B y_{n-2 q+1} x_{n-q+1}}, q=1,2, \ldots \tag{1.1}
\end{equation*}
$$

where $A$ and $B$ are arbitrary constants and with arbitrary nonzero initial conditions $x_{-2 q+1}=a_{-2 q+1}, x_{-2 q+2}=a_{-2 q+2}, \ldots, x_{0}=a_{0}, y_{-2 q+1}=b_{-2 q+1}, y_{-2 q+2}=$
$b_{-2 q+2}, \ldots, y_{0}=b_{0}$. We give a detailed analytical study of qualitative behavior of this class, where we obtain the solution expressions for this system and provide a complete analysis of their convergence. We also determine all equilibrium points and discuss their stability. This study improves and surpasses results about many forms of systems of difference equations such as the ones studied in $[24,16,35,37]$. The generalization is obtained by keeping the order of this system, namely $2 q$, as an arbitrary parameter.

In the next section we introduce some basic definitions and primary results that will be needed in later sections. In Section 3, we obtain the solution expressions of system (1.1). In Section 4, the equilibrium points are determined and their stability is investigated. In Section 5, we present a detailed study for the convergence of the solutions of System (1.1). Numerical examples are given in Section 6 to illustrate the analytical results.

## 2. Preliminaries

A system of two difference equations of order $k$ consists of two equations of the form

$$
\begin{aligned}
& x_{n+1}=F\left(x_{n}, x_{n-1}, \ldots, x_{n-(k-1)}, y_{n}, y_{n-1}, \ldots, y_{n-(l-1)}\right), \\
& y_{n+1}=G\left(x_{n}, x_{n-1}, \ldots, x_{n-(k-1)}, y_{n}, y_{n-1}, \ldots, y_{n-(l-1)}\right),
\end{aligned}
$$

$$
n=0,1, \cdots,(2.1)
$$

where

$$
F=F\left(u_{0}, u_{1}, \ldots, u_{k-1},, v_{0}, v_{1}, \ldots, v_{l-1}\right)
$$

and

$$
G=G\left(u_{0}, u_{1}, \ldots, u_{k-1},, v_{0}, v_{1}, \ldots, v_{l-1}\right)
$$

are functions that map some set $I^{k+l}$ into $I$. The set $I$ is usually an interval of real numbers, or a union of intervals. A solution of System (2.1) is a sequence $X_{n}=\left(x_{n}, y_{n}\right)_{n \geq-k+1}$ that satisfies System (2.1) for all $n \geq 0$.
Definition 1. A point $(\bar{x}, \bar{y}) \in \mathbb{R} \times \mathbb{R}$ is called an equilibrium point of System (2.1), if $\bar{x}=F(\bar{x}, \bar{x}, \ldots, \bar{x}, \bar{y}, \bar{y}, \ldots, \bar{y})$, and $\bar{y}=G(\bar{x}, \bar{x}, \ldots, \bar{x}, \bar{y}, \bar{y}, \ldots, \bar{y})$.

Definition 2. Let $(\bar{x}, \bar{y})$ be an equilibrium point of System (2.1). Then
(1) $(\bar{x}, \bar{y})$ is called locally stable if, for every $\varepsilon>0$, there exists $\delta>0$ such that, for any initial condition $\left(x_{i}, y_{i}\right) \in \mathbb{R}^{2},(i=-k, \ldots, 0, j=-l, \ldots, 0)$, with $\sum_{i=-k}^{0}\left|x_{i}-\bar{x}\right|<\delta, \sum_{i=-l}^{0}\left|y_{i}-\bar{y}\right|<\delta$, we have $\left|x_{n}-\bar{x}\right|<\varepsilon,\left|y_{n}-\bar{y}\right|<\varepsilon$ for all $n \geq 0$. If $(\bar{x}, \bar{y})$ is not locally stable then it is called unstable.
(2) $(\bar{x}, \bar{y})$ is called attractor if $\lim _{n}\left(x_{n}, y_{n}\right)=(\bar{x}, \bar{y})$ for any initial conditions $\left(x_{i}, y_{i}\right) \in \mathbb{R}^{2},(i=-k, \ldots, 0, j=-l, \ldots, 0)$.
(3) If $(\bar{x}, \bar{y})$ is both stable and attractor then it is called asymptotically stable.

Definition 3. Let $(\bar{x}, \bar{y})$ be an equilibrium point of the vector map

$$
H\left(F, x_{n}, \ldots, x_{n-k}, G, y_{n}, \ldots, y_{n-l}\right)
$$

where $F$ and $G$ are continuously differentiable functions at $(\bar{x}, \bar{y})$. Then the linearized system of (2.1) about $(\bar{x}, \bar{y})$ is

$$
\begin{equation*}
X_{n+1}=H\left(X_{n}\right)=H(\bar{x}, \bar{y}) \cdot X_{n}, \tag{2.2}
\end{equation*}
$$

where $H(\bar{x}, \bar{y})$ is the Jacobian matrix of the System (2.1) about $(\bar{x}, \bar{y})$ and $X_{n}=$ $\left(x_{n}, \ldots, x_{n-k}, y_{n}, \ldots, y_{n-l}\right)$.

Lemma 2.1. Let $H$ be the Jacobian matrix of a system of difference equations $X_{n+1}=H\left(X_{n}\right), n=0,1, \cdots$ about an equilibrium point $\bar{X}$. Then we have
(1) If all eigenvalues of $H$ lie inside the open unit disk, then $\bar{X}$ is locally asymptotically stable.
(2) If one of the eigenvalues of $H$ lie outside the closed unit disk, then $\bar{X}$ is unstable.

Definition 4. Let $H$ be the Jacobian matrix of a system of difference equations $X_{n+1}=H\left(X_{n}\right), n=0,1, \cdots$ about an equilibrium point $\bar{X}$, and let $\Lambda$ be the set of all eigenvalues of $H$. Then
(1) $\bar{X}$ is said to be hyperbolic if $|\lambda| \neq 1$ for all $\lambda \in \Lambda$, otherwise it is called nonhyperbolic;
(2) $\bar{X}$ is called saddle if there are two eigenvalues $\lambda_{1}, \lambda_{2} \in \Lambda$ such that $\left|\lambda_{1}\right|<1$ and $\left|\lambda_{2}\right|>1$;
(3) $\bar{X}$ is called repeller if $|\lambda|>1$ for all $\lambda \in \Lambda$.

## 3. Solution Expressions

In this section, we give the solution expressions for System (1.1).
Theorem 3.1. Let $\left(x_{n}, y_{n}\right)_{n \geq-2 q+1}$ be a sequence given by System (1.1), then for each $r \in\{2 q-1,2 q-2, \ldots, q\}$ and for all $n \geq 0$, we have

$$
\begin{gather*}
x_{2 q n-r}=a_{-r} \frac{\prod_{p=1}^{n-1} S_{2 p}}{\prod_{p=0}^{n-1} S_{2 p+1}}  \tag{3.1}\\
y_{2 q n-r}=b_{-r} \frac{\prod_{p=1}^{n-1} T_{2 p}}{\prod_{p=0}^{n-1} T_{2 p+1}}  \tag{3.2}\\
x_{2 q n-r+q}=a_{-r+q} \prod_{p=0}^{n-1}\left(\frac{T_{2 p+1}}{T_{2 p+2}}\right)  \tag{3.3}\\
y_{2 q n-r+q}=b_{-r+q} \prod_{p=0}^{n-1}\left(\frac{S_{2 p+1}}{S_{2 p+2}}\right) \tag{3.4}
\end{gather*}
$$

where, for each non negative integer $l$,

$$
\begin{equation*}
S_{l}=A^{l}+B a_{-r} b_{-r+q} \sum_{k=0}^{l-1} A^{k}, \quad \text { and } \quad T_{l}=A^{l}+B b_{-r} a_{-r+q} \sum_{k=0}^{l-1} A^{k} \tag{3.5}
\end{equation*}
$$

Proof. We prove this result by induction on $n$. It is evident that the results hold for $n=0$. Let $n \geq 0$ be an integer, and suppose that the equations (3.1), (3.2), (3.3), (3.4) hold for all non-negative integers $k \leq n$. We shall now prove that they hold for the step $n+1$.

$$
x_{2 q(n+1)-r}=\frac{x_{2 q n-r}}{A+B x_{2 q n-r} y_{2 q n-r+q}}
$$

$$
\begin{aligned}
& =\frac{a_{-r}\left(\prod_{p=1}^{n-1} S_{2 p}\right) /\left(\prod_{p=0}^{n-1} S_{2 p+1}\right)}{A+B b_{-r+q} a_{-r} \prod_{p=0}^{n-1}\left(\frac{S_{2 p+1}}{S_{2 p+2}}\right)\left(\prod_{p=1}^{n-1} S_{2 p}\right) /\left(\prod_{p=0}^{n-1} S_{2 p+1}\right)} \\
& =\frac{a_{-r} \prod_{p=1}^{n-1} S_{2 p} \prod_{p=1}^{n} S_{2 p}}{A \prod_{p=1}^{n} S_{2 p} \prod_{p=0}^{n-1} S_{2 p+1}+B a_{-r} b_{-r+q} \prod_{p=0}^{n-1} S_{2 p+1} \prod_{p=1}^{n-1} S_{2 p}} \\
& =\frac{a_{-r} \prod_{p=1}^{n} S_{2 p}}{\left(A S_{2 n}+B a_{-r} b_{-r+q}\right) \prod_{p=0}^{n-1} S_{2 p+1}} \\
& =\frac{a_{-r} \prod_{p=1}^{n} S_{2 p}}{\left(A\left(A^{2 n}+B a_{-r} b_{-r+q} \sum_{k=0}^{2 n-1} A^{k}\right)+B a_{-r} b_{-r+q}\right) \prod_{p=0}^{n-1} S_{2 p+1}} \\
& =\frac{a_{-r} \prod_{p=1}^{n} S_{2 p}}{\prod_{p=0}^{n} S_{2 p+1}}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
y_{2 q(n+1)-r} & =\frac{y_{2 q n-r}}{A+B y_{2 q n-r} x_{2 q n-r+q}} \\
& =\frac{b_{-r}\left(\prod_{p=1}^{n-1} T_{2 p}\right) /\left(\prod_{p=0}^{n-1} T_{2 p+1}\right)}{A+B a_{-r+q} b_{-r} \prod_{p=0}^{n-1}\left(\frac{T_{2 p+1}}{T_{2 p+2}}\right)\left(\prod_{p=1}^{n-1} T_{2 p}\right) /\left(\prod_{p=0}^{n-1} T_{2 p+1}\right)} \\
& =\frac{b_{-r} \prod_{p=1}^{n-1} T_{2 p} \prod_{p=1}^{n} T_{2 p}}{A \prod_{p=1}^{n} T_{2 p} \prod_{p=0}^{n-1} T_{2 p+1}+B b_{-r} a_{-r+q} \prod_{p=0}^{n-1} T_{2 p+1} \prod_{p=1}^{n-1} T_{2 p}} \\
& =\frac{b_{-r} \prod_{p=1}^{n} T_{2 p}}{\left(A T_{2 n}+B b_{-r} a_{-r+q}\right) \prod_{p=0}^{n-1} T_{2 p+1}} \\
& =\frac{b_{-r} \prod_{p=1}^{n} T_{2 p}}{\left(A\left(A^{2 n}+B b_{-r} a_{-r+q} \sum_{k=0}^{2 n-1} A^{k}\right)+B b_{-r} a_{-r+q}\right) \prod_{p=0}^{n-1} T_{2 p+1}} \\
& =\frac{b_{-r} \prod_{p=1}^{n} T_{2 p}}{\prod_{p=0}^{n} T_{2 p+1}} .
\end{aligned}
$$

Now we confirm that (3.3) holds for $n+1$.

$$
\begin{aligned}
x_{2 q(n+1)-r+q} & =\frac{x_{2 q n-r+q}}{A+B y_{2 q n-r+2 q} x_{2 q n-r-q}} \\
& =\frac{x_{2 q n-r+q}}{A+B y_{2 q(n+1)-r} x_{2 q n-r-q}} \\
& =\frac{a_{-r+q} \prod_{p=0}^{n-1}\left(\frac{T_{2 p+1}}{T_{2 p+2}}\right)}{A+B b_{-r} a_{-r+q} \prod_{p=0}^{n-1}\left(\frac{T_{2 p+1}}{T_{2 p+2}}\right)\left(\prod_{p=1}^{n} T_{2 p}\right) /\left(\prod_{p=0}^{n} T_{2 p+1}\right)} \\
& =\frac{a_{-r+q} \prod_{p=0}^{n-1} T_{2 p+1}}{\left.\left(\prod_{p=0}^{n-1} T_{2 p+1}\right)\left(A T_{2 n+1} \prod_{p=0}^{n-1} T_{2 p+2}+B b_{-r} a_{-r+q} \prod_{p=1}^{n} T_{2 p}\right)\right) /\left(\prod_{p=0}^{n} T_{2 p+1}\right)}
\end{aligned}
$$

$$
\begin{aligned}
& =\frac{a_{-r+q} \prod_{p=0}^{n} T_{2 p+1}}{A T_{2 n+1} \prod_{p=1}^{n} T_{2 p}+B b_{-r} a_{-r+q} \prod_{p=1}^{n} T_{2 p}} \\
& =\frac{a_{-r+q} \prod_{p=0}^{n} T_{2 p+1}}{\left(A T_{2 n+1}+B b_{-r} a_{-r+q}\right) \prod_{p=1}^{n} T_{2 p}} \\
& =\frac{a_{-r+q} \prod_{p=0}^{n} T_{2 p+1}}{\left(A\left(A^{2 n+1}+B b_{-r} a_{-r+q} \sum_{k=0}^{2 n} A^{k}\right)+B b_{-r} a_{-r+q}\right) \prod_{p=1}^{n} T_{2 p}} \\
& =\frac{a_{-r+q} \prod_{p=0}^{n} T_{2 p+1}}{T_{2 n+2} \prod_{p=1}^{n} T_{2 p}} .
\end{aligned}
$$

By similarity we obtain identity (3.4).
Next, we obtain simplified expressions of Eq. (3.1), (3.2), (3.3) and (3.4) when $A=1$ and $A \neq 1$, which will be used in the next section

Corollary 3.2. Let $\left(x_{n}, y_{n}\right)_{n \geq-2 q+1}$ be a sequence defined by System (1.1). If $A=1$, then for each $r \in\{2 q-1,2 q-2, \ldots, q\}$,

$$
\begin{gather*}
x_{2 q n-r}=a_{-r} \frac{\prod_{p=1}^{n-1}\left(1+2 p B a_{-r} b_{-r+q}\right)}{\prod_{p=0}^{n-1}\left(1+(2 p+1) B a_{-r} b_{-r+q}\right)},  \tag{3.6}\\
y_{2 q n-r}=b_{-r} \frac{\prod_{p=1}^{n-1}\left(1+2 p B b_{-r} a_{-r+q}\right)}{\prod_{p=0}^{n-1}\left(1+(2 p+1) B b_{-r} a_{-r+q}\right)} \tag{3.7}
\end{gather*}
$$

and for each $s \in\{q-1, q-2, \ldots, 0\}$,

$$
\begin{align*}
& x_{2 q n-s}=a_{-s} \prod_{p=0}^{n-1}\left(\frac{1+(2 p+1) B a_{-s} b_{-s-q}}{1+(2 p+2) B a_{-s} b_{-s-q}}\right) .  \tag{3.8}\\
& y_{2 q n-s}=b_{-s} \prod_{p=0}^{n-1}\left(\frac{1+(2 p+1) B b_{-s} a_{-s-q}}{1+(2 p+2) B b_{-s} a_{-s-q}}\right) . \tag{3.9}
\end{align*}
$$

Corollary 3.3. Let $\left(x_{n}, y_{n}\right)_{n \geq-2 q+1}$ be a sequence defined by the equation (1.1) with $A \neq 1$. Then for $r \in\{2 q-1,2 q-2, \ldots, q\}$,

$$
\begin{equation*}
x_{2 q n-r}=a_{-r}(1-A) \frac{\prod_{p=1}^{n-1}\left(B a_{-r} b_{-r+q}-\left(A-1+B a_{-r} b_{-r+q}\right) A^{2 p}\right)}{\prod_{p=0}^{n-1}\left(B a_{-r} b_{-r+q}-\left(A-1+B a_{-r} b_{-r+q}\right) A^{2 p+1}\right)}, \tag{3.10}
\end{equation*}
$$

$$
\begin{equation*}
y_{2 q n-r}=b_{-r}(1-A) \frac{\prod_{p=1}^{n-1}\left(B b_{-r} a_{-r+q}-\left(A-1+B b_{-r} a_{-r+q}\right) A^{2 p}\right)}{\prod_{p=0}^{n-1}\left(B b_{-r} a_{-r+q}-\left(A-1+B b_{-r} a_{-r+q}\right) A^{2 p+1}\right)} \tag{3.11}
\end{equation*}
$$

and for $s \in\{q-1, q-2, \ldots, 0\}$,

$$
\begin{align*}
& x_{2 q n-s}=a_{-s} \prod_{p=1}^{n-1}\left(\frac{B b_{-s-q} a_{-s}-\left(A-1+B b_{-s-q} a_{-s}\right) A^{2 p+1}}{B b_{-s-q} a_{-s}-\left(A-1+B b_{-s-q} a_{-s}\right) A^{2 p}}\right)  \tag{3.12}\\
& y_{2 q n-s}=b_{-s} \prod_{p=1}^{n-1}\left(\frac{B a_{-s-q} b_{-s}-\left(A-1+B a_{-s-q} b_{-s}\right) A^{2 p+1}}{B a_{-s-q} b_{-s}-\left(A-1+B a_{-s-q} b_{-s}\right) A^{2 p}}\right) \tag{3.13}
\end{align*}
$$

Proof. Since $|A| \neq 1$ we get that for any integer $l$,

$$
\sum_{k=0}^{l-1} A^{k}=\frac{1-A^{l}}{1-A}
$$

Substituting this in Eq. (3.5) yields
$S_{l}=\frac{B a_{-r} b_{-r+q}-\left(A-1+B a_{-r} b_{-r+q}\right) A^{l}}{1-A}, T_{l}=\frac{B b_{-r} a_{-r+q}-\left(A-1+B b_{-r} a_{-r+q}\right) A^{l}}{1-A}$.
Then the result follows by substituting these expressions in Eqs. (3.1), (3.2), (3.3), and (3.4).

## 4. Analysis of the equilibrium points

We start this section by the following theorem which determines the equilibrium points of System (1.1).
Theorem 4.1. Let $\left(x_{n}, y_{n}\right)_{n \geq-2 q+1}$ be a solution of System (1.1).
(1) If $A=1$, then the equilibrium points of System (1.1) are $(s, 0),(0, t)$, $s, t \in \mathbb{R}$.
(2) If $A \neq 1$, then the equilibrium points of System (1.1) are $(0,0),\left(r, \frac{1-A}{B r}\right)$, $r \in \mathbb{R}^{*}$.

Proof. Let $(\bar{x}, \bar{y})$ be an equilibrium point of System (1.1). Then we have

$$
\bar{x}(A+B \bar{x} \bar{y}-1)=0 \text { and } \bar{y}(A+B \bar{x} \bar{y}-1)=0
$$

If $A=1$, then we get $B \bar{x}^{2} \bar{y}=0$ and $B \bar{x} \bar{y}^{2}=0$, which yields $\bar{x}=0$ or $\bar{y}=0$. Thus the equilibrium points of System (1.1) are $(s, 0)$ and $(0, t), s, t \in \mathbb{R}$. Suppose $A \neq 1$. If $\bar{x}=0$, then we get $\bar{y}(A-1)=0$. Since $A \neq 1, \bar{y}=0$, and hence we obtain the equilibrum point $(0,0)$. Assume $\bar{x} \neq 0$. Then $A+B \bar{x} \bar{y}-1=0$, which implies that $\bar{x}=r$ and $\bar{y}=\frac{1-A}{B r}$ where $r$ can be any nonzero real number.

To construct the linearized form of the nonlinear System (1.1), we consider the transformation

$$
\begin{equation*}
\left(x_{n}, \ldots, x_{n-2 q+1}, y_{n}, \ldots, y_{n-2 q+1}\right) \longmapsto\left(f, \ldots, f_{2 q-1}, g, \ldots, g_{2 q-1}\right) \tag{4.1}
\end{equation*}
$$

where

$$
f=\frac{x_{n-2 q+1}}{A+B x_{n-2 q+1} y_{n-q+1}}, g=\frac{y_{n-2 q+1}}{A+B y_{n-2 q+1} x_{n-q+1}}
$$

$$
f_{i}=x_{n-i+1}, g_{i}=y_{n-i+1}, i=1, \cdots, 2 q-1 .
$$

The Jacobian matrix about an equilibrium point $(\bar{x}, \bar{y})$ under this transformation is the $4 q \times 4 q$ matrix $H(\bar{x}, \bar{y})=\left[h_{i j}\right]$ whose entries are given by

$$
\begin{gathered}
h_{1,2 q}=h_{2 q, 4 q}=\frac{A}{(A+B \bar{x} \bar{y})^{2}}, h_{1,3 q}=\frac{-B \bar{x}^{2}}{(A+B \bar{x} \bar{y})^{2}}, h_{2 q+1, q}=\frac{-B \bar{y}^{2}}{(A+B \bar{x} \bar{y})^{2}}, \\
h_{i, i-1}=h_{2 q+i, 2 q+i-1}=1, i=2, \ldots, 2 q,
\end{gathered}
$$

and $h_{i j}=0$ otherwise. Let $\alpha=\frac{A}{(A+B \bar{x} \bar{y})^{2}}, \rho=\frac{-B \bar{x}^{2}}{(A+B \bar{x} \bar{y})^{2}}, \sigma=\frac{-B \bar{y}^{2}}{(A+B \bar{x} \bar{y})^{2}}$, and $D$ and $C(\omega)$ be the $q \times q$ matrices of the forms

$$
D=\left[\begin{array}{cccc}
-\lambda & 0 & \cdots & 0 \\
1 & -\lambda & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 1 & -\lambda
\end{array}\right], C(\omega)=\left[\begin{array}{cccc}
0 & \cdots & 0 & \omega \\
0 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0
\end{array}\right]
$$

Then $H-\lambda I$ can be written in the form of the $4 \times 4$ block matrix

$$
H-\lambda I=\left[\begin{array}{cccc}
D & C(\alpha) & C(\rho) & O \\
O & D & O & O \\
C(\sigma) & O & D & C(\alpha) \\
O & O & O & D
\end{array}\right]
$$

Now with some tedious calculations we get that the characteristic polynomial of the Jacobian matrix is

$$
\begin{equation*}
p(\lambda)=\lambda^{4 q}-(2 \alpha+\beta) \lambda^{2 q}+\alpha^{2}, \tag{4.2}
\end{equation*}
$$

where

$$
\begin{equation*}
\alpha=\frac{A}{(A+B \bar{x} \bar{y})^{2}} \quad, \quad \beta=\rho \sigma=\frac{B^{2} \bar{x}^{2} \bar{y}^{2}}{(A+B \bar{x} \bar{y})^{4}} \tag{4.3}
\end{equation*}
$$

Theorem 4.2. The following are true about the equilibrium points of System (1.1).
(1) The equilibrium point $(0,0)$ is
(a) locally asymptotically stable when $|A|>1$;
(b) nonhyperbolic when $|A|=1$;
(c) repeller and unstable when $|A|<1$;
(2) All equilibrium points other than $(0,0)$ are nonhyperbolic.

Proof. (1) By substituting $(\bar{x}, \bar{y})=(0,0)$ in (4.3), we get $\alpha=\frac{1}{A}, \beta=0$. So the characteristic equation (4.2) of the Jacobian matrix of System (1.1) about (0, 0) can be written as

$$
\begin{equation*}
\lambda^{4 q}-\frac{2}{A} \lambda^{2 q}+\frac{1}{A^{2}}=\left(\lambda^{2 q}-\frac{1}{A}\right)^{2}=0 . \tag{4.4}
\end{equation*}
$$

Hence the roots of Eq. (4.4) satisfy $|\lambda|=\left|\frac{1}{A}\right|^{1 / 2 q}$. If $|A|>1$ then all roots of Eq. (4.4) lie inside the open unit disk, which implies that $(0,0)$ is locally asymptotically stable. If $|A|=1$, then all eigenvalues satisfy $|\lambda|=1$, which yields $(0,0)$ is nonhyperbolic. If $|A|<1$ then all roots of Eq. (4.4) lie outside the open unit disk, which implies $(0,0)$ is repeller and unstable.
(2) If $A=1$, then by Theorem (4.1), the equilibrium points of System (1.1) are $(s, 0),(0, t), s, t \in \mathbb{R}$. For each one of these points we have $\alpha=1$ and $\beta=0$. Thus all roots of Eq. (4.4) satisfy $|\lambda|=1$, which implies that each one of these equilibrium points is nonhyperbolic. If $A \neq 1$, then the equilibrium points other
than $(0,0)$ are $\left(r, \frac{1-A}{B r}\right), r \in \mathbb{R}$. Substituting $(\bar{x}, \bar{y})=\left(r, \frac{1-A}{B r}\right)$ in (4.3) yields $\alpha=A$, and $\beta=(1-A)^{2}$. Then from (4.2) we get that $\lambda^{4 q}-\left(A^{2}+1\right) \lambda^{2 q}+$ $A^{2}=\left(\lambda^{2 q}-A^{2}\right)\left(\lambda^{2 q}-1\right)=0$. So some eigenvalues satisfy $|\lambda|=1$, and hence $(\bar{x}, \bar{y})=\left(r, \frac{1-A}{B r}\right)$ is nonhyperbolic.

## 5. Convergence

In this section, we study the asymptotic behavior of the solutions of System (1.1).

Theorem 5.1. Let $\left(x_{n}, y_{n}\right)_{n \geq-2 q+1}$ be a solution of System (1.1). If $|A|<1$, then for each $r \in\{0,1, \ldots, 2 q-1\}$, the subsequence $\left(x_{2 q n-r}, y_{2 q n-r}\right)_{n \geq 0}$ converges.
Proof. We divide the proof into two cases.
Case 1: $r \in\{2 q-1, \ldots, q\}$.
Since $|A| \neq 1$, from Corollary (3.3), we get

$$
x_{2 q n-r}=a_{-r}(1-A) \frac{\prod_{p=1}^{n-1}\left(B a_{-r} b_{-r+q}-\left(A-1+B a_{-r} b_{-r+q}\right) A^{2 p}\right)}{\prod_{p=0}^{n-1}\left(B a_{-r} b_{-r+q}-\left(A-1+B a_{-r} b_{-r+q}\right) A^{2 p+1}\right)}
$$

which can be written as

$$
x_{2 q n-r}=\frac{a_{-r}}{B a_{-r} b_{-r+q}+A} \prod_{p=1}^{n-1} \frac{1-\alpha A^{2 p}}{1-\alpha A^{2 p+1}}
$$

where $\alpha=\frac{A-1+B a_{-r} b_{-r+q}}{B a_{-r} b_{-r+q}}$. If $A-1+B a_{-r} b_{-r+q}=0$, then $\alpha=0$. This yields $x_{2 q n-r}=a_{-r}$, for all $n \geq 0$, i.e. $\left(x_{2 q n-r}\right)_{n \geq 0}$ is constant. Now assume $A-1+$ $B a_{-r} b_{-r+q} \neq 0$. Then, the Taylor expansion of $\frac{1-\alpha A^{2 p}}{1-\alpha A^{2 p+1}}$ gives that

$$
\frac{1-\alpha A^{2 p}}{1-\alpha A^{2 p+1}}=1+\alpha(A-1) A^{2 p}+o\left(A^{2 p}\right)
$$

which yields $\frac{1-\alpha A^{2 p}}{1-\alpha A^{2 p+1}} \sim 1+\alpha(A-1) A^{2 p}$. Now, depending on the sign of $\alpha$, we can choose an integer $N$ sufficiently large so that either $\frac{1-\alpha A^{2 p}}{1-\alpha A^{2 p+1}}>1$, for all $p \geq N$, or $0<\frac{1-\alpha A^{2 p}}{1-\alpha A^{2 p+1}}<1$, for all $p \geq N$. Since $\prod_{p \geq 1}\left(1+\alpha(A-1) A^{2 p}\right)$ converges, by equivalence criterion it follows that $\prod_{p \geq 1} \frac{1-\alpha A^{2 p}}{1-\alpha A^{2 p+1}}$ converges. Therefore $\left(x_{2 q n-r}\right)_{n \geq 0}$ converges. Using a very similar argument we obtain that $\left(y_{2 q n-r}\right)_{n \geq 0}$ converges.

Case 2: $s \in\{q-1, \ldots, 0\}$.
By Eq. (3.12) we get

$$
\begin{equation*}
x_{2 q n-s}=a_{-s} \prod_{p=1}^{n-1}\left(\frac{B b_{-s-q} a_{-s}-\left(A-1+B b_{-s-q} a_{-s}\right) A^{2 p+1}}{B b_{-s-q} a_{-s}-\left(A-1+B b_{-s-q} a_{-s}\right) A^{2 p}}\right) \tag{5.1}
\end{equation*}
$$

If $A-1+B b_{-s-q} a_{-s}=0$, then again $\left(x_{2 q n-s}\right)_{n \geq 0}$ is constant, and hence it converges. Now assume $A-1+B b_{-s-q} a_{-s} \neq 0$. Then Eq. (5.1) can be written as

$$
x_{2 q n-s}=a_{-s} \prod_{p=1}^{n-1} \frac{1-\beta A^{2 p+1}}{1-\beta A^{2 p}}
$$

where $\beta=\frac{A-1+B b_{-s-q} a_{-s}}{B b_{-s-q} a_{-s}}$. Similar to the argument in Case 1, we obtain that $\frac{1-\beta A^{2 p+1}}{1-\beta A^{2 p}} \sim 1+\beta(1-A) A^{2 p}$. Also depending on the sign of $\beta$, we can find an integer $N$ so that either $\frac{1-\beta A^{2 p+1}}{1-\beta A^{2 p}}>1$, for all $p \geq N$, or $0<\frac{1-\beta A^{2 p+1}}{1-\beta A^{2 p}}<1$, for all $p \geq N$. Since $\prod_{p \geq 1}\left(1+\beta(1-A) A^{2 p}\right)$ converges, by equivalence criterion $\prod_{p \geq 1} \frac{1-\beta A^{2 p+1}}{1-\beta A^{2 p}}$ converges, and hence the subsequence $\left(x_{2 q n-s}\right)_{n \geq 0}$ converges.
Theorem 5.2. Let $\left(x_{n}, y_{n}\right)_{n \geq-2 q+1}$ be a solution of the equation (1.1). If $A=-1$, then the followings hold:
(1) For each $r=2 q-1,2 q-2, \ldots, q$, the subsequence $\left(x_{2 q n-r}, y_{2 q n-r}\right)$ converges if and only if both $a_{-r} b_{-r+q}$ and $b_{-r} a_{-r+q}$ belong to

$$
\left(-\infty, \min \left(0, \frac{2}{B}\right)\right) \cup\left(\max \left(0, \frac{2}{B}\right), \infty\right) \cup\left\{\frac{2}{B}\right\}
$$

(2) For each $r=q-1, q-2, \ldots, 0$, the subsequence $\left(x_{2 q n-r}, y_{2 q n-r}\right)$ converges if and only if both $a_{-r-q} b_{-r}$ and $b_{-r-q} a_{-r}$ belong to

$$
\left(\min \left(0, \frac{2}{B}\right), \max \left(0, \frac{2}{B}\right)\right) \cup\left\{\frac{2}{B}\right\} .
$$

Proof.

1) Let $r \in\{2 q-1,2 q-2, \ldots, q\}$. Then by substituting $A=-1$ in Equations (3.10) and (3.11), we get

$$
x_{2 q n-r}=\frac{a_{-r}}{\left(B a_{-r} b_{-r+q}-1\right)^{n}}, y_{2 q n-r}=\frac{b_{-r}}{\left(B b_{-r} a_{-r+q}-1\right)^{n}} .
$$

Therefore, $\left(x_{2 q n-r}\right)_{n}$ converges if and only if $\left|B a_{-r} b_{-r+q}-1\right|>1$ or $B a_{-r} b_{-r+q}-$ $1=1$ if and only if $a_{-r} b_{-r+q} \in\left(-\infty, \min \left(0, \frac{2}{B}\right)\right) \cup\left(\max \left(0, \frac{2}{B}\right), \infty\right) \cup\left\{\frac{2}{B}\right\}$. Similarly, $\left(y_{2 q n-r}\right)_{n}$ converges if and only if $\left|B b_{-r} a_{-r+q}-1\right|>1$ or $B b_{-r} a_{-r+q}-1=1$ if and only if $b_{-r} a_{-r+q} \in\left(-\infty, \min \left(0, \frac{2}{B}\right)\right) \cup\left(\max \left(0, \frac{2}{B}\right), \infty\right) \cup\left\{\frac{2}{B}\right\}$.
2) Let $r \in\{q-1, q-2, \ldots, 0\}$. Then by substituting $A=-1$ in Equations (3.12) and (3.13), yields

$$
x_{2 q n-r}=a_{-r}\left(B a_{-r-q} b_{-r}-1\right)^{n-1}, y_{2 q n-r}=a_{-r}\left(B b_{-r-q} a_{-r}-1\right)^{n-1}
$$

So $\left(x_{2 q n-r}, y_{2 q n-r}\right)_{n}$ converges if and only if both $a_{-r-q} b_{-r}$ and $b_{-r-q} a_{-r}$ belong to $\left(\min \left(0, \frac{2}{B}\right), \max \left(0, \frac{2}{B}\right)\right) \cup\left\{\frac{2}{B}\right\}$.
Theorem 5.3. Let $\left(x_{n}, y_{n}\right)_{n \geq-2 q+1}$ be a solution of System (1.1). If $A=1$, then the sequence $\left(x_{n}, y_{n}\right)_{n \geq-2 q+1}$ converges to $(0,0)$.
Proof. Let $r \in\{2 q-1,2 q-2, \ldots, q\}$. From Corollary (3.2), we get

$$
\begin{aligned}
& x_{2 q n-r}=\frac{a_{-r}}{1+B a_{-r} b_{-r+q}} \prod_{p=1}^{n-1}\left(1-\frac{B a_{-r} b_{-r+q}}{1+(2 p+1) B a_{-r} b_{-r+q}}\right), \\
& y_{2 q n-r}=\frac{b_{-r}}{1+B b_{-r} a_{-r+q}} \prod_{p=1}^{n-1}\left(1-\frac{B b_{-r} a_{-r+q}}{1+(2 p+1) B b_{-r} a_{-r+q}}\right),
\end{aligned}
$$

which can be written as

$$
x_{2 q n-r}=\frac{a_{-r}}{1+B a_{-r} b_{-r+q}} \exp \left(\sum_{p=1}^{n-1} \ln V_{p}\right)
$$

$$
y_{2 q n-r}=\frac{b_{-r}}{1+B b_{-r} a_{-r+q}} \exp \left(\sum_{p=1}^{n-1} \ln U_{p}\right),
$$

where $V_{p}=1-\frac{B a_{-r} b_{-r+q}}{1+(2 p+1) B a_{-r} b_{-r+q}}$ and $U_{p}=1-\frac{B b_{-r} a_{-r+q}}{1+(2 p+1) B b_{-r} a_{-r+q}}$. Now,

$$
\lim _{p \longrightarrow \infty} \frac{\ln V_{p}}{1-V_{p}}=\infty \text { and } \sum_{p \geq 1}\left(1-V_{p}\right)=\sum_{p \geq 1}\left(\frac{B a_{-r} b_{-r+q}}{1+(2 p+1) B a_{-r} b_{-r+q}}\right)
$$

which is divergent. Moreover, There exists $N \in \mathbb{N}$ such that $\ln V_{p}<0$ for all $p \geq N$. Hence $\sum_{p \geq 1} \ln V_{p}$ diverges to $-\infty$. Thus the subsequence $\left(x_{2 q n-r}\right)$ converges to zero. Similarly we deduce that $\sum_{p \geq 1} \ln U_{p}$ diverges to $-\infty$, which yields $\left(y_{2 q n-r}\right)$ converges to zero. Using similar argument we can show that for all $r=q-1, q-2, \ldots, 0$, the subsequence $\left(x_{2 q n-r}, y_{2 q n-r}\right)_{n \geq 0}$ converges to $(0,0)$. Therefore the whole sequence $\left(x_{n}, y_{n}\right)_{n \geq-2 q+1}$ converges to $(0,0)$.

Theorem 5.4. Let $\left(x_{n}, y_{n}\right)_{n \geq-2 q+1}$ be a solution of System (1.1). If $|A|>1$, then for each $r \in\{2 q-1,2 q-2, \ldots, 0\}$, the subsequence $\left(x_{2 q n-r}, y_{2 q n-r}\right)$ converges. Moreover, for each $r \in\{2 q-1,2 q-2, \ldots, q\}$, one of the following statements is true
(1) If $A-1+B a_{-r} b_{-r+q}=0$, then the subsequences $\left(x_{2 q n-r}\right)_{n \geq 0}$ and $\left(y_{2 q n-r+q}\right)_{n \geq 0}$ are constants.
(2) If $A-1+B b_{-r} a_{-r+q}=0$, then the subsequences $\left(y_{2 q n-r}\right)_{n \geq 0}$ and $\left(x_{2 q n-r+q}\right)_{n \geq 0}$ are constants.
(3) If $A-1+B a_{-r} b_{-r+q} \neq 0$, then the subsequences $\left(x_{2 q n-r}\right)_{n \geq 0}$ and $\left(y_{2 q n-r+q}\right)_{n \geq 0}$ converge to zero.
(4) If $A-1+B b_{-r} a_{-r+q} \neq 0$, then the subsequences $\left(y_{2 q n-r}\right)_{n \geq 0}$ and $\left(x_{2 q n-r+q}\right)_{n \geq 0}$ converge to zero.

Proof. Let $r \in\{2 q-1,2 q-2, \ldots, q\}$. We will only show the convergence of the subsequences $\left(x_{2 q n-r}, y_{2 q n-r}\right)_{n \geq 0}$. The convergence of the subsequence $\left(x_{2 q n-r+q}, y_{2 q n-r+q}\right)_{q \geq 0}$ can be established using a similar argument.
If $A-1+B a_{-r} b_{-r+q}=0$, then the subsequence $\left(x_{2 q n-r}\right)_{n \geq 0}$ is constant equal $a_{-r}$, and the subsequence $\left(y_{2 q n-r+q}\right)_{n \geq 0}$ is constant equal $b_{-r+q}$. Similarly if $A-1+B b_{-r} a_{-r+q}=0$ then $\left(y_{2 q n-r}\right)_{n \geq 0}$ is constant equal to $b_{-r}$ and $\left(x_{2 q n-r+q}\right)_{n \geq 0}$ is constant equal to $a_{-r+q}$. Assume $A-1+B a_{-r} b_{-r+q} \neq 0$ and $A-1+B b_{-r} a_{-r+q} \neq$ 0 . So we can write Equations (3.10) and (3.12) as

$$
x_{2 q n-r}=\frac{a_{-r}}{\left(B a_{-r} b_{-r+q}+A\right) A^{n-1}} \prod_{p=1}^{n-1} \frac{1-\frac{1}{\alpha A^{2 p}}}{1-\frac{1}{\alpha A^{2 p+1}}},
$$

and

$$
y_{2 q n-r}=\frac{b_{-r}}{\left(B b_{-r} a_{-r+q}+A\right) A^{n-1}} \prod_{p=1}^{n-1} \frac{1-\frac{1}{\beta A^{2 p}}}{1-\frac{1}{\beta A^{2 p+1}}}
$$

where $\alpha=\frac{A-1+B a_{-r} b_{-r+q}}{B a_{-r} b_{-r+q}}$ and $\beta=\frac{A-1+B b_{-r} a_{-r+q}}{B b_{-r} a_{-r+q}}$. Since $A^{1-n}$ converges to zero, it suffices to show that

$$
\prod_{p=1}^{n-1}\left(1-\frac{1}{\alpha A^{2 p}}\right) /\left(1-\frac{1}{\alpha A^{2 p+1}}\right) \text { and } \prod_{p=1}^{n-1}\left(1-\frac{1}{\beta A^{2 p}}\right) /\left(1-\frac{1}{\beta A^{2 p+1}}\right)
$$

converge. Using the Taylor expansion of $\left(1-\frac{1}{\alpha A^{2 p}}\right) /\left(1-\frac{1}{\alpha A^{2 p+1}}\right)$ we obtain that

$$
\left(1-\frac{1}{\alpha A^{2 p}}\right) /\left(1-\frac{1}{\alpha A^{2 p+1}}\right) \sim 1+\frac{1}{\alpha}\left(\frac{1}{A}-1\right) \frac{1}{A^{2 p}}
$$

Now there exists $N \in \mathbb{N}$ such that $\left(1-\frac{1}{\alpha A^{2 p}}\right) /\left(1-\frac{1}{\alpha A^{2 p+1}}\right)>1$, for all $p \geq N$, or $0<\left(1-\frac{1}{\alpha A^{2 p}}\right) /\left(1-\frac{1}{\alpha A^{2 p+1}}\right)<1$, for all $p \geq N$. Since $\prod_{p \geq 1}\left(1+\frac{1}{\alpha}\left(\frac{1}{A}-1\right) \frac{1}{A^{2 p}}\right)$ converges, by equivalence criterion $\prod_{p \geq 1}\left(1-\frac{1}{\alpha A^{2 p}}\right) /\left(1-\frac{1}{\alpha A^{2 p+1}}\right)$ converges. Similarly, we deduce that $\prod_{p=1}^{n-1}\left(1-\frac{1}{\beta A^{2 p}}\right) /\left(1-\frac{1}{\beta A^{2 p+1}}\right)$ converges. This complete the proof.

## 6. Numerical simulation

(1) The case $|A|<1$, is studied using the parameter values $q=2, A=\frac{1}{2}$, $B=-3$, and the initial data $a_{-3}=3, a_{-2}=-2, a_{-1}=-4, a_{0}=-2$, $b_{-3}=4, b_{-2}=-5, b_{-1}=5, b_{0}=-1$. In Figure 1, it is shown that the subsequences $\left(x_{4 n-i}, y_{4 n-i}\right), i=0,1,2,3$ converge and the whole sequence $\left(x_{n}, y_{n}\right)$ is bounded, which is coherent with Theorem 5.1.


Figure 1. The Case $|A|<1$
(2) The case $A=-1$, is investigated using the parameter values $q=1, A=-1$, $B=-2$, and the initial data $a_{-1}=-1, a_{0}=-3, b_{-1}=1, b_{0}=4$. In Figure 2 , we notice that the subsequences $\left(x_{2 n-1}, y_{2 n-1}\right)_{n}$ converge, however the subsequences $\left(x_{2 n}, y_{2 n}\right)_{n}$ diverge to infinity. This is justified analytically in the proof of Theorem 5.2. The sequences $\left(x_{n}\right)$ and $\left(y_{n}\right)$ are unbounded.
(3) The case $A=1$, is studied using the parameter values $q=3, A=1, B=4$, and the initial data $a_{-5}=3, a_{-4}=-2, a_{-3}=-4, a_{-2}=-2, a_{-1}=5$, $a_{0}=1, b_{-5}=4, b_{-4}=-5, b_{-3}=5, b_{-2}=-1, b_{-1}=-4, b_{0}=-2$. In


Figure 2. The Case $A=-1$
Figure 3, it is clear that the solution converges to zero. This is justified analytically in the proof of Theorem 5.3.


Figure 3. The Case $A=1$
(4) Figure 4 illustrates the case $A>1$ using the parameter values $q=2$, $A=-3, B=1$, and the initial data $a_{-3}=3, a_{-2}=1, a_{-1}=-2, a_{0}=3$, $b_{-3}=5, b_{-2}=4, b_{-1}=-1, b_{0}=4$. Since $A-1+B a_{-2} b_{0}=0$, the subsequences $\left(x_{4 n-2}\right)_{n}$ and $\left(y_{4 n}\right)_{n}$ are constants. However $A-1+B a_{-3} b_{-1}, A-$
$1+B b_{-3} a_{-1}, A-1+B b_{-2} a_{0}$ are all nonzero. So all of the subsequences
$\left(x_{4 n-3}\right),\left(x_{4 n-1}\right),\left(x_{4 n}\right),\left(y_{4 n-3}\right),\left(y_{4 n-2}\right)$, and $\left(y_{4 n-1}\right)$ converge to zero. These observations are coherent with the result in Theorem 5.4.


Figure 4. The Case $|A|>1$.

## 7. Conclusion

We have presented a complete study of a large class of systems of rational difference equations with arbitrary parameters and initial conditions, namely the system

$$
x_{n+1}=\frac{x_{n-2 q+1}}{A+B x_{n-q+1} y_{n-2 q+1}}, y_{n+1}=\frac{y_{n-2 q+1}}{A+B y_{n-q+1} x_{n-2 q+1}}
$$

where $A$ and $B$ are arbitrary parameters and $q$ is an arbitrary non-negative integer. The local and global stability of the equilibrium points of this system have been investigated. A detailed analytical study of the convergence of the solutions including their dependence on parameters and initial conditions has been presented. The local and global stability of the equilibrium points have been investigated. Numerical simulations have been done to confirm the correctness of the analytical results.

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