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INERTIAL ITERATIVE METHODS FOR GENERAL QUASI VARIATIONAL INEQUALITIES AND DYNAMICAL SYSTEMS

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Abstract. In this paper, we consider the technique of inertial methods for solving a class of general quasi-variational inequalities. We associate a dynamical system with general quasi variational inequalities. This equivalent formulation is used to prove that these dynamical systems converge asymptotically to the unique solution of general quasi-variational inequalities. We suggest and investigate a new class of inertial iterative methods for solving the general quasi-variational inequalities. Convergence analysis of these iterative methods is analyzed under some appropriate conditions. Several interesting special cases are obtained as applications of the results. The ideas and techniques of this paper may inspire further research.

1. Introduction

Quasi variational inequalities were introduced and studied by Bensoussan and Lions [9] in impulse control problems. It is a well-known result that the set involved in the these inequalities depends upon the solution explicitly or implicitly. If the involved set does not depend on the solution, then this inequality reduces to the variational inequality, the beginning of which can be followed back to Stampacchia [34]. Variational inequalities and quasi variational inequalities provide us a unified framework to study various interrelated and unrelated problems which emerge in various branches of applied and pure sciences. However quasi variational inequalities are more difficult and challenging as compared with variational inequalities. It is still a difficult task to propose some effective strategies for solving quasi variational inequalities. The most common way for solving quasi-variational inequalities is to show that the these inequalities are equivalent to the fixed point problems. This alternative equivalent formulation was used to suggest some projection type methods for solving the quasi-variational inequalities; see [19] and references therein. Noor [21] used this fixed point formulation to suggest and investigate the implicit dynamical system for quasi variational inequality. For more details, see [18, 19, 20, 21, 22, 23, 24, 25, 27, 28, 29, 30, 31] and the references therein.

We want to mention that all the works carried out in this direction assumed that

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the fundamental set is a convex set. In many applications, the underlying set may not be a convex set. In such situation, the set may be made convex set with respect to an arbitrary function. Motivated by these facts, Noor [22] introduced the general convex sets and general convex functions involving an arbitrary function ψ. Noor [22] and Cristescu et al [10] have studied the basic properties of ψ-convex sets. Noor [23] has proved that the minimum value of a differentiable general convex function on the general convex set can be characterized by a type of variational inequalities, which is called the general variational inequality. Recently, projection operator techniques have been used to study several dynamical systems associated with variational inequalities, which is due to Dupuis and Nagurney [11], Friesz et al [12] and Noor [20]. The dynamical approach is more attractive because of their wide applicability, flexibility and numerical efficiency. In this method, the problem of variational inequality is reformulated as an initial value problem. It enables us to study the asymptotic stability of unique solutions to variational inequality problems. There are two types of the projected dynamical systems. The first category is attributed to Dupuis and Nagurney [11], which is known as local projected dynamical systems, while the second category is attributed to Friesz et al [12], is called global projected dynamical systems. In the field of continuous optimization, inertial-type algorithms have attracted much attention in recent years due to their convergence. The idea comes from the field of second-order dissipative dynamical system [6, 5]. The results show that the inertial term can accelerate the convergence speed of existing algorithms, see [1, 2, 3, 4, 5, 6, 7, 8, 15, 16, 17, 32, 33] and reference therein.

Motivated by the ongoing research activities in this direction, we consider a new class of quasi-variational inequalities involving two operators, called the general quasi variational inequality. This class contains the classical quasi-variational inequalities and general variational inequalities as special cases. We consider the dynamical system associated with general quasi-variational inequality and study the asymptotically stability of the trajectory to the solution. Discretizing the dynamical system, we introduce several continuous inertial-type iterative methods for solving general quasi variational inequalities and related optimization problems. Convergence of these proposed methods is investigated under some appropriate conditions. As applications of the main results, some special cases are discussed. It is an interesting problem to implement these methods and compare with other techniques

2. Basic Definitions and Results

Let a real Hilbert space $\mathcal{H}$ with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let $\mathcal{K}$ be any convex closed set in $\mathcal{H}$.

Defintion 2.1. [22] A set $\mathcal{K}_\psi$ in $\mathcal{H}$ is called a general convex set, if there exists an arbitrary function ψ, such that

$$(1-t)p + t\,\psi(q) \in \mathcal{K}_\psi, \qquad \forall\, p, q \in \mathcal{H} : p, \psi(q) \in \mathcal{K}_\psi,\ t \in [0,1].$$

Clearly every convex set is general convex, but converse is not true. For the properties of general convex sets, see Noor [22] and Cristescu et al [7].

Defintion 2.2. [22] A function F is called general convex function with respect to an arbitrary function ψ, if

$$F((1-t)p + t\,\psi\,(q)) \leq (1-t)F(p) + tF(\psi(q)), \qquad \forall\, p,\, \psi\,(q) \;\in\; \mathcal{K}_{\psi},\, t \in [0,1].$$

The general convex functions were introduced by Noor [22]. Noor [22] proved that the minimum $p \in \mathcal{H}$, $\psi(p) \in \mathcal{K}_{\psi}$ of the differentiable general convex functions F can be characterized by a class of variational inequities of the form:

$$\langle F^{'}(p),\, \psi(q) - p\rangle \geq 0. \quad \forall\, q \,\in\, \mathcal{H} \,:\, \psi(q) \,\in\, \mathcal{K}_{\psi}, \tag{2.1}$$

which is known as general variational inequalities.

In many important applications, the inequality of the type (2.1) may not arise as a result of optimality conditions and the underlying set is a set-valued convex set. These facts motivated us to consider the more general quasi-variational inequality, which includes the inequalities of the type (2.1). To be more precise, we consider the following problem:

Let $\mathcal{K} \,:\, \mathcal{H} \longrightarrow \mathcal{H}$ be a set-valued mapping, which for any element $p \,\in\, \mathcal{H}$, associates a closed and convex set $\mathcal{K}(p) \subset \mathcal{H}$. For the given two nonlinear operators $\mathcal{T}, \psi : \mathcal{H} \longrightarrow \mathcal{H}$, find $p \,\in\, \mathcal{H} : \psi\,(p) \in\, \mathcal{K}\,(p)$, such that

$$\langle\, \rho\, \mathcal{T}\, p + p - \psi\,(p)\;\; , \psi\,(q) - p\,\rangle \geq 0, \qquad \forall\, q \,\in\, \mathcal{H} \,:\, \psi\,(q) \,\in\, \mathcal{K}\,(p)\,, \tag{2.2}$$

where $\rho > 0$ is a constant. Inequality (2.2) is called general quasi-variational inequality involving two operators, which was introduced and studied by Noor [22, 23].

Now we present some special cases of problem (2.2).

I. If $p = \psi(p)$, then problem (2.2) is equivalent to find $p \,\in\, \mathcal{H} : \psi\,(p) \in\, \mathcal{K}\,(p)$ such that

$$\langle\; \mathcal{T}\,(\psi(p)),\, \psi\,(q) - \psi(p)\,\rangle \geq 0, \quad \forall\, q \,\in\, \mathcal{H} \,:\, \psi\,(q) \,\in\, \mathcal{K}\,(p)\,, \tag{2.3}$$

which is known as general quasi-variational inequality involving two nonlinear operators, see [29].

II. If $\psi(p) \;=\; p$, then problem (2.2) is equivalent to find $p \,\in\, \mathcal{H} : p \in\, \mathcal{K}\,(p)$ such that

$$\langle\; \mathcal{T}\, p\,, \psi\,(q) - p\,\rangle \geq 0, \qquad \forall\, q \,\in\, \mathcal{H} \,:\, \psi\,(q) \,\in\, \mathcal{K}\,(p)\,, \tag{2.4}$$

which is called a general quasi-variational inequality.

III. For $\psi \;= I$, problem (2.2) is equivalent to find $p \,\in\, \mathcal{K}(p)$ such that

$$\langle\, \mathcal{T}p\,, q - p\,\rangle \geq 0, \qquad \forall\, q \,\in\, \mathcal{K}\,(p)\,, \tag{2.5}$$

this problem is called quasi-variational inequality, which was introduced by Bensoussan and Lions, see [9].

IV. If $\mathcal{K}\,(p) = \mathcal{K}$, closed convex set and $\psi \;= I$, then problem (2.2) is equivalent to finding $p \,\in\, \mathcal{H} : \psi(p) \in\, \mathcal{K}$ this

$$\langle\, \mathcal{T}p\,, \psi\,(q) - \psi(p)\,\rangle \geq 0, \qquad \forall\, q \,\in\, \mathcal{H} \,:\, \psi\,(q) \,\in\, \mathcal{K}, \tag{2.6}$$

which is called a general variational inequality, which was introduced by Noor [19]. For the applications, numerical methods and other aspects of general variational inequalities, see Noor[25, 26] and the references therein.
We need following basic definitions and lemmas to prove our results.

Defintion 2.3. For a constant $\xi_1 > 0$, a nonlinear operator $\mathcal{T} : \mathcal{H} \longrightarrow \mathcal{H}$ is said to be strongly monotone, if

$$\langle \mathcal{T} p - \mathcal{T} q , p - q \rangle \geq \xi_1 \; \| p - q \|^2, \qquad \forall p, q \in \mathcal{H}. \tag{2.7}$$

Defintion 2.4. For a constant $\eta_1 > 0$, a nonlinear operator $\mathcal{T} : \mathcal{H} \longrightarrow \mathcal{H}$ is called Lipschitz continuous, if

$$\| \mathcal{T} p - \mathcal{T} q \| \leq \eta_1 \; \| p - q \|, \qquad \forall p, q \in \mathcal{H}. \tag{2.8}$$

From (2.7) and (2.8), it can be noted that $\xi_1 \leq \eta_1$.

Lemma 2.5. [29] For a given $w \in \mathcal{H}$, $p \in \mathcal{K}(p)$, satisfies the inequality

$$\langle p - w, q - p \rangle \geq 0, \;\; \forall\, q \in \mathcal{K}(p),$$

if and only if,

$$p = \Pi_{\mathcal{K}(p)} [w],$$

where $\Pi_{\mathcal{K}(p)}$ is the implicit projection of $\mathcal{H}$ onto the closed convex-valued set $\mathcal{K}(p)$ in $\mathcal{H}$.

Assumption 2.6. [21]. The implicit projection operator $\mathcal{P}_{\mathcal{K}(p)}$ is not nonexpansive but satisfies the condition

$$\| \Pi_{\mathcal{K}(p)} [\omega] - \Pi_{\mathcal{K}(q)} [\omega] \| \leq \; \upsilon \; \| p - q \| \qquad \forall\, p,\, q\,,\, \omega \in \mathcal{H}, \tag{2.9}$$

where $\upsilon > 0$, is a constant.

Lemma 2.7. [35]. Consider a sequence of non-negative real numbers $\{\varrho_n\}$, which satisfy the inequality:

$$\varrho_{n+1} \leq (1 - \Upsilon_n)\varrho_n + \Upsilon_n \, \sigma_n + \varsigma_n, \;\; \forall\, n \geq 1,$$

where

i. $\{\Upsilon_n\} \subset [0, 1]$, $\quad \sum\limits_{n=1}^{\infty} \Upsilon_n = \infty$;
ii. $\lim \sup \sigma_n \leq 0$;
iii. $\varsigma_n \geq 0 \; (n \geq 1)$, $\quad \sum\limits_{n=1}^{\infty} \varsigma_n < \infty$.

Then, $\varrho_n \longrightarrow 0$ as $n \longrightarrow \infty$.

Lemma 2.8. [22]. Let $\mathcal{K}(p)$ be a closed and convex-valued set in $\mathcal{H}$. Then $p \in \mathcal{H} : \psi(p) \in \mathcal{K}(p)$ is the solution of problem (2.2), if and only if,

$$p = \Pi_{\mathcal{K}(p)} [\, \psi(p) - \rho \mathcal{T}(p)] , \tag{2.10}$$

where $\Pi_{\mathcal{K}(p)}$ is the implicit projection of $\mathcal{H}$ onto the closed and convex-valued set $\mathcal{K}(p)$ in $\mathcal{H}$ and $\rho > 0$ is a constant.

We define a mapping $\mathfrak{F}(p)$ associated with (2.10) as:

$$\mathfrak{F}(p) = \Pi_{\mathcal{K}(p)} [\, \psi(p) - \rho \mathcal{T}(p)] . \tag{2.11}$$

The following theorem provides the sufficient conditions, under which the mapping defined by (2.11), is a contraction mapping. Thus there exists a solution of the inequality (2.2).

Theorem 2.9. Let the operators $\mathcal{T}, \psi : \mathcal{H} \longrightarrow \mathcal{H}$ be strongly monotone with constant $\xi_1 > 0$, $\xi_2 > 0$ and Lipschitz continuous with constant $\eta_1 > 0$, $\eta_2 > 0$, respectively. If Assumption P holds and $\rho > 0$ satisfies the condition

$$\left| \rho - \frac{\xi_1}{\eta_1^2} \right| < \frac{\sqrt{\xi_1^2 - \eta_1^2\, \kappa\, (2-\kappa)}}{\eta_1^2} \qquad \xi_1 > \eta_1 \sqrt{\kappa(2-\kappa)}, \quad \kappa < 1, \tag{2.12}$$

where

$$\kappa = \upsilon + \sqrt{1 - 2\,\xi_2 + \eta_2^2}\,,$$

then there exists a solution $p \in \mathcal{H} : \psi(p) \in \mathcal{K}(p)$, which satisfies the inequality (2.2).

Proof. See [31] and reference therein. □

We now discuss some special of Theorem 2.1.

Special cases.

- **i.** For $\psi = I$, Theorem 2.1 provides the existence of a solution for inequality (2.5).
- **ii.** If $\mathcal{K}(p) = \mathcal{K}$ and $\psi = I$, then from Theorem 2.1, we can find the existence of a solution for variational inequality.
- **iii.** If $\mathcal{K}(p) = \mathcal{K}$, closed convex set and $\psi \neq I$, then Theorem 2.1 gives the existence of a solution for inequality (2.6).

3. Dynamical System

In this section, we study the basic properties of dynamical systems and investigate the asymptotically convergence of the system.

Firstly, we define the residue vector $\mathfrak{R}(p)$ as:

$$\mathfrak{R}(p) = p \,-\, \Pi_{\mathcal{K}(p)}\left[\,\psi\,(p) - \rho\mathcal{T}\,(p)\right]. \tag{3.1}$$

It can be observed from Lemma 2.3 that problem (2.2) has solution $p \in \mathcal{H} : \psi(p) \in \mathcal{K}(p)$, if and only if $p \in \mathcal{H} : \psi(p) \in \mathcal{K}(p)$ is a zero of the equation

$$\mathfrak{R}(p) = 0.$$

Remark. Equation (2.10) can be written in the following equivalent form

$$p = (1 - \mathsf{a}(t))p + \mathsf{a}(t)\Pi_{\mathcal{K}(p)}\left[\,\psi(p) - \rho\mathcal{T}(p)\right], \tag{3.2}$$

where $0 < \mathsf{a}(t) \leq 1,\ t \geq 0$.

Consider the dynamical system associated with (2.2) as

$$\begin{aligned} \frac{d\,p}{d\,t} &= -\lambda\,\mathfrak{R}(p) \\ &= \lambda\left(-\mathsf{a}\,(t)\,p\,(t) + \mathsf{a}(t)\Pi_{\mathcal{K}(p(t))}\left[\,\psi(p(t)) - \rho\mathcal{T}(p(t))\,\right]\right), \quad p(t_0) = p_0, \end{aligned} \tag{3.3}$$

where $\lambda > 0$, $\mathsf{a}(t) > 0$ and $\rho > 0$ are parameters.
Note that, if $\dfrac{d\,p}{d\,t} = 0$, then p is the solution of (2.2) and vice versa.

Special cases. Here we discuss some special cases of (3.3):

- **i.** For $\mathsf{a}(t) = 1$ and $\psi = I$, dynamical system (3.3) reduces to continuous gradient dynamical system [15].

ii. If we take $\mathcal{K}(p) = \mathcal{K}$, $\mathsf{a}(t) = 1$ and $\psi = I$, dynamical system associated with (3.1) reduces to continuous gradient dynamical system [1].
iii. For $\mathcal{K}(p) = \mathcal{K}$ and $\psi = I$, dynamical system (3.3) reduces to gradient-type projection method [17].

We now discuss the existence and uniqueness of dynamical system (3.3).

Theorem 3.1. Let that the operators $\mathcal{T}$, $\psi : \mathcal{H} \longrightarrow \mathcal{H}$ are Lipschitz continuous with constant $\eta_1 > 0$, $\eta_2 > 0$ respectively. If Assumption (2.6) holds, then for arbitrary initial point $p_0 \in \mathcal{H}$, the dynamical system (3.3) has a unique continuous solution $p(t)$ for all $t \geq 0$.

Proof. Let

$$F(p,t) \;=\; \lambda \left(-\mathsf{a}\,(t)\, p\,(t) + \;\mathsf{a}(t)\, \Pi_{\mathcal{K}(p(t))}\, [\, \psi(p(t)) - \rho \mathcal{T}(p(t))\,]\,\right).$$

Firstly, we have to show that $F(p,t)$ is Lipschitz continuous. Using (2.6) and Lipschitz continuity of operators $\mathcal{T}, \psi$ with constants $\eta_1 > 0$, $\eta_2 > 0$, respectively. For $p_1 \neq p_2$, consider

$$\begin{aligned}
\| \, F(p_1,t) - F(p_2,t) \, \| &= \lambda \; \| -\mathsf{a}(t)p_1 + \mathsf{a}(t)\Pi_{\mathcal{K}(p_1)}[\, \psi(p_1) - \rho\mathcal{T}p_1 \,] \\
&\quad + \mathsf{a}(t)p_2 - \mathsf{a}(t)\Pi_{\mathcal{K}(p_2)}[\, \psi(p_2) - \rho\mathcal{T}p_2 \,] \, \| \\
&\leq \lambda\, \mathsf{a}(t) \, \| \, p_1 - p_2 \, \| + \lambda\, \mathsf{a}(t) \, \| \, \Pi_{\mathcal{K}(p_1)}[\, \psi(p_1) - \rho\mathcal{T}p_1 \,] - \\
&\quad \Pi_{\mathcal{K}(p_2)}[\, \psi(p_2) - \rho\mathcal{T}p_2 \,] \, \| \\
&\leq \lambda\, \mathsf{a}(t) \, \| \, p_1 - p_2 \, \| + \lambda\, \mathsf{a}(t) \, \| \, \Pi_{\mathcal{K}(p_1)}[\psi(p_1) - \rho\mathcal{T}p_1] - \\
&\quad \Pi_{\mathcal{K}(p_1)}[\, \psi(p_2) - \rho\mathcal{T}p_2 \,] \, \| + \lambda\, \mathsf{a}(t) \, \| \, \Pi_{\mathcal{K}(p_1)}[\, \psi(p_2) - \rho\mathcal{T}p_2 \,] - \\
&\quad \Pi_{\mathcal{K}(p_2)}[\, \psi(p_2) - \rho\mathcal{T}p_2 \,] \, \| \\
&\leq \lambda\, \mathsf{a}(t) \, \| \, p_1 - p_2 \, \| + \lambda\, \mathsf{a}(t) \, \| \, \psi(p_1) - \rho\mathcal{T}p_1 - \psi(p_2) + \rho\mathcal{T}p_2 \, \| \\
&\quad + \lambda\, \mathsf{a}(t)\, \upsilon \;\, \| \, p_1 - p_2 \, \| \\
&\leq \lambda \; (1 + \eta_2 + \rho\, \eta_1 + \upsilon) \;\, \mathsf{a}(t) \;\, \| \, p_1 - p_2 \, \|,
\end{aligned}$$

From this, it follows that $F(p,.)$ is Lipschitz continuous for all $p \in \mathcal{H}$ and dynamical system (3.3), for arbitrary initial point p_0, has a unique solution, in interval $t_0 \leq t < \mathfrak{T}_1$ with initial condition $p(t_0) = p_0$. Let its maximal interval of existence be $\big[t_0, \mathfrak{T}_1\big)$. We show that $\mathfrak{T}_1 = \infty$. Consider

$$\begin{aligned}
\Big|\Big|\frac{dp}{dt}\Big|\Big| &= \lambda\,\mathsf{a}\,(t)\,\|\,\Pi_{\mathcal{K}(p)}\,[\,\psi\,(p) - \rho\mathcal{T}p\,]\,-p\,\| \\
&= \lambda\,\mathsf{a}\,(t)\,\|\,\Pi_{\mathcal{K}(p)}\,[\,\psi\,(p) - \rho\mathcal{T}p\,] - \Pi_{\mathcal{K}(p)}\,[\,\psi\,(p)] \\
&\quad + \Pi_{\mathcal{K}(p)}\,[\,\psi\,(p)] - \Pi_{\mathcal{K}(p^\star)}\,[\,\psi\,(p^\star)] + \Pi_{\mathcal{K}(p^\star)}\,[\,\psi\,(p^\star)] - p\,\| \\
&\leq \lambda\,\mathsf{a}\,(t)\,\Big[\,\|\,\Pi_{\mathcal{K}(p)}\,[\,\psi\,(p) - \rho\mathcal{T}p\,] - \Pi_{\mathcal{K}(p)}\,[\,\psi\,(p)]\,\| \\
&\quad + \|\,\Pi_{\mathcal{K}(p)}\,[\,\psi\,(p)] - \Pi_{\mathcal{K}(p)}\,[\,\psi\,(p^\star)]\,\| \\
&\quad + \|\,\Pi_{\mathcal{K}(p)}\,[\,\psi\,(p^\star)] - \Pi_{\mathcal{K}(p^\star)}\,[\,\psi\,(p^\star)]\,\| + \|\,\Pi_{\mathcal{K}(p^\star)}\,[\,\psi\,(p^\star)]\,\| + \|\,p\,\|\,\Big] \\
&= \lambda\,\mathsf{a}\,(t)\,\Big[\rho\,\eta_1\,\|\,p\,\| + \big(\eta_2 + \upsilon\big)\,\|\,p - p^\star\,\| + \|\,\Pi_{\mathcal{K}(p^\star)}\,[\,\psi\,(p^\star)]\,\| + \|\,p\,\|\,\Big] \\
&\leq \lambda\,\mathsf{a}\,(t)\,\Big[\rho\,\eta_1\,\|\,p\,\| + (\eta_2 + \upsilon)\big(\,\|\,p\,\| + \|\,p^\star\,\|\,\big) + \|\,\Pi_{\mathcal{K}(p^\star)}\,[\,\psi\,(p^\star)]\,\| + \|\,p\,\|\,\Big] \\
&= \lambda\,\mathsf{a}\,(t)\,(\rho\,\eta_1 + \eta_2 + \upsilon + 1)\,\|\,p\,\| \\
&\quad + \lambda\,\mathsf{a}\,(t)\,\Big((\eta_2 + \upsilon)\,\|\,p^\star\,\| + \|\,\Pi_{\mathcal{K}(p^\star)}\,[\,\psi\,(p^\star)]\,\|\,\Big),
\end{aligned} \tag{3.4}$$

where we have used the Assumption (2.6), Lipschitz continuity of operator $\mathcal{T}$ and ψ with constant $\eta_1 > 0, \eta_2 > 0$. respectively.
Integrating (3.4) from t_0 to t, we have

$$\begin{aligned}
\|\,p(t)\,\| - \|\,p(t_0)\,\| &\leq c_1\int_{t_0}^{t} 1\,\mathrm{ds} + c_2\int_{t_0}^{t}\|\,p(s)\,\|\,\mathrm{ds} \\
&\leq c_1(t - t_0) + c_2\int_{t_0}^{t}\|\,p(s)\,\|\,\mathrm{ds},
\end{aligned}$$

Using Gronwall Lemma [12] , we have

$$\begin{aligned}
\|\,p(t)\,\| &\leq \Big(\,\|\,p(t_0)\,\| + c_1(t - t_0)\Big) + c_2\int_{t_0}^{t}\|\,p(s)\,\|\,\mathrm{ds} \\
&\leq \Big(\,\|\,p(t_0)\,\| + c_1(t - t_0)\Big)\,e^{c_2(t_0 - t)}, \quad t \in \big[t_0,\,\mathfrak{T}\big),
\end{aligned}$$

where

$$\begin{aligned}
c_1 &= \lambda\,\mathsf{a}\,(t)\,\Big((\eta_2 + \upsilon)\,\|\,p^\star\,\| + \|\,\Pi_{\mathcal{K}(p^\star)}\,[\,\psi\,(p^\star)]\,\|\,\Big), \\
c_2 &= \mathsf{a}\,(t)\,(\rho\,\eta_1 + \eta_2 + \upsilon + 1).
\end{aligned}$$

Consequently, solution $p(t)$ is bounded on $\big[t_0,\,\infty\big)$. Thus $\mathfrak{T} = \infty$. □

We now investigate the asymptotically stability of the trajectory to the solution of dynamical system (3.3) with $\lambda = 1$.

Theorem 3.2. Let the following assumptions be satisfied:

i. The operators $\mathcal{T},\,\psi : \mathcal{H} \longrightarrow \mathcal{H}$ be strongly monotone with constant $\xi_1 > 0,\ \xi_1 > 0$ and Lipschitz continuous with constant $\eta_1 > 0,\ \eta_2 > 0$ respectively.

ii. Assumption (2.6) and (2.12) hold.

iii. Parameter $\mathsf{a}(t) \in C\,(\,[\,0,\infty\,))$ and $\rho > 0$ satisfy

$$0 < \mathsf{a}(t) \leq 1, \quad t \geq 0, \qquad \int_0^\infty \mathsf{a}(\zeta)\, d\zeta = \infty,$$

$$\left|\rho - \frac{\xi_1}{\eta_1^2}\right| < \frac{\sqrt{\xi_1^2 - \eta_1^2\, \kappa\,(2-\kappa)}}{\eta_1^2}, \qquad \xi_1 > \eta_1 \sqrt{\kappa(2-\kappa)}, \quad \kappa < 1, \tag{3.5}$$

where

$$\kappa = \upsilon + \sqrt{1 - 2\,\xi_2 + \eta_2^2}\,.$$

Then, for every initial approximation $p_0 \in \mathcal{H}$, trajectory $p(t),\ t \geq 0$ defined by method (3.3). converges to the unique solution $p^\star \in \mathcal{K}(p^\star)$ of problem (2.2) with convergence rate:

$$\|\, p(t) - p^\star \,\| \leq e^{\int_0^\infty \Omega(\zeta)\, d\zeta} \,\|\, p_0 - p^\star \,\|,$$

where

$$\Omega(t) = -\frac{1}{2}\mathsf{a}(t) + \frac{1}{2}\mathsf{a}(t)\left(\kappa + \sqrt{1 - 2\,\rho\,\xi_1 + \rho^2\,\eta_1^2}\right)^2.$$

Proof. Since all the conditions from Theorem 2.1 are fulfilled , so unique solution $p^\star \in \mathcal{K}(p^\star)$ exists. We now define the Lyapunov function as

$$\mathfrak{L}(t) = \frac{1}{2}\,\|\, p(t) - p^\star \,\|^2.$$

For convergence, we have to show that $\mathfrak{L}(t) \longrightarrow 0$ as $t \longrightarrow \infty$.
Now,

$$\frac{d\,\mathfrak{L}}{d\,t} = \left\langle p(t) - p^\star\,, \frac{d\,p}{d\,t} \right\rangle. \tag{3.6}$$

From Remark(3.2), (3.3), and (3.6), we have

$$\begin{aligned}
\frac{d\,\mathfrak{L}}{d\,t} &= \Big\langle\, p(t) - p^{\star}\,, -\mathsf{a}\,(t)\,p\,(t) + \,\mathsf{a}(t)\Pi_{\mathcal{K}(p(t))}\,[\,\psi(p(t)) - \rho\mathcal{T}(p(t))\,]\,\Big\rangle \\
&= \Big\langle\, (\,p(t) - p^{\star}\,) + \mathsf{a}(t)\,p^{\star} - \mathsf{a}(t)\Pi_{\mathcal{K}(p^{\star})}\,[\psi(p^{\star}) - \rho\mathcal{T}(p^{\star})]\,, \\
&: -\mathsf{a}(t)\,p(t) + \mathsf{a}(t)\Pi_{\mathcal{K}(p(t))}\,[\psi(p) - \rho\,\mathcal{T}p]\,\Big\rangle \\
&= \frac{1}{2}\parallel (1-\mathsf{a}(t))\,(\,p(t) - p^{\star}\,) \\
&+ \mathsf{a}(t)\left(\Pi_{\mathcal{K}(p(t))}\,[\psi(p(t)) - \rho\mathcal{T}p(t)] - \Pi_{\mathcal{K}(p^{\star})}\,[\psi(p^{\star}) - \rho\mathcal{T}p^{\star}]\,\right)\parallel^2 \\
&- \frac{1}{2}\parallel (\,p(t) - p^{\star}\,) + \mathsf{a}(t)\,p^{\star} - \mathsf{a}(t)\Pi_{\mathcal{K}(p^{\star})}\,[\psi(p^{\star}) - \rho\mathcal{T}(p^{\star})]\parallel^2 \\
&\quad - \frac{1}{2}\parallel -\mathsf{a}(t)\,p(t) + \mathsf{a}(t)\,\Pi_{\mathcal{K}(p(t))}\,[\psi(p(t)) - \rho\mathcal{T}p(t)]\parallel^2 \\
&\le \frac{1}{2}\parallel (1-\mathsf{a}(t))\,(\,p(t) - p^{\star}\,) \\
&+ \mathsf{a}(t)\left(\Pi_{\mathcal{K}(p(t))}\,[\psi(p(t)) - \rho\mathcal{T}p(t)] - \Pi_{\mathcal{K}(p^{\star})}\,[\psi(p^{\star}) - \rho\mathcal{T}p^{\star}]\,\right)\parallel^2 \\
&\quad - \frac{1}{2}\parallel p(t) - p^{\star}\parallel^2 \\
&= \frac{1}{2}\,(1-\mathsf{a}(t))^2\parallel p(t) - p^{\star}\parallel^2 \\
&+ \frac{1}{2}\mathsf{a}^2(t)\parallel \Pi_{\mathcal{K}(p(t))}\,[\psi(p(t)) - \rho\mathcal{T}p(t)] - \Pi_{\mathcal{K}(p^{\star})}\,[\psi(p^{\star}) - \rho\mathcal{T}p^{\star}]\parallel^2 \\
&\quad + \mathsf{a}(t)\,(1-\mathsf{a}(t))\,\Big\langle p(t) - p^{\star}, \Pi_{\mathcal{K}(p(t))}\,[\psi(p(t)) - \rho\mathcal{T}p(t)] - \Pi_{\mathcal{K}(p^{\star})}\,[\psi(p^{\star}) - \rho\mathcal{T}p^{\star}]\Big\rangle \\
&\quad - \frac{1}{2}\parallel p(t) - p^{\star}\parallel^2 \\
&= -\frac{1}{2}\,\mathsf{a}(t)\parallel p(t) - p^{\star}\parallel^2 \\
&+ \frac{1}{2}\,\mathsf{a}(t)\parallel \Pi_{\mathcal{K}(p(t))}\,[\psi(p(t)) - \rho\mathcal{T}p(t)] - \Pi_{\mathcal{K}(p^{\star})}\,[\psi(p^{\star}) - \rho\mathcal{T}p^{\star}]\parallel^2, \qquad (3.7)
\end{aligned}$$

we have used the relation $2\,\big\langle\, \delta,\, \mu\,\big\rangle = \parallel \delta + \mu\parallel^2 - \parallel \delta\parallel^2 - \parallel \mu\parallel^2, \quad \forall\,\delta,\,\mu \in \mathcal{H}$. Using Assumption (2.6), strongly monotonicity and Lipschitz continuity of operators $\mathcal{T}$ and ψ, we get

$$\begin{aligned}
&\parallel \Pi_{\mathcal{K}(p(t))}\,[\psi(p(t)) - \rho\mathcal{T}p(t)] - \Pi_{\mathcal{K}(p^{\star})}\,[\psi(p^{\star}) - \rho\mathcal{T}p^{\star}]\parallel \\
&\le \parallel \Pi_{\mathcal{K}(p(t))}\,[\psi(p(t)) - \rho\mathcal{T}p(t)] - \Pi_{\mathcal{K}(p(t))}\,[\psi(p^{\star}) - \rho\mathcal{T}p^{\star}] \\
&\quad + \parallel \Pi_{\mathcal{K}(p(t))}\,[\psi(p^{\star}) - \rho\mathcal{T}p^{\star}] - \Pi_{\mathcal{K}(p^{\star})}\,[\psi(p^{\star}) - \rho\mathcal{T}p^{\star}]\parallel \\
&\le \parallel [\psi(p(t)) - \rho\mathcal{T}p(t)] - [\psi(p^{\star}) - \rho\mathcal{T}p^{\star}]\parallel \\
&\quad + \upsilon\parallel p(t) - p^{\star}\parallel \\
&\le \parallel p(t) - p^{\star} - (\psi(p(t) - \psi(p^{\star}))\parallel + \parallel p(t) - p^{\star} - \rho\,(\mathcal{T}p(t) - \mathcal{T}p^{\star})\parallel \\
&\quad + \upsilon\parallel p(t) - p^{\star}\parallel \\
&\le \left(\upsilon + \sqrt{1 - 2\,\xi_2 + \eta_2^2} + \sqrt{1 - 2\,\rho\,\xi_1 + \rho^2\,\eta_1^2}\right)\parallel p(t) - p^{\star}\parallel. \qquad (3.8)
\end{aligned}$$

From (3.7) and (3.8), we have

$$\begin{aligned}
\frac{d\,\mathfrak{L}}{d\,t} \leq & -\frac{1}{2}\,\mathsf{a}(t)\ \parallel p(t)-p^{\star}\parallel^2 \\
& +\frac{1}{2}\,\mathsf{a}(t)\ \left(\upsilon+\sqrt{1-2\,\xi_2+\eta_2^2}+\sqrt{1-2\,\rho\,\xi_1+\rho^2\,\eta_1^2}\right)^2\ \parallel p(t)-p^{\star}\parallel^2 \\
& \leq \frac{1}{2}\left[-\mathsf{a}(t)+\mathsf{a}(t)\ \left(\kappa+\sqrt{1-2\rho\,\xi_1+\rho^2\,\eta_1^2}\right)^2\right]\ \parallel p(t)-p^{\star}\parallel^2 \\
& = \frac{1}{2}\Omega(t)\parallel p(t)-p^{\star}\parallel^2,
\end{aligned} \tag{3.9}$$

Consequently, we obtain the following estimate

$$\parallel p(t)-p^{\star}\parallel \leq e^{\int_0^{\infty}\Omega(\zeta)\,d\zeta}\ \parallel p_0-p^{\star}\parallel,$$

where

$$\Omega(\zeta)=-\frac{1}{2}\mathsf{a}(\zeta)+\frac{1}{2}\mathsf{a}(\zeta)\left(\kappa+\sqrt{1-2\rho\,\xi_1+\rho^2\,\eta_1^2}\right)^2.$$

Since $\int_0^{\infty}\mathsf{a}(\zeta)\,d\,\zeta=\infty$ and $\kappa+\sqrt{1-2\rho\,\xi_1+\rho^2\,\eta_1^2}<1$, we have $\int_0^{\infty}\Omega(\zeta)\,d\,\zeta=-\infty$, thus $e^{\int_0^{\infty}\Omega\,(\zeta)\,d\zeta}=0$.

It follows that the trajectory $p(t)$ converges asymptotically to unique solution $p^{\star}$ satisfies the inequality (2.2), the desired result. □

For $\psi=I$, one can obtain the result of [17]. The above result is quite different from the result studied in [17].

4. Iterative Methods

In this section, we suggest some inertial type methods for solving general quasi variational inequalities. Discretizing dynamical system (3.3) by using forward difference scheme, we have

$$\frac{p_{n+1}\ -\ p_n}{h}=-\alpha_n(t)\,p_n+\ \alpha_n(t)\,\Pi_{\mathcal{K}(p_n)}\left[\,\psi(p_n)-\rho\mathcal{T}p_n\,\right],$$

where $0<\alpha_n(t)\leq 1$, and h is the step size.
We now introduce an iterative scheme by taking step size $h=1$ and $\alpha_n(t)=\alpha_n$.

Algorithm 1. For given $p_0\in\mathcal{H}:\psi(p_0)\in\mathcal{K}(p)$, compute p_{n+1} through iterative scheme

$$p_{n+1}=(1-\alpha_n)p_n+\ \alpha_n\,\Pi_{\mathcal{K}(p_n)}\left[\,\psi(p_n)-\rho\mathcal{T}(p_n)\right],\ \ n=0,1,\cdots$$

where $0\leq\ \alpha_n\ \leq 1$ and $\rho>\ 0$ are parameters, which is called Mann iterative process.

Again discretizing dynamical system (3.3)by using forward difference scheme, we have the following iterative scheme

Algorithm 2. For given $p_0 \in \mathcal{H} : \psi(p_0) \in \mathcal{K}(p)$, compute p_{n+1} by the iterative scheme

$$p_{n+1} = (1-\alpha_n)p_n + \alpha_n \Pi_{\mathcal{K}(p_n)} [\psi(p_n) - \rho\mathcal{T}(p_{n+1})], \quad n = 0, 1, \cdots$$

where $0 \leq \alpha_n \leq 1$ and $\rho > 0$ are parameters.

This method is an implicit method and can be viewed as analogues to the extragradient method of Korpelevich [14]. To implement Algorithm 2, we use the inertial-type predictor and corrector technique.

Algorithm 3. For given $p_0, p_1 \in \mathcal{H} : \psi(p_0), \psi(p_1) \in \mathcal{K}(p)$, compute p_{n+1} by the iterative scheme

$$w_n = p_n + \Theta_n (p_n - p_{n-1}),$$
$$p_{n+1} = (1-\alpha_n)p_n + \alpha_n \Pi_{\mathcal{K}(p_n)} [\psi(p_n) - \rho\mathcal{T}(w_n)], \quad n = 1, 2, \cdots$$

where $0 \leq \alpha_n \leq 1$ and $\rho > 0$ are parameters.

Algorithm 4. For given $p_0 \in \mathcal{H} : \psi(p_0) \in \mathcal{K}(p)$, compute p_{n+1} by the iterative scheme

$$p_{n+1} = (1-\alpha_n)p_{n+1} + \alpha_n \Pi_{\mathcal{K}(p_{n+1})} [\psi(p_{n+1}) - \rho\mathcal{T}(p_{n+1})], \quad n = 0, 1, \cdots$$

where $0 \leq \alpha_n \leq 1$ and $\rho > 0$ are parameters.

For $\alpha_n) = 1$ and $\mathcal{K}(p) = \mathcal{K}$, the convex set, then Algorithm 4 is due to Noor[26]. Algorithm 4 is an implicit method. To implement this method, we use the inertial-type predictor and corrector technique. Consequently, Algorithm 4 can be written in the following form.

Algorithm 5. For given $p_0, p_1 \in \mathcal{H} : \psi(p_0), \psi(p_1) \in \mathcal{K}(p)$, compute p_{n+1} through iterative scheme

$$w_n = p_n + \Theta_n (p_n - p_{n-1}),$$
$$p_{n+1} = (1-\alpha_n)w_n + \alpha_n \Pi_{\mathcal{K}(w_n)} [\psi(w_n) - \rho\mathcal{T}(w_n)], \quad n = 1, 2, \cdots$$

where $0 \leq \alpha_n, \Theta_n \leq 1$ and $\rho > 0$ are parameters.

If we take $\alpha_n = 1$, Algorithm 5 reduces to a new inertial-type iterative scheme for solving inequality (2.2).

Algorithm 6. For given $p_0, p_1 \in \mathcal{H} : \psi(p_0), \psi(p_1) \in \mathcal{K}(p)$, compute p_{n+1} through iterative scheme

$$w_n = p_n + \Theta_n (p_n - p_{n-1}),$$
$$p_{n+1} = \Pi_{\mathcal{K}(w_n)} [\psi(w_n) - \rho\mathcal{T}(w_n)], \quad n = 1, 2, \cdots$$

where $0 \leq \Theta_n \leq 1$ and $\rho > 0$ are parameters.

For $\psi = I$, Algorithm 5 collapses to the following new method for solving inequality (2.5).

Algorithm 7. [33] For given $p_0, p_1 \in \mathcal{H}$, compute p_{n+1} through iterative scheme

$$w_n = p_n + \Theta_n (p_n - p_{n-1}),$$
$$p_{n+1} = (1-\alpha_n)w_n + \alpha_n \Pi_{\mathcal{K}(w_n)} [w_n - \rho\mathcal{T}(w_n)], \quad n = 1, 2, \cdots$$

where $0 \leq \alpha_n, \Theta_n \leq 1$ and $\rho > 0$ are parameters.

The advantage of these methods is that only one projection operator is used. We now introduce a three-step inertial-projection iterative method for solving the general quasi variational inequality (2.2).

Algorithm 8. For given $p_0,\, p_1 \in \mathcal{H} : \psi(p_0),\, \psi(p_1) \in \mathcal{K}(p)$, compute p_{n+1} through iterative scheme

$$\begin{aligned}
w_n &= p_n + \Theta_n \left(p_n - p_{n-1}\right), && (4.1)\\
y_n &= (1-\gamma_n)w_n + \, \gamma_n \, \Pi_{\mathcal{K}(w_n)} \left[\, \psi(w_n) - \rho \mathcal{T}(w_n)\right], && (4.2)\\
z_n &= (1-\beta_n)y_n + \, \beta_n \, \Pi_{\mathcal{K}(y_n)} \left[\, \psi(y_n) - \rho \mathcal{T}(y_n)\right], && (4.3)\\
u_{n+1} &= (1-\alpha_n)z_n + \, \alpha_n \, \Pi_{\mathcal{K}(z_n)} \left[\, \psi(z_n) - \rho \mathcal{T}(z_n)\right], \;\; n = 1, 2, \cdots && (4.4)
\end{aligned}$$

where $0 \leq \, \alpha_n,\, \beta_n,\, \gamma_n, \;\, \Theta_n \; \leq 1$ and $\rho > \; 0$ are parameters.

If we take $\mathcal{K}(p) = \mathcal{K}$, closed convex set, then Algorithm 8 reduces to a new inertial-type iterative scheme for solving general variational inequality.

Algorithm 9. For given $p_0 \; \in \mathcal{H} \, : \, \psi(p_0) \; \in \mathcal{K}$, compute p_{n+1} through iterative scheme

$$\begin{aligned}
w_n &= p_n + \Theta_n \left(p_n - p_{n-1}\right),\\
y_n &= (1-\gamma_n)p_n + \, \gamma_n \, \Pi_{\mathcal{K}} \left[\, \psi(p_n) - \rho \mathcal{T}(p_n)\right],\\
z_n &= (1-\beta_n)y_n + \, \beta_n \, \Pi_{\mathcal{K}} \left[\, \psi(y_n) - \rho \mathcal{T}(y_n)\right],\\
p_{n+1} &= (1-\alpha_n)z_n + \, \alpha_n \, \Pi_{\mathcal{K}} \left[\, \psi(z_n) - \rho \mathcal{T}(z_n)\right], \;\; n = 1, 2, \cdots
\end{aligned}$$

where $0 \leq \, \alpha_n,\, \beta_n,\, \gamma_n, \;\, \Theta_n \; \leq 1$ and $\rho > \; 0$ are parameters.

For different and suitable choice of suitable operators and spaces, one can obtain new and previous iterative schemes for solving inequality (2.2) and related problems.

We now analyze convergence analysis for Algorithm 8 under some appropriate conditions.

Theorem 4.1. Let the operators $\mathcal{T},\, \psi \, : \, \mathcal{H} \longrightarrow \mathcal{H}$ be strongly monotone with constants $\xi_1 > 0,\, \xi_2 > 0$ and Lipschitz continuous with constants $\eta_1 > 0,\, \eta_2 \, > 0$ respectively.
Let $0 \leq \, \Theta_n,\, \alpha_n,\, \beta_n,\, \gamma_n \leq 1$, for all $n \geq 1$ such that

$$\sum_{n=1}^{\infty} \alpha_n = \infty, \;\; \sum_{n=1}^{\infty} \Theta_n \parallel p_n - p_{n-1} \parallel < \infty.$$

If Assumption P holds and $\rho \; > \; 0$ satisfies (2.12). Then p_{n+1}, an approximate solution obtained through the iterative scheme defined in Algorithm 8 converges to unique solution $p^\star \in \mathcal{K}(p^\star)$ of problem (2.2).

Proof. Since all the conditions of Theorem 2.1 are satisfied, so $p^\star \in \mathcal{H} \; : \psi(p^\star) \in \mathcal{K}(p^\star)$ is a solution of inequality (2.2). Then from Lemma 2.3, we have

$$\begin{aligned}
p^\star &= (1-\alpha_n)p^\star + \, \alpha_n \, \Pi_{\mathcal{K}(p^\star)} \left[\psi(p^\star) - \rho \mathcal{T}(p^\star)\right], && (4.5)\\
&= (1-\beta_n)p^\star + \, \beta_n \, \Pi_{\mathcal{K}(p^\star)} \left[\psi(p^\star) - \rho \mathcal{T}(p^\star)\right], && (4.6)\\
&= (1-\gamma_n)p^\star + \, \gamma_n \, \Pi_{\mathcal{K}(p^\star)} \left[\psi(p^\star) - \rho \mathcal{T}(p^\star)\right], && (4.7)
\end{aligned}$$

where $0 \, \leq \, \alpha_n,\, \beta_n,\, \gamma_n \leq \, 1$, are constants.

Using Assumption (2.6), from (3.8), (4.4) and (4.5), we have

$$\begin{aligned}
\| p_{n+1} - p^\star \| =& \| \big((1-\alpha_n)z_n + \alpha_n \Pi_{\mathcal{K}(z_n)}[\psi(z_n) - \rho\mathcal{T}(z_n)]\big) \\
& - \big((1-\alpha_n)p^\star + \alpha_n \Pi_{\mathcal{K}(p^\star)}[\psi(p^\star) - \rho\mathcal{T}(p^\star)]\big) \| \\
\leq& (1-\alpha_n) \| z_n - p^\star \| + \alpha_n \| \Pi_{\mathcal{K}(z_n)}[\psi(z_n) - \rho\mathcal{T}z_n] - \Pi_{\mathcal{K}(p^\star)}[\psi(p^\star) - \rho\mathcal{T}p^\star] \| \\
\leq& (1-\alpha_n) \| z_n - p^\star \| + \alpha_n \| \Pi_{\mathcal{K}(z_n)}[\psi(z_n) - \rho\mathcal{T}z_n] - \Pi_{\mathcal{K}(z_n)}[\psi(p^\star) - \rho\mathcal{T}p^\star] \| \\
& + \alpha_n \| \Pi_{\mathcal{K}(z_n)}[\psi(p^\star) - \rho\mathcal{T}p^\star] - \Pi_{\mathcal{K}(p^\star)}[\psi(p^\star) - \rho\mathcal{T}p^\star] \| \\
\leq& (1-\alpha_n) \| z_n - p^\star \| + \alpha_n \| [\psi(z_n) - \rho\mathcal{T}z_n] - [\psi(p^\star) - \rho\mathcal{T}p^\star] \| + \alpha_n \upsilon \| z_n - p^\star \| \\
\leq& (1-\alpha_n) \| z_n - p^\star \| + \alpha_n \left(\upsilon + \sqrt{1 - 2\,\xi_2 + \eta_2^2} + \sqrt{1 - 2\rho\,\xi_1 + \rho^2\,\eta_1^2}\right) \| z_n - p^\star \| \\
=& (1-\alpha_n) \| z_n - p^\star \| + \alpha_n \left(\kappa + \sqrt{1 - 2\rho\,\xi_1 + \rho^2\,\eta_1^2}\right) \| z_n - p^\star \| \\
=& \big[1 - \alpha_n(1-\vartheta)\big] \| z_n - p^\star \| . \qquad (4.8)
\end{aligned}$$

Similarly, from (4.3) and (4.6), we have

$$\begin{aligned}
\| z_n - p^\star \| =& \| \big((1-\beta_n)y_n + \beta_n \Pi_{\mathcal{K}(y_n)}[\psi(y_n) - \rho\mathcal{T}(y_n)]\big) \\
& - \big((1-\beta_n)p^\star + \beta_n \Pi_{\mathcal{K}(p^\star)}[\psi(p^\star) - \rho\mathcal{T}(p^\star)]\big) \| \\
\leq& (1-\beta_n) \| y_n - p^\star \| + \beta_n \vartheta \| y_n - p^\star \| \\
=& \big[1 - \beta_n(1-\vartheta)\big] \| y_n - p^\star \| \\
\leq& \| y_n - p^\star \| . \qquad (4.9)
\end{aligned}$$

From (4.2) and (4.7), we have

$$\begin{aligned}
\| y_n - p^\star \| =& \| \big((1-\gamma_n)w_n + \gamma_n \Pi_{\mathcal{K}(w_n)}[\psi(w_n) - \rho\mathcal{T}(w_n)]\big) \\
& - \big((1-\gamma_n)p^\star + \gamma_n \Pi_{\mathcal{K}(p^\star)}[\psi(p^\star) - \rho\mathcal{T}(p^\star)]\big) \| \\
=& \big[1 - \gamma_n(1-\vartheta)\big] \| w_n - p^\star \| \\
\leq& \| w_n - p^\star \| . \qquad (4.10)
\end{aligned}$$

Now, from (4.1), we have

$$\begin{aligned}
\| w_n - p^\star \| =& \| p_n - p^\star + \Theta_n (p_n - p_{n-1}) \| \\
\leq& \| p_n - p^\star \| + \Theta_n \| p_n - p_{n-1} \| . \qquad (4.11)
\end{aligned}$$

From (4.8), (4.9), (4.10), and (4.11), we have

$$\begin{aligned}
\| p_{n+1} - p^\star \| \leq& \big[1 - \alpha_n(1-\vartheta)\big]\big[\| p_n - p^\star \| + \Theta_n \| p_n - p_{n-1} \| \big] \\
\leq& \big[1 - \alpha_n(1-\vartheta)\big] \| p_n - p^\star \| + \Theta_n \| p_n - p_{n-1} \| .
\end{aligned}$$

From (2.12), we have $\vartheta < 1$. Since $\sum_{n=1}^{\infty} \Theta_n \| p_n - p_{n-1} \| < \infty$, using Lemma 2.2, we get that
$p_n \longrightarrow p^\star$, $n \longrightarrow \infty$. Hence the sequence $\{p_n\}$ obtained from the Algorithm 8 converges to a unique solution $p^\star \in \mathcal{K}(p^\star)$ satisfying the inequality (2.2), which is the desired result. □

(I). If $\mathcal{K}(p) = \mathcal{K}$, then following result can be obtained from Theorem 4.1.

Theorem 4.2. Let the operators $\mathcal{T}, \psi : \mathcal{H} \longrightarrow \mathcal{H}$ be strongly monotone with constants $\xi_1 > 0$, $\xi_2 > 0$ and Lipschitz continuous with constants $\eta_1 > 0$, $\eta_2 > 0$ respectively.
Let $0 \leq \Theta_n, \alpha_n, \beta_n, \gamma_n \leq 1$, for all $n \geq 1$ such that

$$\sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} \Theta_n \parallel p_n - p_{n-1} \parallel < \infty.$$

If $\rho > 0$ satisfies the condition

$$\left| \rho - \frac{\xi_1}{\eta_1^2} \right| < \frac{\sqrt{\xi_1^2 - \eta_1^2 \, \kappa \, (2-\kappa)}}{\eta_1^2} \qquad \xi_1 > \eta_1 \sqrt{\kappa(2-\kappa)}, \quad \kappa < 1,$$

where

$$\kappa = \sqrt{1 - 2\,\xi_2 + \eta_2^2}\,,$$

Then p_{n+1}, an approximate solution obtained through the iterative scheme defined in Algorithm 9 converges to unique solution $p^\star \in \mathcal{K}$ of problem (2.6).

(II).If we take $\psi = I$, then following result can be obtained from Theorem 4.1.

Theorem 4.3. Let the operators $\mathcal{T} : \mathcal{H} \longrightarrow \mathcal{H}$ be strongly monotone with constants $\xi > 0$, and Lipschitz continuous with constants $\eta > 0$ respectively.
Let $0 \leq \Theta_n, \alpha_n, \beta_n, \gamma_n \leq 1$, for all $n \geq 1$ such that

$$\sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} \Theta_n \parallel p_n - p_{n-1} \parallel < \infty.$$

If $\rho > 0$ satisfies the condition

$$\left| \rho - \frac{\xi}{\eta^2} \right| < \frac{\sqrt{\xi^2 - \eta^2 \, \upsilon \, (2-\upsilon)}}{\eta^2} \qquad \xi > \eta \sqrt{\upsilon(2-\upsilon)}, \quad \upsilon < 1.$$

Then an approximate solution p_{n+1} obtain from

$$\begin{aligned}
w_n &= p_n + \Theta_n \,(p_n - p_{n-1})\,, \\
y_n &= (1-\gamma_n) w_n + \gamma_n \, \Pi_{\mathcal{K}(w_n)} \left[w_n - \rho \mathcal{T}(w_n) \right], \\
z_n &= (1-\beta_n) y_n + \beta_n \, \Pi_{\mathcal{K}(y_n)} \left[y_n - \rho \mathcal{T}(y_n) \right], \\
u_{n+1} &= (1-\alpha_n) z_n + \alpha_n \, \Pi_{\mathcal{K}(z_n)} \left[z_n - \rho \mathcal{T}(z_n) \right], \ \ n = 1, 2, \cdots
\end{aligned}$$

converges to unique solution $p^\star \in \mathcal{K}(p^\star)$ of problem (2.5).

Conclusion

In this article, we have introduced a dynamical system and studied the unique existence of the solutions of dynamical system associated with general quasi variational inequalities. We have developed a continuous three-step inertial method for solving quasi variational inequalities. We have estimated the convergence rate of the proposed method. We have proved that the approximate solution obtained by the inertial projection iterative schemes converges to exact solution. It is an interesting problem to compare the efficiency of the proposed methods with other known methods. Results obtained from the current study may stimulate future research in this area.

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