

GENERALIZED OSTROWSKI INTEGRAL INEQUALITY WITH WEIGHTS FOR SECOND ORDER DIFFERENTIABLE MAPPINGS

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ABSTRACT. We have proved a generalized integral inequality of Ostrowski's type using weights with parameters for differentiable mapping whose second order derivatives are bounded and first order derivatives are absolutely continuous. Further, we obtained number of new results as special cases. We also stated applications of our work in Numerical Quadrature rules and Probability Theory.

1. INTRODUCTION

A famous inequality known as Ostrowski inequality was revealed by Ukrainian Mathematician Alexandar Markowich Ostrowski in 1938. It states that [17]:

Proposition 1.1. *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I^0 (I^0 is the interior of I) and let $a, b \in I^0$ with $a < b$. If $f' : (a, b) \rightarrow \mathbb{R}$ is bounded on (a, b) , i.e.,*

$$\|f'\|_{\infty} = \sup_{t \in (a, b)} |f'(t)| < \infty,$$

then the following inequality holds

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \|f'\|_{\infty}$$

$\forall x \in [a, b]$. The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

Ostrowski's inequality estimates the absolute deviation of functional value from its integral mean. It has number of applications in Probability Theory and Numerical Integration. It also deals with Special Means.

In 1976, Milovanović and Pečarić proved generalized Ostrowski's inequality for n -time differentiable mappings [15] from which we reproduce the case of second order differentiable mappings [15, p. 470].

2010 *Mathematics Subject Classification.* 26D10, 26D20, 26D99.

Key words and phrases. Ostrowski's inequality, Numerical integration.

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Submitted December 21, 2019. Published February 23, 2020.

Communicated by N.L. Braha.

Proposition 1.2. *Let $g : [a, b] \rightarrow \mathbb{R}$ be a second order differentiable mapping such that $g'' : (a, b) \rightarrow \mathbb{R}$ is bounded on (a, b) . Then the inequality holds.*

$$\begin{aligned} & \left| \frac{1}{2} \left[g(x) + \frac{(x-a)g(a) + (b-x)g(b)}{b-a} \right] - \frac{1}{b-a} \int_a^b g(t)dt \right| \\ & \leq \frac{\|g''\|_\infty}{4} (b-a)^2 \left[\frac{1}{12} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] \end{aligned}$$

$\forall x \in [a, b]$.

In year 1999, Cerone et.al. in [1] proved the following inequality.

Proposition 1.3. *Under the supposition of Proposition 1.2, following inequality holds*

$$\begin{aligned} & \left| g(x) - \left(x - \frac{a+b}{2} \right) g'(x) - \frac{1}{b-a} \int_a^b g(t)dt \right| \\ & \leq \left[\frac{(b-a)^2}{24} + \frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 \right] \|g''\|_\infty \leq \frac{(b-a)^2}{6} \|g''\|_\infty. \end{aligned}$$

$\forall x \in [a, b]$.

In the same year, Dragomir et.al. [3] proved the following result.

Proposition 1.4. *Under the supposition of Proposition 1.2, following inequality holds*

$$\begin{aligned} & \left| g(x) - \frac{g(b) - g(a)}{b-a} \left(x - \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b g(t)dt \right| \\ & \leq \frac{(b-a)^2}{2} \left\{ \left[\left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 + \frac{1}{4} \right]^2 + \frac{1}{12} \right\} \|g''\|_\infty \leq \frac{(b-a)^2}{6} \|g''\|_\infty. \end{aligned}$$

$\forall x \in [a, b]$.

In [2], Dragomir et. al, stated an Ostrowski type inequality which is as under:

Proposition 1.5. *Let $g : [a, b] \rightarrow \mathbb{R}$ be a mapping whose first order derivative is absolutely continuous on $[a, b]$ and assume that the second order derivative $g'' \in L_\infty [a, b]$. Then the following inequality holds*

$$\begin{aligned} & \left| \int_a^b g(t) - \frac{1}{2} \left[g(x) + \frac{g(a) + g(b)}{2} \right] (b-a) + \frac{(b-a)}{2} \left(x - \frac{a+b}{2} \right) g'(x) \right| \\ & \leq \|g''\|_\infty \left(\frac{1}{3} \left| x - \frac{a+b}{2} \right|^3 + \frac{(b-a)^3}{48} \right) \end{aligned} \quad (1.1)$$

$\forall x \in [a, b]$.

Now onward we assume that $\alpha = a + \lambda \frac{b-a}{2}$ and $\beta = b - \lambda \frac{b-a}{2}$ where λ is a parameter such that $\lambda \in [0, 1]$.

Zafar and Mir generalized aforementioned result in [22] which is stated as under:

Proposition 1.6. *Under the assumptions of Proposition 1.5, we have the inequality*

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b g(t) dt - \frac{1}{2} [(1-\lambda)(g(x) + g(a+b-x)) + (1+\lambda) \right. \\ & \left. \left(\frac{g(a)+g(b)}{2} \right) - (1-\lambda) \left(x - \frac{a+b}{2} \right) g'(x) - \lambda \frac{b-a}{4} (g'(b) - g'(a))] \right| \\ & \leq \|g''\|_{\infty} \frac{1}{(b-a)} \left[\frac{1}{3} \left| x - \frac{a+b}{2} \right|^3 + \frac{(b-a)^3}{48} \Psi(\lambda) \right] \end{aligned} \quad (1.2)$$

$\forall x \in [\alpha, \beta]$, where $\Psi(\lambda) = (1-\lambda)[2(1-\lambda)^2 - 1] + 2\lambda$ and $\lambda \in [0, 1]$.

Pečarić and Savić [18] introduced 1st weighted Ostrowski inequality in 1983. Due to importance and significance of weighted Ostrowski's inequality, in the recent years, researchers are working continuously on this inequality in order to get better bounds. For recent work related to the topic we refer the reader following articles [5, 6, 7, 8, 9, 10, 13, 14, 12, 20, 21].

We would use weights and parameters in (1.1) to improve results stated by Zafar and Mir in [22]. We also plan to state some applications in Numerical Integration and Probability Theory. The weighted Peano kernels approach with parameters and weight is proved helpful to generalized Ostrowski type inequalities.

Present paper is arranged in the following manner: The first section is based on preliminaries where as the second section states new results involving weights with parameters in inequality (1.1) here weights used are probability density function. In the final section we have stated some of the applications in composite quadrature rule and Probability Theory.

2. MAIN RESULTS

Theorem 2.1. *Under the assumptions of Proposition 1.5, we have the inequality for $x \in [\alpha, \beta]$*

$$\begin{aligned} & \left| \int_a^b g(t) \omega(t) dt - B(g(a)\omega(a) + g(b)\omega(b)) \right. \\ & - \frac{1}{2} \left[g(b) \int_{\beta}^b \omega(t) dt - g(a) \int_{\alpha}^a \omega(t) dt \right. \\ & \left. - B \left(g'(a) \int_{\alpha}^a \omega(t) dt - g'(b) \int_{\beta}^b \omega(t) dt \right) \right. \\ & + \int_{\alpha}^A \omega(t) dt \left(g(x) - g'(x)(x-A) \right) \\ & + \int_A^{\beta} \omega(t) dt \left(g(a+b-x) + g'(a+b-x)(x-A) \right) \\ & \left. \left. - \int_a^b g(t)(t-A)\omega'(t) dt \right] \right| \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\|g''\|_\infty}{4} \left[\int_\alpha^\beta (x-t) \left[(a+b-x) - t \right] \omega(t) dt \right. \\
&\quad + 2 \int_{a+b-x}^A (x-t) \left[(a+b-x) - t \right] \omega(t) dt \\
&\quad \left. - \int_a^\alpha (t-a)(t-b) \omega(t) dt - \int_\beta^b (t-a)(t-b) \omega(t) dt \right] \quad (2.1)
\end{aligned}$$

where $\omega : [a, b] \rightarrow [0, \infty)$ is a probability density function, i.e., it is a positive integrable function satisfying $\int_a^b \omega(t) dt = 1$, $A = \frac{a+b}{2}$ and $B = \frac{b-a}{2}$.

Proof. It is easy to see that

$$\begin{aligned}
&f(x) \int_\alpha^{\frac{a+b}{2}} \omega(t) dt + f(a+b-x) \int_{\frac{a+b}{2}}^\beta \omega(t) dt \\
&= \int_a^b \omega(t) f(t) dt + f(a) \int_\alpha^a \omega(t) dt \\
&\quad + f(b) \int_\beta^b \omega(t) dt + \int_a^b P_\omega(x, t) f'(t) dt \quad (2.2)
\end{aligned}$$

$\forall x \in [\alpha, \beta]$, provided that f is absolutely continuous on $[a, b]$ and the kernel $P_\omega : [\alpha, \beta] \times [a, b] \rightarrow \mathbb{R}$ is given by:

$$P_\omega(x, t) = \begin{cases} \int_\alpha^t \omega(u) du, & \text{if } t \in [a, x], \\ \int_{\frac{a+b}{2}}^t \omega(u) du, & \text{if } t \in (x, a+b-x], \\ \int_\beta^t \omega(u) du, & \text{if } t \in (a+b-x, b], \end{cases}$$

Let us put

$$f(x) = \left(x - \frac{a+b}{2} \right) g'(x).$$

in (2.2) we get

$$\begin{aligned}
&\left(x - \frac{a+b}{2} \right) \left(g'(x) \int_\alpha^{\frac{a+b}{2}} \omega(t) dt - g'(a+b-x) \int_{\frac{a+b}{2}}^\beta \omega(t) dt \right) \\
&= \int_a^b \omega(t) \left(t - \frac{a+b}{2} \right) g'(t) dt - \frac{b-a}{2} \left(g'(a) \int_\alpha^a \omega(t) dt - g'(b) \int_\beta^b \omega(t) dt \right) \\
&\quad + \int_a^b P_\omega(x, t) \left[g'(t) + \left(t - \frac{a+b}{2} \right) g''(t) \right] dt. \quad (2.3)
\end{aligned}$$

Using integration by parts, we have

$$\begin{aligned} \int_a^b \left(t - \frac{a+b}{2} \right) \omega(t) g'(t) dt &= \frac{b-a}{2} [g(a)\omega(a) + g(b)\omega(b)] \\ &\quad - \int_a^b g(t)\omega(t) dt - \int_a^b g(t) \left(t - \frac{a+b}{2} \right) \omega'(t) dt, \end{aligned} \quad (2.4)$$

also

$$\begin{aligned} \int_a^b P_\omega(x, t) g'(t) dt &= g(b) \int_\beta^b \omega(t) dt - g(a) \int_\alpha^a \omega(t) dt \\ &\quad + g(x) \int_\alpha^{\frac{a+b}{2}} \omega(t) dt + g(a+b-x) \int_{\frac{a+b}{2}}^\beta \omega(t) dt - \int_a^b g(t)\omega(t) dt. \end{aligned} \quad (2.5)$$

Now using equations (2.4) and (2.5) in (2.3), we get

$$\begin{aligned} &\left(x - \frac{a+b}{2} \right) \left(g'(x) \int_\alpha^{\frac{a+b}{2}} \omega(t) dt - g'(a+b-x) \int_{\frac{a+b}{2}}^\beta \omega(t) dt \right) \\ &= \frac{b-a}{2} (g(a)\omega(a) + g(b)\omega(b)) + g(b) \int_\beta^b \omega(t) dt - g(a) \int_\alpha^a \omega(t) dt \\ &\quad - \frac{b-a}{2} \left(g'(a) \int_\alpha^a \omega(t) dt - g'(b) \int_\beta^b \omega(t) dt \right) + g(x) \int_\alpha^{\frac{a+b}{2}} \omega(t) dt \\ &\quad + g(a+b-x) \int_{\frac{a+b}{2}}^\beta \omega(t) dt - 2 \int_a^b g(t)\omega(t) dt - \int_a^b g(t) \left(t - \frac{a+b}{2} \right) \omega'(t) dt \\ &\quad + \int_a^b P_\omega(x, t) \left(t - \frac{a+b}{2} \right) g''(t) dt. \end{aligned}$$

or we can write it as

$$\begin{aligned} \int_a^b \omega(t) g(t) dt &= \frac{b-a}{4} (g(a)\omega(a) + g(b)\omega(b)) + \frac{g(b)}{2} \int_\beta^b \omega(t) dt \\ &\quad - \frac{g(a)}{2} \int_\alpha^a \omega(t) dt - \frac{b-a}{4} \left(g'(a) \int_\alpha^a \omega(t) dt - g'(b) \int_\beta^b \omega(t) dt \right) \\ &\quad + \frac{1}{2} \int_\alpha^{\frac{a+b}{2}} \omega(t) dt \left(g(x) - \left(x - \frac{a+b}{2} \right) g'(x) \right) + \frac{1}{2} \int_{\frac{a+b}{2}}^\beta \omega(t) dt \\ &\quad \times \left(g(a+b-x) + \left(x - \frac{a+b}{2} \right) g'(a+b-x) \right) \\ &\quad - \frac{1}{2} \int_a^b g(t) \left(t - \frac{a+b}{2} \right) \omega'(t) dt + \frac{1}{2} \int_a^b P_\omega(x, t) \left(t - \frac{a+b}{2} \right) g''(t) dt, \end{aligned}$$

$\forall x \in [\alpha, \beta]$ which gives us

$$\begin{aligned}
& \left| \int_a^b g(t)\omega(t)dt - \frac{b-a}{2}(g(a)\omega(a) + g(b)\omega(b)) \right. \\
& - \frac{1}{2} \left(g(b) \int_\beta^b \omega(t)dt - g(a) \int_\alpha^a \omega(t)dt \right) \\
& + \frac{b-a}{4} \left(g'(a) \int_\alpha^a \omega(t)dt - g'(b) \int_\beta^b \omega(t)dt \right) \\
& - \frac{1}{2} \int_\alpha^{\frac{a+b}{2}} \omega(t)dt \left(g(x) - g'(x)(x - \frac{a+b}{2}) \right) \\
& - \frac{1}{2} \int_{\frac{a+b}{2}}^\beta \omega(t)dt \left(g(a+b-x) + g'(a+b-x) \left(x - \frac{a+b}{2} \right) \right) \\
& \left. + \frac{1}{2} \int_a^b g(t) \left(t - \frac{a+b}{2} \right) \omega'(t)dt \right| = \left| \frac{1}{2} \int_a^b P_\omega(x, t) \left(t - \frac{a+b}{2} \right) g''(t)dt \right| \\
& \leq \frac{1}{2} \int_a^b |P_\omega(x, t)| \left| t - \frac{a+b}{2} \right| |g''(t)| dt
\end{aligned} \tag{2.6}$$

But

$$\int_a^b |P_\omega(x, t)| \left| t - \frac{a+b}{2} \right| |g''(t)| dt \leq \|g''\|_\infty \int_a^b |P_\omega(x, t)| \left| t - \frac{a+b}{2} \right| dt, \tag{2.7}$$

Now we define

$$I = \int_a^b |P_\omega(x, t)| \left| t - \frac{a+b}{2} \right| dt$$

which in turn becomes

$$\begin{aligned}
I &= \int_a^x \left| \int_\alpha^t \omega(u)du \right| \left| t - \frac{a+b}{2} \right| dt + \int_x^{a+b-x} \left| \int_{\frac{\alpha+\beta}{2}}^t \omega(u)du \right| \left| t - \frac{a+b}{2} \right| \\
&+ \int_x^b \left| \int_\beta^t \omega(u)du \right| \left| t - \frac{a+b}{2} \right| dt.
\end{aligned}$$

Now, we have only one case: Due to symmetry either $x \in [\alpha, \frac{a+b}{2}]$ and $a+b-x \in [\frac{a+b}{2}, \beta]$ or $x \in [\frac{a+b}{2}, \beta]$ and $a+b-x \in [\alpha, \frac{a+b}{2}]$, we obtain

$$\begin{aligned}
I &= \int_a^\alpha \left(\int_t^\alpha \omega(u)du \right) \left(\frac{a+b}{2} - t \right) dt + \int_\alpha^x \left(\int_\alpha^t \omega(u)du \right) \left(\frac{a+b}{2} - t \right) dt \\
&+ \int_x^{\frac{a+b}{2}} \left(\int_t^{\frac{\alpha+\beta}{2}} \omega(u)du \right) \left(t - \frac{a+b}{2} \right) dt + \int_{\frac{a+b}{2}}^{a+b-x} \left(\int_{\frac{\alpha+\beta}{2}}^t \omega(u)du \right) \\
&\times \left(t - \frac{a+b}{2} \right) dt + \int_{a+b-x}^\beta \left(\int_t^\beta \omega(u)du \right) \left(t - \frac{a+b}{2} \right) dt \\
&+ \int_\beta^b \left(\int_\beta^t \omega(u)du \right) \left(t - \frac{a+b}{2} \right) dt.
\end{aligned}$$

Using integration by parts and after some simplification we finally get

$$\begin{aligned}
I = & \left[\int_{\alpha}^{\beta} \omega(t) dt \left(\frac{(a+b)x}{2} - \frac{x^2}{2} \right) - \int_{\alpha}^{\beta} \left(\frac{(a+b)t}{2} - \frac{t^2}{2} \right) \omega(t) dt \right. \\
& + \int_{\alpha}^a \left(\frac{(a+b)t}{2} - \frac{t^2}{2} \right) \omega(t) dt + \int_b^{\beta} \left(\frac{(a+b)t}{2} - \frac{t^2}{2} \right) \omega(t) dt \\
& + 2 \int_{a+b-x}^{\frac{a+b}{2}} \left(\frac{(a+b)x}{2} - \frac{x^2}{2} \right) \omega(t) dt - 2 \int_{a+b-x}^{\frac{a+b}{2}} \left(\frac{(a+b)t}{2} - \frac{t^2}{2} \right) \omega(t) dt \\
& \left. + \left(\int_{\alpha}^a \omega(t) dt + \int_b^{\beta} \omega(t) dt \right) \frac{ab}{2} \right] \quad (2.8)
\end{aligned}$$

$\forall x \in [\alpha, \beta]$.

After further simplification of (2.8) and using inequality (2.6) we get our required result. \square

Special Case 1. If we put $\omega(t) \equiv \frac{1}{b-a}$ in (2.1), then we get

$$\begin{aligned}
& \left| \int_a^b g(t) dt - \frac{B}{2} \left[(2+\lambda)(g(a) + g(b)) - \lambda(g'(a) + g'(b)) \right. \right. \\
& \quad \left. \left. - (1-\lambda)(g(x) - g'(x)(x-A) + g(a+b-x) + g'(a+b-x)(x-A)) \right] \right| \\
& \leq \frac{\|g''\|_{\infty}}{4} \left[\int_{\alpha}^{\beta} (x-t) [(a+b-x) - t] dt \right. \\
& \quad + 2 \int_{a+b-x}^A (x-t) [(a+b-x) - t] dt \\
& \quad \left. + \int_a^{\alpha} (t-a)(t-b) dt + \int_{\beta}^b (t-a)(t-b) dt \right] \quad (2.9)
\end{aligned}$$

Remark 2.2. If we put $\lambda = 0$, then $\alpha = a$ and $\beta = b$ in (2.1), then we get:

$$\begin{aligned}
& \left| \int_a^b g(t) \omega(t) dt - B(g(a)\omega(a) + g(b)\omega(b)) \right. \\
& \quad - \frac{1}{2} \left[\int_a^A \omega(t) dt (g(x) - g'(x)(x-A)) \right. \\
& \quad + \int_A^b \omega(t) dt (g(a+b-x) - g'(a+b-x)(x-A)) \\
& \quad \left. \left. - \int_a^b g(t)(t-A) \omega'(t) dt \right] \right| \\
& \leq \frac{\|g''\|_{\infty}}{4} \left[\int_a^b (x-t) [(a+b-x) - t] w(t) dt \right. \\
& \quad \left. + 2 \int_{a+b-x}^A (x-A) [(a+b-x) - t] w(t) dt \right] \quad (2.10)
\end{aligned}$$

Remark 2.3. If we put $x = \frac{a+b}{2}$ in 2.10, then we get

$$\begin{aligned} & \left| \int_a^b w(t)g(t)dt - B(g(a)w(a) + g(b)w(b)) \right. \\ & \quad \left. - \frac{1}{2} \left[g\left(\frac{a+b}{2}\right) - \int_a^b g(t)(t-a)w'(t)dt \right] \right| \\ & \leq \frac{\|g''\|_\infty}{4} \int_a^b \left(t - \frac{a+b}{2}\right)^2 w(t)dt. \end{aligned} \quad (2.11)$$

Remark 2.4. If we put $w(t) = \frac{1}{b-a}$ in (2.11), then we get

$$\begin{aligned} & \left| \int_a^b g(t)dt - B(g(a) + g(b)) - \frac{(b-a)}{2} g\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{\|g''\|_\infty}{4} \int_a^b \left(t - \frac{a+b}{2}\right)^2 dt. \end{aligned} \quad (2.12)$$

If we put the midpoint $x = \frac{a+b}{2}$ in (2.1), then we get the better estimate result from midpoint, so from inequality in main theorem, we have

$$\begin{aligned} & \left| \int_a^b g(t)\omega(t)dt - B(g(a)\omega(a) + g(b)\omega(b)) \right. \\ & \quad - \frac{1}{2} \left[g(b) \int_\beta^b \omega(t)dt - g(a) \int_\alpha^a \omega(t)dt \right. \\ & \quad \left. - B\left(g'(a) \int_\alpha^a \omega(t)dt - g'(b) \int_\beta^b \omega(t)dt\right) \right. \\ & \quad \left. + g\left(\frac{a+b}{2}\right) \int_\alpha^\beta \omega(t)dt - \int_a^b g(t)(t-A)\omega'(t)dt \right] \right| \\ & \leq \frac{\|g''\|_\infty}{4} \left[\int_\alpha^\beta (A-t)^2 \omega(t)dt + \int_a^\alpha (t-a)(t-b)\omega(t)dt \right. \\ & \quad \left. + \int_\beta^b (t-a)(t-b)\omega(t)dt \right] \end{aligned} \quad (2.13)$$

Special Case 2. If we put $\omega(t) \equiv \frac{1}{b-a}$ in (2.10), then we get the midpoint rule which gives us the better estimate.

$$\begin{aligned} & \left| \int_a^b g(t) dt - \frac{B}{2} [(2g(a) + g(b)) + g(a+b-x) - g'(a+b-x)(x-A)] \right| \\ & \leq \frac{\|g''\|_\infty}{4(b-a)} \left[\int_a^b (x-t)[(a+b-x)-t] dt \right. \\ & \quad \left. + 2 \int_{a+b-x}^A (x-A)[(a+b-x)-t] dt \right] \end{aligned}$$

Remark 2.5. Let $\lambda = 0$, then $\alpha = a$ and $\beta = b$ in (2.11), we get the inequality

$$\begin{aligned} & \left| \int_a^b g(t)\omega(t) dt - B(g(a)\omega(a) + g(b)\omega(b)) - \frac{1}{2} \int_a^b g(t)(t-A)\omega'(t) dt \right| \\ & \leq \frac{\|g''\|_\infty}{4} \left[\int_a^b (A-t)^2 \omega(t) dt \right] \end{aligned} \quad (2.14)$$

Special Case 4. If we put $\omega(t) \equiv \frac{1}{b-a}$ in (2.14), then we get the midpoint rule which gives us the better estimate.

$$\left| \int_a^b g(t) dt - B(g(a) + g(b)) \right| \leq \frac{\|g''\|_\infty}{48} (b-a)^3.$$

Remark 2.6. Let $\lambda = 1$, then $\alpha = \beta = \frac{a+b}{2}$ in (2.11), we get the inequality

$$\begin{aligned} & \left| \int_a^b g(t)\omega(t) dt - B(g(a)\omega(a) + g(b)\omega(b)) \right. \\ & \quad - \frac{1}{2} \left[g(b) \int_A^b \omega(t) dt - g(a) \int_A^a \omega(t) dt \right. \\ & \quad \left. - B \left(g'(a) \int_A^a \omega(t) dt - g'(b) \int_A^b \omega(t) dt \right) \right. \\ & \quad \left. \left. - \int_a^b g(t)(t-A)\omega'(t) dt \right] \right| \\ & \leq \frac{\|g''\|_\infty}{4} \left[\int_a^b (t-a)(t-b)\omega(t) dt \right] \end{aligned} \quad (2.15)$$

Special Case 5. If we put $\omega(t) \equiv \frac{1}{b-a}$ in (2.15), then we get perturbed trapezoid inequality which gives us the better estimate for $\|\cdot\|_\infty$ norm.

$$\begin{aligned} & \left| \int_a^b g(t)\omega(t) dt - \frac{3B}{2}(g(a) + g(b)) + \frac{B^2}{2}(g'(a) + g'(b)) \right| \\ & \leq \frac{\|g''\|_\infty(a-b)^3}{24}. \end{aligned}$$

3. APPLICATIONS IN NUMERICAL INTEGRATION

Let $I_n : a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ be a division of the interval $[a, b]$, $h_k = x_{k+1} - x_k$, $\alpha_k \leq \xi_k \leq \beta_k$, where $\alpha_k = x_k + \lambda \frac{h_k}{2}$, $\beta_k = x_k - \lambda \frac{h_k}{2}$, $k \in \{0, \dots, n-1\}$, also $A_k = \frac{x_k + x_{k+1}}{2}$, $B_k = \frac{h_k}{2}$.

Consider the general quadrature formula where

$$\begin{aligned}
Q(g, g'', I_n) = & \frac{1}{2} \sum_{i=0}^{n-1} [2B_k(g(x_k)w(x_k) + g(x_{k+1})w(x_{k+1})) \\
& + g(x_{k+1}) \int_{\beta_k}^{x_{k+1}} w(t)dt - g(x_k) \int_{\beta_k}^{x_k} w(t)dt \\
& - B_k \left(g'(x_k) \int_{\alpha_k}^{x_k} w(t)dt - g'(x_{k+1}) \int_{\beta_k}^{x_{k+1}} w(t)dt \right) \\
& + \int_{\alpha_k}^{A_k} w(t)dt(g(x_k) - g'(x_k)(\xi_k - A_k)) \\
& + \int_{A_k}^{\beta_k} w(t)dt(g(x_k + x_{k+1} - \xi_k) - g'(x_k + x_{k+1} - \xi_k)(\xi_k - A_k)) \\
& - \int_{x_k}^{x_{k+1}} (t - A_k)w'(t)dt]
\end{aligned} \tag{3.1}$$

Theorem 3.1. *Under the assumptions of Proposition 1.5, we have*

$$\int_a^b g(t)\omega(t)dt = Q(g, g'', I_n) + R(g, g'', I_n)$$

where $Q(g, g'', I_n)$ is defined in 3.1 and the remainder satisfies the estimates

$$\begin{aligned}
& |R(g, g'', I_n)| \\
& \leq \frac{\|g''\|_\infty}{4} \sum_{k=0}^{n-1} \left[\int_{\alpha_k}^{\beta_k} (\xi_k - t)((x_k + x_{k+1} - \xi) - t)w(t)dt \right. \\
& + 2 \int_{x_k}^{\alpha_k} (t - \xi_{k+1})((x_k + x_{k+1} - \xi) - t)w(t)dt \\
& \left. + \int_{x_k}^{\alpha_k} (t - \xi_k)(t - \xi_{k+1})w(t)dt + \int_{\beta_k}^{x_{k+1}} (t - \xi_k)(t - \xi_{k+1})w(t)dt \right]
\end{aligned} \tag{3.2}$$

Proof. We may apply inequality (2.1) on $[x_k, x_{k+1}]$,

$$\begin{aligned}
R(g, g'', I_n) &= \int_a^b g(t) \omega(t) dt - \frac{1}{2} [2B_k(g(x_k)w(x_k) + g(x_{k+1})w(x_{k+1})) \\
&+ g(x_{k+1}) \int_{\beta_k}^{x_{k+1}} w(t) dt - g(x_k) \int_{\beta_k}^{x_k} w(t) dt \\
&- B_k \left(g'(x_k) \int_{\alpha_k}^{x_k} w(t) dt - g'(x_{k+1}) \int_{\beta_k}^{x_{k+1}} w(t) dt \right) \\
&+ \int_{\alpha_k}^{A_k} w(t) dt (g(x_k) - g'(x_k)(\xi_k - A_k)) \\
&+ \int_{A_k}^{\beta} w(t) dt (g(x_k + x_{k+1} - \xi_k) - g'(x_k + x_{k+1} - \xi_k)(\xi_k - A_k)) \\
&- \int_{x_k}^{x_{k+1}} (t - A_k) w'(t) dt \Big].
\end{aligned}$$

Summing the above inequality over k from 0 to $n-1$, we get

$$\begin{aligned}
R(g, g'', I_n) &= \sum_{i=0}^{n-1} \int_{x_k}^{x_{k+1}} f(t) dt - \frac{1}{2} \sum_{i=0}^{n-1} [2B_k(g(x_k)w(x_k) + g(x_{k+1})w(x_{k+1})) \\
&+ g(x_{k+1}) \int_{\beta_k}^{x_{k+1}} w(t) dt - g(x_k) \int_{\beta_k}^{x_k} w(t) dt \\
&- B_k \left(g'(x_k) \int_{\alpha_k}^{x_k} w(t) dt - g'(x_{k+1}) \int_{\beta_k}^{x_{k+1}} w(t) dt \right) \\
&+ \int_{\alpha_k}^{A_k} w(t) dt (g(x_k) - g'(x_k)(\xi_k - A_k)) \\
&+ \int_{A_k}^{\beta} w(t) dt (g(x_k + x_{k+1} - \xi_k) - g'(x_k + x_{k+1} - \xi_k)(\xi_k - A_k)) \\
&- \int_{x_k}^{x_{k+1}} (t - A_k) w'(t) dt \Big]
\end{aligned}$$

According to (2.1), we have

$$\begin{aligned}
&|R(g, g'', I_n)| \\
&\leq \frac{\|g''\|_{\infty}}{4} \sum_{k=0}^{n-1} \left[\int_{\alpha_k}^{\beta_k} (\xi_k - t)((x_k + x_{k+1} - \xi) - t) w(t) dt \right. \\
&+ 2 \int_{x_k}^{\alpha_k} (t - \xi_{k+1})((x_k + x_{k+1} - \xi) - t) w(t) dt + \int_{x_k}^{\alpha_k} (t - \xi_k)(t - \xi_{k+1}) w(t) dt \\
&\left. + \int_{\beta_k}^{x_{k+1}} (t - \xi_k)(t - \xi_{k+1}) w(t) dt \right]
\end{aligned}$$

□

4. APPLICATION TO PROBABILITY DENSITY FUNCTION

From [8], suppose X be a continuous random variable with function of probability density: $g : [a, b] \rightarrow \mathbb{R}_+$ and the cumulative distribution function $G : [a, b] \rightarrow [0, 1]$, i.e.,

$$G(x) = \int_a^x g(t)dt, \quad x \in [\alpha, \beta] \subset [a, b]$$

and $E_\omega(X) = \int_a^b tg(t)\omega(t)dt,$

is the weighted expectation of the random variable X on the interval $[a, b]$. Now we state main theorem of this section.

Theorem 4.1. *Under the assumption of Proposition 1.5 we have*

$$\begin{aligned} & \left| b\omega(b) - E_w(x) - \int_a^b \omega'(t)tG(t)dt \right. \\ & - \frac{1}{2} \left[2B\omega(b) + \int_\beta^b \omega(t)dt + (G(x) - g(x)(x - A)) \int_\alpha^A \omega(t)dt \right. \\ & + (G(a + b - x) + G'(a + b - x)(x - A)) \int_A^\beta \omega(t)dt \\ & \left. \left. - \int_a^b G(t)(t - A)\omega'(t)dt \right] \right| \\ & \leq \frac{\|g'\|_\infty}{4} \left[\int_\alpha^\beta (x - t) \left[(a + b - x) - t \right] \omega(t)dt \right. \\ & + 2 \int_{a+b-x}^A (x - t) \left[(a + b - x) - t \right] \omega(t)dt \\ & \left. + \int_a^\alpha (t - a)(t - b)\omega(t)dt + \int_\beta^b (t - a)(t - b)\omega(t)dt \right] \end{aligned} \quad (4.1)$$

for all $x \in [a, b]$.

Proof. Put $g = G$ in (2.1) we get (4.1) by simply using following identities

$$\int_a^b G(t)\omega(t)dt = b\omega(b) - E_\omega(X) - \int_a^b tG(t)\omega'(t)dt$$

□

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