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SOME NEW REFINEMENTS OF HARDY-TYPE INEQUALITIES

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Abstract. We obtain some further refinements of Hardy-type inequalities via superqudraticity technique. Our results both unify and further generalize several results on refinements of Hardy-type inequalities in the literature.

1. Introduction

The classical Hardy integral inequality states: if $p > 1$ and f is a nonnegative integrable function on $(0, \infty)$, then

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)dt \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^\infty f^p(x)dx. \tag{1.1}$$

Inequality (1.1) was announced in a note published in 1920 by Hardy [2] (see also [5]) and later proved in 1925 by Hardy in his famous paper [3]. In 1928, Hardy himself (see [4]) obtained the the first weighted generalization of (1.1) as follows:

$$\int_0^\infty \left(\int_0^x f(t)dt \right)^p x^{-k} dx \leq \left(\frac{p}{k-1} \right)^p \int_0^\infty f^p(x) x^{p-k} dx, \tag{1.2}$$

for $p > 1, k > 1$ and $f \in L_p(0, \infty)$. After that it has an almost unbelievable development of these inequalities to what today is called Hardy-type inequalities. See e.g. the books [7] and [8] and the references cited therein.

For example, in 1971 Shum [12] obtained the first refinement of (1.2). Furthermore, Imoru [6] generalized Shum's results via a convexity argument. In 2008, Persson et al. [11] generalized Shum's result by presenting an elementary proof using another convexity approach. In 2008, Oguntuase et al. [9] using mainly the concept of superquadratic or subquadratic functions (introduced by Abramovich et al. [1]) and the corresponding Jensen type inequality, proved some results that showed that it is possible to include another refinement term to the left hand side of the inequality (1.2) completely different from that of the Shum result. In particular, in this case the crucial "breaking point" was $p = 2$ (and not $p = 1$ as in the classical

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case). Specifically, the following inequality was obtained:

$$\begin{aligned}
&\int_0^b \frac{1}{x^k}\left(\int_0^x f(t)dt\right)^p dx\\
&+\frac{k-1}{p}\int_0^b\int_t^b\left(\left|\left(\frac{p}{k-1}\right)\left(\frac{t}{x}\right)^{1-\frac{k-1}{p}} f(t)-\frac{1}{x}\int_0^x f(t)dt\right|\right)^p x^{p-k-\frac{k-1}{p}}dxt^{\frac{k-1}{p}-1}dt\\
&\leq\left(\frac{p}{k-1}\right)^p\int_0^b\left[1-\left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right]x^{p-k}f^p(x)dx,
\end{aligned} \tag{1.3}$$

where $1<k<p-1, p\geq 2$ and $0<b\leq\infty$. In a recent paper, Oguntuase et al. [10] proved the following refined Hardy-type inequality: Let $g(x)>0,\ 0<x\leq\infty, 0<a<1$ and let the function f be a non-negative measurable function such that $f\in L^p(0,\infty)$. If $p\geq 2,\ q>p-a(p-1),\ \frac{x}{g(x)}$ is non-increasing and $F(x)=\int_0^x f(t)dt$, then

$$\begin{aligned}
&\int_0^\infty \frac{F^p(x)}{g^q(x)}dx\\
&+(1-a)\int_0^\infty\int_t^\infty\left(\left|\left(\frac{1}{1-a}\right)\left(\frac{t}{x}\right)^a f(t)-\frac{1}{x}\int_0^x f(t)dt\right|\right)^p\frac{x^{p+a-1}}{g^q(x)}dxt^{-a}dt\\
&\leq\frac{1}{[(a-1)(p-1)+q-1](1-a)^{p-1}}\int_0^\infty\frac{(xf(x)^p)}{g^q(x)}dx.
\end{aligned} \tag{1.4}$$

If $\frac{x}{g(x)}$ is non-decreasing and $1<p\leq 2$, then (1.4) holds in the reversed direction. Moreover, the authors pointed out that the following inequality:

$$\begin{aligned}
&\int_0^\infty [g(x)]^{-k}\left(\int_0^x f(t)dt\right)^p dx\\
&+\frac{k-1}{p}\int_0^\infty\int_t^\infty\left(\left|\left(\frac{p}{k-1}\right)\left(\frac{t}{x}\right)^{1-\frac{k-1}{p}} f(t)-\frac{1}{x}\int_0^x f(t)dt\right|\right)^p \frac{x^{p-\frac{k-1}{p}}}{g^k(x)}dxt^{\frac{k-1}{p}-1}dt\\
&\leq\left(\frac{p}{k-1}\right)^p\int_0^\infty x^p[g(x)]^{-k}f^p(x)dx,
\end{aligned} \tag{1.5}$$

is a special case of (1.4) and that the inequalities (1.4) and (1.3) coincide when $b=\infty$ in (1.3) and $g(x)=x,\ x\in(0,\infty)$ in (1.5). However, finding a corresponding generalization of (1.3) is still an open problem, since (1.5) cannot be regarded as such a generalization.

In this paper, we obtain some new refinements of Hardy-type inequalities via superquadraticity. In particular, our results both generalize and unify several results concerning refined Hardy-type inequalities in the literature including (1.3) to (1.5).

In Section 2, we give some preliminaries including some basic results on superquadraticity. In Section 3, we derive and prove the announced new refined Hardy-type inequalities, which in particular, both unify and generalize the recent results of Oguntuase et al. [10] and the inequalities (1.3) and (1.4) (see Theorem 3.3 and Theorem 3.5).

2. Definition including some basic results on superquadraticity

We present here the Definition and some basic results on superquadraticity needed to understand and prove our main results in the next Section.

Definition 2.1 ([1]). *A function $\varphi : [0,\infty) \to \mathbb{R}$ is said to be superquadratic provided for each $x \geq 0$ there exists a constant $C_\varphi \in \mathbb{R}$ such that*

$$\varphi(y) - \varphi(x) - C_x(y-x) - \varphi(|y-x|) \geq 0 \tag{2.1}$$

for all $y \in [0,\infty)$. φ is subquadratic if $-\varphi$ is superquadratic.

Lemma 2.2 ([1]). *Let $\varphi(x)$ be a superquadratic function with C_φ as in Definition 2.1. Then*

(1) $\varphi(0) \leq 0$.
(2) *If $\varphi(0) = \varphi'(0) = 0$, then $C_x = \varphi'(x)$ whenever φ is differentiable at $x > 0$.*
(3) *If $\varphi(x) \geq 0$ for $x \in [0,\infty)$, then φ is convex and $\varphi(0) = \varphi'(0) = 0$.*

The next result gives a refined Jensen's inequality for superquadratic and subquadratic functions.

Theorem 2.3 ([1]). *Let (Ω, Σ, μ) be a probability measure space. Then the inequality*

$$\varphi\left(\int_\Omega f(x)d\mu(x)\right) \leq \int_\Omega \left[\varphi\left(f(x)\right) - \varphi\left(\left|f(x) - \int_\Omega f(y)d\mu(y)\right|\right)\right] d\mu(x) \tag{2.2}$$

holds for all probability measures μ and all non-negative integrable functions f if and only if $\varphi : [0,\infty) \to \mathbb{R}$ is superquadratic. Moreover, (2.2) holds in the reversed direction if and only if φ is subquadratic.

Remark. *By setting $\varphi(u) = u^p, p \geq 2$, in Theorem 2.3, we obtain*

$$\left(\int_\Omega f(x)d\mu(x)\right)^p + \int_\Omega \left(\left|f(x) - \int_\Omega f(x)d\mu(x)\right|\right)^p d\mu(x) \leq \int_\Omega f^p(x)d\mu(x), \tag{2.3}$$

while the sign of inequality (2.3) is reversed if $1 < p \leq 2$.

3. Main results

The following lemma will be needed in the proof of our main results.

Lemma 3.1 ([10]). *Let $p > 1, 0 < a < 1$ and $f \geq 0$ be a measurable function. Then, for $p \geq 2$ and $l > 0$,*

$$\begin{aligned} &\left(\int_0^l f(t)dt\right)^p \\ &+ (1-a)l^{a-1+p}\int_0^l t^{-a}\left(\left|\left(\frac{1}{1-a}\right)\left(\frac{t}{l}\right)^a f(t) - \frac{1}{l}\int_0^l f(t)dt\right|\right)^p dt \\ &\leq \frac{l^{(p-1)(1-a)}}{(1-a)^{p-1}}\int_0^l t^{a(p-1)} f^p(t)dt. \end{aligned} \tag{3.1}$$

If $1 < p \leq 2$, then the sign of (3.1) is reversed.

Before we state our main results in this section, first we state and prove the following useful Lemma, which will be needed in the proof of Theorem 3.5 and also of independent interest.

Lemma 3.2. *Let $p > 1, a > 1$, and $f \geq 0$ be a measurable function. Then, for $p \geq 2$ and $0 < l < b \leq \infty$,*

$$\begin{aligned}
&\left(\int_l^b f(t)dt\right)^p \\
&+ \frac{(a-1)l^{a-1+p}}{\left(1-\left(\frac{l}{b}\right)^{a-1}\right)} \int_l^b t^{-a} \left(\left|\frac{1-\left(\frac{l}{b}\right)^{a-1}}{a-1}\left(\frac{t}{l}\right)^a f(t) - \frac{1}{l}\int_l^b f(t)dt\right|\right)^p dt \qquad (3.2)\\
\leq &\frac{l^{(p-1)(1-a)}}{(a-1)^{p-1}}\left(1-\left(\frac{l}{b}\right)^{a-1}\right)^{p-1}\int_l^b t^{a(p-1)} f^p(t)dt.
\end{aligned}$$

If $1 < p \leq 2$, then the inequality (3.2) holds in the reversed direction.

Proof. Let $p \geq 2$ and assume that $b < \infty$. Define the probability measure $d\mu$ on (l, b) by

$$d\mu = \frac{(a-1)t^{-a}l^{a-1}}{1-\left(\frac{l}{b}\right)^{a-1}}dt.$$

Then by using Theorem 2.3 (c.f. Remark 2) we obtain that

$$\begin{aligned}
&\left(\int_l^b f(t)dt\right)^p = \left(\int_l^b l^{1-a}t^a \frac{\left(1-\left(\frac{l}{b}\right)^{a-1}\right)}{a-1} f(t)d\mu\right)^p \\
&= l^{p(1-a)}\frac{\left(1-\left(\frac{l}{b}\right)^{a-1}\right)^p}{(a-1)^p}\left(\int_l^b t^a f(t)d\mu\right)^p \\
&\leq l^{p(1-a)}\frac{\left(1-\left(\frac{l}{b}\right)^{a-1}\right)^p}{(a-1)^{p-1}} \\
&\times \left[\int_l^b (t^a f(t))^p\, d\mu - \int_l^b \left(\left|t^a f(t) - \int_l^b t^a f(t)d\mu\right|\right)^p d\mu\right] \qquad (3.3)\\
&\leq l^{(p-1)(1-a)}\frac{\left(1-\left(\frac{l}{b}\right)^{a-1}\right)^{p-1}}{(a-1)^{p-1}} \\
&\times \left[\int_l^b t^{a(p-1)} f^p(t)dt - \int_l^b t^{-a}\left(\left|t^a f(t) - \frac{(a-1)l^{a-1}}{1-\left(\frac{l}{b}\right)^{a-1}}\int_l^b f(t)dt\right|\right)^p dt\right] \\
&\leq l^{(p-1)(1-a)}\frac{\left(1-\left(\frac{l}{b}\right)^{a-1}\right)^p}{(a-1)^{p-1}}\int_l^b t^{a(p-1)} f^p(t)dt \\
&- \frac{(a-1)l^{a-1+p}}{\left(1-\left(\frac{l}{b}\right)^{a-1}\right)}\int_l^b t^{-a}\left(\left|\frac{1-\left(\frac{l}{b}\right)^{a-1}}{a-1}\left(\frac{t}{l}\right)^a f(t) - \frac{1}{l}\int_l^b f(t)dt\right|\right)^p dt.
\end{aligned}$$

The proof of the case $1 < p \leq 2$ is similar to the proof above except that the inequalities signs are reversed. □

Remark. *By setting $b = \infty$ in Lemma 3.2, we obtain Lemma 3.6 in* [10].

Our first main result is the following generalization of (1.4):

Theorem 3.3. *(a). Let $f(x) \geq 0$ be a measurable function, $g(x) > 0, 0 < x < b \leq \infty, 0 < a < 1, p \geq 2$ and $q > p - a(p-1)$. If $\frac{x}{g(x)}$ is non-increasing and $F(x) := \int_0^x f(t)dt$, then*

$$\begin{aligned}&\int_0^b \frac{F^p(x)}{g^q(x)}dx \\ &+ (1-a)\int_0^b \int_t^b \left(\left|\left(\frac{1}{1-a}\right)\left(\frac{t}{x}\right)^a f(t) - \frac{1}{x}\int_0^x f(t)dt\right|\right)^p \frac{x^{a-1+p}}{g^q(x)} dx t^{-a} dt \\ &\leq \frac{1}{[(a-1)(p-1)+q-1](1-a)^{p-1}} \int_0^b \frac{(xf(x))^p}{g^q(x)}\left[1 - \left(\frac{x}{b}\right)^{q-1+(a-1(p-1))}\right] dt.\end{aligned} \tag{3.4}$$

(b). If $\frac{x}{g(x)}$ is non-decreasing and $1 < p \leq 2$, then (3.4) holds in the reversed direction.

Remark. *In particular, Theorem 3.3 shows that (1.4) holds also if $\int_0^\infty$ and $\int_t^\infty$ in (1.4) are replaced by $\int_0^b$ and $\int_t^b$, respectively, for any b, $0 < b < \infty$. However, in this case, it can be replaced by the strictly better inequality (3.4).*

Proof. (a) By using Lemma 3.1 with $l = x$ and Fubini's theorem we find that

$$\begin{aligned}&\int_0^b \frac{F^p(x)}{g^q(x)}dx = \int_0^b g^{-q}(x)\left(\int_0^x f(t)dt\right)^p dx \\ &\leq \int_0^b g^{-q}(x)\frac{x^{(p-1)(1-a)}}{(1-a)^{p-1}}\int_0^x t^{a(p-1)} f^p(t) dt dx \\ &- \int_0^b g^{-q}(x)\frac{x^{a-1+p}}{(1-a)^{-1}}\int_0^x t^a \left(\left|\left(\frac{1}{1-a}\right)\left(\frac{t}{x}\right)^a f(t) - \frac{1}{x}\int_0^x f(t)dt\right|\right)^p dt dx \\ &= \frac{1}{(1-a)^{p-1}}\int_0^b t^{a(p-1)} f^p(t)\int_t^b x^{(1-a)(p-1)} g^{-q}(x) dx dt \\ &- (1-a)\int_0^b \int_t^b \frac{x^{a-1+p}}{g^q(x)}\left(\left|\left(\frac{1}{1-a}\right)\left(\frac{t}{x}\right)^a f(t) - \frac{1}{x}\int_0^x f(t)dt\right|\right) dx t^{-a} dt \\ &\leq \frac{1}{(1-a)^{p-1}}\int_0^b t^{a(p-1)} f^p(t)\left(\frac{t}{g(t)}\right)^q \int_t^b x^{(1-a)(p-1)-q} dx dt \\ &- (1-a)\int_0^b \int_t^b \frac{x^{a-1+p}}{g^q(x)}\left(\left|\left(\frac{1}{1-a}\right)\left(\frac{t}{x}\right)^a f(t) - \frac{1}{x}\int_0^x f(t)dt\right|\right) dx t^{-a} dt \\ &= \frac{1}{[(a-1)(p-1)+q-1](1-a)^{p-1}}\int_0^b \frac{(tf(t)^p)}{g^q(t)}\left[1 - \left(\frac{t}{b}\right)^{q-1+(a-1)(p-1)}\right] dt \\ &- (1-a)\int_0^b \int_t^b t^{-a}\left(\left|\left(\frac{1}{1-a}\right)\left(\frac{t}{x}\right)^a f(t) - \frac{1}{x}\int_0^x f(t)dt\right|\right)^p \frac{x^{p+a-1}}{g^q(x)} dx t^{-a} dt.\end{aligned} \tag{3.5}$$

(b) The proof of (b) is similar to the proof of (a) except that the signs of the inequalities are reversed. □

Example 3.4. *Assume that $p \geq 2, q = k, a = 1 - \frac{k-1}{p}$ and $1 < k < p-1$. Then, for $0 < b \leq \infty$ inequality (3.4) yields*

$$\begin{aligned}
&\int_0^b g(x)^{-k}\left(\int_0^x f(t)dt\right)^p dx \\
&+ \frac{k-1}{p}\int_0^b\int_t^b \left(\left|\left(\frac{p}{k-1}\right)\left(\frac{t}{x}\right)^{1-\frac{k-1}{p}} f(t) - \frac{1}{x}\int_0^x f(t)dt\right|\right)^p \frac{x^{p-\frac{k-1}{p}}}{g^k(x)}dxt^{\frac{k-1}{p}-1}dt \\
&\leq \left(\frac{p}{k-1}\right)^p \int_0^b x^p g(x)^{-k} f^p(x)\left[1-\left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right]dx,
\end{aligned} \tag{3.6}$$

Remark. *By setting $g(x) = x$ in inequality (3.6), then our result coincide with inequality (1.3). Hence, Theorem 3.3 gives a generalization of inequality (1.3), which is a refinement of inequality (1.2).*

Remark. *Observe that inequality (3.6) coincides with inequality (3.3) obtained by Oguntuase et al. in [10], when $b = \infty$.*

Finally, we consider the case $a > 1$.

Theorem 3.5. *(a) Let $0 < x \leq b < \infty, a > 1, p \geq 2$ and $f(x) \geq 0$ be a measurable function, $g(x) > 0$ and $q < p - a(p-1)$. If $\frac{x}{g(x)}$ is non-decreasing and $F_1(x) = \int_x^\infty f(t)dt$, then*

$$\begin{aligned}
&\int_0^b \frac{F_1^p(x)}{g^q(x)}dx \\
&+ (a-1)\int_0^b\int_0^t \left(\left|\left(\frac{\left(1-\left(\frac{x}{b}\right)^{a-1}\right)}{a-1}\right)\left(\frac{t}{x}\right)^a f(t) - \frac{1}{x}\int_x^b f(t)dt\right|\right)^p \\
&\times \frac{x^{p+a-1}}{g^q(x)\left(1-\left(\frac{x}{b}\right)^{a-1}\right)}dxt^{-a}dt \\
&\leq \frac{1}{\left[-q+1+(p-1)(1-a)\right](a-1)^{p-1}}\int_0^b \frac{(xf(x))^p}{g^q(x)}dx.
\end{aligned} \tag{3.7}$$

(b) If $\frac{x}{g(x)}$ is non-increasing and $1 < p \leq 2$, then (3.7) holds in the reversed direction.

Proof. (a) Let $p \geq 2$. By using Proposition 3.2 with $l = x$ and Fubini's theorem, we get

$$\begin{aligned}
&\int_0^b \frac{F_1^p(x)}{g^q(x)}dx = \int_0^b g^{-q}(x)\left(\int_x^b f(t)dt\right)^p dx \\
&\le \int_0^b g^{-q}(x)\frac{x^{(p-1)(1-a)}}{(a-1)^{p-1}}\left(1-\left(\frac{x}{b}\right)^{a-1}\right)^{p-1}\int_x^b t^{a(p-1)}f^p(t)dt \\
&-(a-1)\int_0^b \frac{x^{p+a-1}}{g^q(x)\left(1-\left(\frac{x}{b}\right)^{a-1}\right)} \\
&\times \int_x^b t^{-a}\left(\left|\left(\frac{1-\left(\frac{x}{b}\right)^{a-1}}{a-1}\right)\left(\frac{t}{x}\right)^a f(t)-\frac{1}{x}\int_x^b f(t)dt\right|\right)^p dtdx \qquad (3.8) \\
&= (a-1)^{1-p}\int_0^b t^{a(p-1)}f^p(t)\int_0^t x^{(1-a)(p-1)}g^{-q}(x)\left(1-\left(\frac{x}{b}\right)^{a-1}\right)^{p-1}dxdt \\
&-(a-1)\int_0^b\int_0^t\left(\left|\left(\frac{1-\left(\frac{x}{b}\right)^{a-1}}{a-1}\right)\left(\frac{t}{x}\right)^a f(t)-\frac{1}{x}\int_x^b f(t)dt\right|\right)^p \\
&\times \frac{x^{p+a-1}}{g^q(x)\left(1-\left(\frac{x}{b}\right)^{a-1}\right)}dxt^{-a}dt := I.
\end{aligned}$$

Moreover, the simple fact that both

$$\left(\frac{x}{g(x)}\right)^q \text{ and } \left(1-\left(\frac{x}{b}\right)^{a-1}\right)^{p-1}$$

are non-increasing on $(0,b)$ yields that

$$\begin{aligned}
I &\le (a-1)^{1-p}\int_0^b t^{a(p-1)}f^p(t)\left(\frac{t}{g(t)}\right)^q\int_0^t x^{(1-a)(p-1)-q}dt \\
&-(a-1)\int_0^b\int_0^t\left(\left|\left(\frac{1-\left(\frac{x}{b}\right)^{a-1}}{a-1}\right)\left(\frac{t}{x}\right)^a f(t)-\frac{1}{x}\int_x^b f(t)dt\right|\right)^p \\
&\times \frac{x^{p+a-1}}{g^q(x)\left(1-\left(\frac{x}{b}\right)^{a-1}\right)}dxt^{-a}dt \\
&= \frac{(a-1)^{1-p}}{[(p-1)(1-a)+1-q]}\int_0^b \frac{(tf(t))^p}{g^q(t)}dt \qquad (3.9) \\
&-(a-1)\int_0^b\int_0^t\left(\left|\left(\frac{\left(1-\left(\frac{x}{b}\right)^{a-1}\right)}{a-1}\right)\left(\frac{t}{x}\right)^a f(t)-\frac{1}{x}\int_x^b f(t)dt\right|\right)^p \\
&\times \frac{x^{p+a-1}}{g^q(x)\left(1-\left(\frac{x}{b}\right)^{a-1}\right)}dxt^{-a}dt.
\end{aligned}$$

By combining (3.8) and (3.9), the desired result follows.
(b) The proof of (b) is similar to the proof of (a) except that the signs of the inequalities are reversed. □

Example 3.6. *Suppose that $b = \infty$. Then, inequality (3.7) yields*

$$\int_0^\infty \frac{F^p(x)}{g^q(x)}dx + (a-1)\int_0^\infty \int_0^t \left(\left|\left(\frac{1}{a-1}\right)\left(\frac{t}{x}\right)^a f(t) - \frac{1}{x}\int_x^\infty f(t)dt\right|\right)^p \frac{x^{p+a-1}}{g^q(x)}dxt^{-a}dt \leq \frac{1}{\left[-q+1(p-1)(1-a)\right](a-1)^{p-1}}\int_0^\infty \frac{(xf(x))^p}{g^q(x)}dx. \tag{3.10}$$

Remark. *The inequality (3.10) above coincides with the inequality obtained in Theorem 3.7 in* [10]. *Hence, Theorem 3.5 provides a further generalization and unification of the results obtained by Oguntuase et al. in* [10].

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