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SOME APPROXIMATION RESULTS ON MODIFIED q-BERNSTEIN OPERATORS

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Abstract. The main purpose of this paper is to research the approximation features of modified new q-Bernstein operators. We examine the rate of convergence of the modified new q-Bernstein operators by means of moduli of continuity and also describe the Voronovskaja's type asymptotic formula for the related operators. Additionally, we give some numerical tables and graphs to compare the degree of convergence between modified new q-Bernstein operators, the modified Bernstein operators and the generalizations of the q-Bernstein operators.

1. Introduction

The first work on q-Bernstein Polynomials was written by Lupaş [10] in 1987. He explored the approximation and shape-preserving features of q-Bernstein Polynomials. After Lupaş, Phillips [20] introduced and explored in 1996 the most popular generalizations of the $q-$Bernstein polynomials. In this paper these generalizations will be shown in section 5 as $B_n^q(f;x)$. The implementations of the q-calculus of the approximation theory became very famous and inspired many researchers to explore various positive linear operators which are based on the q-calculus are researched. In the last two decades many modifications of Bernstein and other operators have been studied by different authors, for example: M. Mursaleen, F. Khan and A. Khan in ([15, 14]) have studied respectively the approximation properties for modified q-Bernstein-Kantorovich operators and King's type modified q-Bernstein-Kantorovich operators. More information for the developments on the study of other modifications of Bernstein and other operators are listed here for readers (see:[23, 18, 19, 13, 17, 12, 8, 5, 16, 4, 22, 1, 3, 9, 11, 21]). In 2012 Aydın Izgi defined in [8] a modified new Bernstein operator as below,

$$U_n(f;x) = \left(\frac{n}{n-2}\right)^n \sum_{k=0}^{n} f\left(\frac{(n-2)k+n}{n^2}\right)\binom{n}{k}\left(x-\frac{1}{n}\right)^k\left((1-\frac{1}{n})-x\right)^{n-k}$$

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where $f \in C[\frac{1}{[n]_q}, \frac{[n-1]_q}{[n]_q}]$ and $n \geq 3$. Inspired by this research, we introduce in this article $U_n(f;x)$ operators based the on q-calculus and investigate the approximation features of this new q-Bernstein operators. We describe the Voronovskaja's type result theorem and show some graphics and numerical results to compare the convergence of the modified q-Bernstein operators with the $U_n(f;x)$ and $B_n^q(f;x)$ operators. As an introduction to this study we give the definitions and symbols of the q-calculus for any constant real number $q > 0$ and positive integer n.

The quantity n depend on q-calculus is described as follows

$$[n]_q := \begin{cases} \frac{1-q^n}{1-q} & q \neq 1 \\ 1, & q = 0 \end{cases}$$

The factorial and binomial parameters which are depending on q-calculus described respectively as follows;

$$[n]_q! := \begin{cases} [n]_q [n-1]_q .. [1]_q, & n = 1, 2, .. \\ 1, & n = 0 \end{cases}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!}{[k]_q! \, [n-k]_q!}$$

As long as n and k are integers and provide $n \geq k \geq 0$, then we have the equality as below

$$[n]_q = [k]_q + q^k [n-k]_q \tag{1.1}$$

The q-analogue of $(x-r)^n$ is described as below

$$(x-r)_q^n = \begin{cases} 1, & n = 0 \\ (x-r)(x-qr)\ldots\left(x-q^{n-1}r\right), & n \geq 1. \end{cases}$$

We now construct for $q \in (0,1)$ and $n \geq 3$, the new modified of the q-Bernstein operators as below

$$U_n^q(f;x) := \left(\frac{[n]_q}{q\,[n-2]_q}\right)^n \sum_{k=0}^{n} f\left(\frac{[n]_q + [n-2]_q\,[k]_q}{[n]_q^2}\right) \mu_{n,k}^q \tag{1.2}$$

where $\mu_{n,k}^q = \left(x - \frac{1}{[n]_q}\right)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \prod_{s=0}^{n-k-1} \left(\frac{[n-1]_q}{[n]_q} - q^s x\right)$ and $f \in C[\frac{1}{[n]_q}, \frac{[n-1]_q}{[n]_q}]$.

In next sections; we give some lemmas, theorems and proofs and we end our study with some graphs and numerical tables.

2. Main Results

Lemma 2.1. *Let $r_i(v) = v^i, i = 0, 1, 2, 3, 4.$ then for $\forall\ x \in [\frac{1}{[n]_q}, \frac{[n-1]_q}{[n]_q}]$ and $n \geq 3$ the modified operators (1.2) are satisfied the following equalities:*

$$\begin{aligned}
i)U_n^q(r_0;x) &= 1\\
ii)U_n^q(r_1;x) &= x(\frac{1}{q}) + \frac{q-1}{q\left[n\right]_q}\\
iii)U_n^q(r_2;x) &= x^2\left(\frac{1}{q^2} - \frac{1}{q^2\left[n\right]_q}\right) + x\left(\frac{2\left[n\right]_q - 2\left[n-1\right]_q + \left[n-2\right]_q}{q\left[n\right]_q^2}\right)\\
&+ \frac{\left[n-1\right]_q - \left[n-2\right]_q + (2-q)\left[n\right]_q}{q\left[n\right]_q^3}\\
iv)U_n^q(r_3;x) &= x^3\left(1 - \frac{(q+2)\left[n\right]_q - q - 1}{\left[n\right]_q^2}\right) + x^2\left(\frac{(5-2q)\left[n\right]_q^2 + (6q-9)\left[n\right]_q - 4q}{q\left[n\right]_q^3}\right)\\
&+ x\left(\frac{3\left[n\right]_q^2 + \left[n-2\right]_q^2 + 3\left[n\right]_q\left[n-2\right]_q - 6\left[n\right]_q\left[n-1\right]_q + (q-4)\left[n-1\right]_q\left[n-2\right]_q}{q\left[n\right]_q^4}\right)\\
&+ \frac{(q-3)\left[n\right]_q^2 + \left[n-2\right]_q^2 + 3\left[n\right]_q\left[n-1\right]_q - 3\left[n\right]_q\left[n-2\right]_q + 2\left[n-1\right]_q\left[n-2\right]_q}{q\left[n\right]_q^5}\\
v)U_n^q(r_4;x) &= x^4\left(\frac{1}{q^4} - \frac{(q^2+2q+3) + (q^3+3q^2+4q+2)\left[n\right]_q - (q^3+q^2+q)\left[n\right]_q^2}{q^4\left[n\right]_q^3}\right)\\
&+ x^3\left(\frac{-4q^2\left[n-1\right]_q\left[n-2\right]_q\left[n-3\right]_q + 4\left[n\right]_q\left[n-1\right]_q\left[n-2\right]_q + (q^2+2q+3)\left[n-1\right]_q\left[n-2\right]_q^2}{\left[n\right]_q^4}\right)\\
&+ x^3\left(\frac{-4q^2\left[n-1\right]_q\left[n-2\right]_q\left[n-3\right]_q + 4\left[n\right]_q\left[n-1\right]_q\left[n-2\right]_q + (q^2+2q+3)\left[n-1\right]_q\left[n-2\right]_q^2}{\left[n\right]_q^4}\right)\\
&+ x^2\left(\frac{\frac{6}{q}\left[n\right]_q^2\left[n-1\right]_q + 8(\frac{1}{q}-1)\left[n\right]_q\left[n-1\right]_q\left[n-2\right]}{\left[n\right]_q^5}\right.\\
&\left.+\frac{6q^2\left[n-1\right]_q\left[n-2\right]_q\left[n-3\right]_q + (\frac{3}{q} - 3q^2 - 5q - 6)\left[n-1\right]_q\left[n-2\right]_q^2}{\left[n\right]_q^5}\right)\\
&+ x\left(\frac{4\left[n\right]_q^3 + \left[n-2\right]_q^3 + (\frac{1}{q})(-12\left[n\right]_q^2\left[n-1\right]_q + 6\left[n\right]_q^2\left[n-2\right]_q + 4\left[n\right]_q\left[n-2\right]_q^2}{\left[n\right]_q^6}\right.\\
&+ \frac{(3q^2+4q+3-\frac{3}{q})\left[n-1\right]_q\left[n-2\right]_q^2 - 12\left[n\right]_q\left[n-1\right]_q\left[n-2\right]_q}{\left[n\right]_q^6}\\
&\left.+\frac{-4q^2\left[n-1\right]_q\left[n-2\right]_q\left[n-3\right]_q}{\left[n\right]_q^6}\right)\\
&+ \frac{(1-\frac{4}{q})\left[n\right]_q^3 + \frac{6}{q}\left[n\right]_q^2\left[n-1\right]_q + \frac{8}{q}\left[n\right]_q\left[n-1\right]_q\left[n-2\right]_q - (q^2+3q-\frac{3}{q})\left[n-1\right]_q\left[n-2\right]_q^2}{\left[n\right]_q^7}\\
&+ \frac{\frac{\left[n-2\right]_q}{q}\left(6\left[n\right]_q^2 - \left[n-2\right]_q^2 + 4\left[n\right]_q\left[n-2\right]_q\right) + q^2\left[n-1\right]_q\left[n-2\right]_q\left[n-3\right]_q}{\left[n\right]_q^7}
\end{aligned}$$

Proof. We will give the proof of $i), ii), iii)$. Because the proofs $iv)$ and $v)$ follow the same reasoning they are not eleborated here.

$$
\begin{aligned}
i) U_n^q(r_0;x) &= \left(\frac{[n]_q}{q\,[n-2]_q}\right)^n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \left(x-\frac{1}{[n]_q}\right)^k \prod_{s=0}^{n-k-1}\left(\frac{[n-1]_q}{[n]_q}-q^s x\right) \\
&= \left(\frac{[n]_q}{q\,[n-2]_q}\right)^n \left(\frac{q\,[n-2]_q}{[n]_q}\right)^n = 1
\end{aligned}
$$

$$
\begin{aligned}
ii) U_n^q(r_1;x) &= \left(\frac{[n]_q}{q\,[n-2]_q}\right)^n \sum_{k=0}^{n} \frac{[n-2]_q\,[k]_q+[n]_q}{[n]_q^2} \begin{bmatrix} n \\ k \end{bmatrix}_q \left(x-\frac{1}{[n]_q}\right)^k \prod_{s=0}^{n-k-1}\left(\frac{[n-1]_q}{[n]_q}-q^s x\right) \\
&= \left(\frac{[n]_q}{q\,[n-2]_q}\right)^n \frac{[n-2]_q}{[n]_q} \sum_{k=0}^{n} \frac{[k]_q}{[n]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q \left(x-\frac{1}{[n]_q}\right)^k \prod_{s=0}^{n-k-1}\left(\frac{[n-1]_q}{[n]_q}-q^s x\right) + \\
&\left(\frac{[n]_q}{q\,[n-2]_q}\right)^n \frac{1}{[n]_q} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \left(x-\frac{1}{[n]_q}\right)^k \prod_{s=0}^{n-k-1}\left(\frac{[n-1]_q}{[n]_q}-q^s x\right) \\
&= \frac{1}{q}\left(\frac{[n]_q}{q\,[n-2]_q}\right)^{n-1} \sum_{k=1}^{n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \left(x-\frac{1}{[n]_q}\right)^k \prod_{s=0}^{n-k-1}\left(\frac{[n-1]_q}{[n]_q}-q^s x\right) + \frac{1}{[n]_q} \\
&= \frac{1}{q}\left(\frac{[n]_q}{q\,[n-2]_q}\right)^{n-1} \sum_{k=0}^{n} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \left(x-\frac{1}{[n]_q}\right)^{k+1} \prod_{s=0}^{n-k-2}\left(\frac{[n-1]_q}{[n]_q}-q^s x\right) + \frac{1}{[n]_q} \\
&= x\left(\frac{1}{q}\right) + \frac{q-1}{q\,[n]_q}
\end{aligned}
$$

$$
\begin{aligned}
iii) U_n^q(r_2;x) &= \left(\frac{[n]_q}{q\,[n-2]_q}\right)^n \sum_{k=0}^{n} \left(\frac{[n-2]_q\,[k]_q+[n]_q}{[n]_q^2}\right)^2 \begin{bmatrix} n \\ k \end{bmatrix}_q \left(x-\frac{1}{[n]_q}\right)^k \prod_{s=0}^{n-k-1}\left(\frac{[n-1]_q}{[n]_q}-q^s x\right) \\
&= \left(\frac{[n]_q}{q\,[n-2]_q}\right)^n \frac{[n-2]_q^2}{[n]_q^2} \sum_{k=0}^{n} \frac{[k]_q^2}{[n]_q^2} \begin{bmatrix} n \\ k \end{bmatrix}_q \left(x-\frac{1}{[n]_q}\right)^k \prod_{s=0}^{n-k-1}\left(\frac{[n-1]_q}{[n]_q}-q^s x\right) + \\
2&\left(\frac{[n]_q}{q\,[n-2]_q}\right)^n \frac{[n-2]_q}{[n]_q^2} \sum_{k=0}^{n} \frac{[k]_q}{[n]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q \left(x-\frac{1}{[n]_q}\right)^k \prod_{s=0}^{n-k-1}\left(\frac{[n-1]_q}{[n]_q}-q^s x\right) + \frac{1}{[n]_q^2} \\
&= \left(\frac{[n]_q}{q\,[n-2]_q}\right)^n \frac{[n-2]_q^2}{[n]_q^2} \sum_{k=1}^{n} \frac{[k]_q}{[n]_q} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \left(x-\frac{1}{[n]_q}\right)^k \prod_{s=0}^{n-k-1}\left(\frac{[n-1]_q}{[n]_q}-q^s x\right) + \\
&\frac{2}{q}\left(\frac{[n]_q}{q\,[n-2]_q}\right)^{n-1} \frac{1}{[n]_q} \left(\frac{q\,[n-2]_q}{[n]_q}\right)^{n-1} \left(x-\frac{1}{[n]_q}\right) + \frac{1}{[n]_q^2} \\
&= \left(\frac{[n]_q}{q\,[n-2]_q}\right)^n \frac{[n-2]_q^2}{[n]_q^2} \sum_{k=1}^{n} \frac{q\,[k-1]_q+1}{[n]_q} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \left(x-\frac{1}{[n]_q}\right)^k \prod_{s=0}^{n-k-1}\left(\frac{[n-1]_q}{[n]_q}-q^s x\right)
\end{aligned}
$$

$$+\frac{2}{q\,[n]_q}\left(x-\frac{1}{[n]_q}\right)+\frac{1}{[n]_q^2}$$

$$=\frac{1}{q^2}\left(\frac{[n]_q}{q\,[n-2]_q}\right)^{n-2}\frac{q\,[n-1]_q}{[n]_q}\sum_{k=0}^{n}\begin{bmatrix}n-2\\k\end{bmatrix}_q\left(x-\frac{1}{[n]_q}\right)^{k+2}\prod_{s=0}^{n-k-3}\left(\frac{[n-1]_q}{[n]_q}-q^sx\right)$$

$$+\frac{1}{q^2}\left(\frac{[n]_q}{q\,[n-2]_q}\right)^{n-2}\frac{1}{[n]_q}\left(\frac{q\,[n-2]_q}{[n]_q}\right)^{n-1}\left(x-\frac{1}{[n]_q}\right)+\frac{2}{q\,[n]_q}\left(x-\frac{1}{[n]_q}\right)+\frac{1}{[n]_q^2}$$

$$=\frac{[n-1]_q}{q\,[n]_q}\left(x-\frac{1}{[n]_q}\right)^2+\left(\frac{[n-2]_q}{q\,[n]_q^2}+\frac{2}{q\,[n]_q}\right)\left(x-\frac{1}{[n]_q}\right)+\frac{1}{[n]_q^2}$$

$$=x^2\left(\frac{1}{q^2}-\frac{1}{q^2\,[n]_q}\right)+x\left(\frac{[n-2]_q+2\,[n]_q-2\,[n-1]_q}{q\,[n]_q^2}\right)$$

$$+\frac{(2-q)\,[n]_q+[n-1]_q-[n-2]_q}{q\,[n]_q^3}$$

□

Lemma 2.2. *Let* $q\in(0,1)$ $\forall$ $x\in[\frac{1}{[n]_q},\frac{[n-1]_q}{[n]_q}]$ *and* $n\geq 3$ *then the following equalities holds*

$$i)U_n^q((t-x);x)=x(\frac{1}{q}-1)+\frac{q-1}{[n]_q\,q}$$

$$ii)U_n^q((t-x)^2;x)=x^2\left(\left(\frac{q-1}{q}\right)^2-\frac{1}{[n]_q\,q^2}\right)$$

$$+x\left(\frac{2(1-q)\,[n]_q+(2q^2-2q+1)\,[n-2]_q+2q}{[n]_q^2\,q}\right)$$

$$+\frac{(q^3-2q^2+q-1)\,[n-2]_q+(q^2-q-1)}{[n]_q^3\,q}$$

$$iii)U_n^q((t-x)^4;x)=x^4\left(\frac{(1-\frac{4}{q})\,[n]_q^3+\frac{6}{q}\,[n]_q^2\,[n-1]_q-4\,[n]_q\,[n-1]_q\,[n-2]_q}{[n]_q^3}\right.$$

$$\left.+\frac{q^2\,[n-1]_q\,[n-2]_q\,[n-3]_q}{[n]_q^3}\right)$$

$$+x^3\left(\frac{(\frac{4-4q}{q})\,[n]_q^3+(16-\frac{8}{q})\,[n]_q\,[n-1]_q\,[n-2]_q+\frac{6}{q}\,[n]_q^2\,[n-2]_q}{[n]_q^4}\right.$$

$$\left.+\frac{(q^2+2q+3)\,[n-1]_q\,[n-2]_q^2-4q^2\,[n-1]_q\,[n-2]_q\,[n-3]_q+2\,[n]_q-4\,[n-1]_q}{[n]_q^4}\right)$$

$$
\begin{aligned}
&+ x^2\left(\frac{\frac{36}{q}\left[n\right]_q^2\left[n-1\right]_q-\frac{18}{q}\left[n\right]_q^2\left[n-2\right]_q-\frac{4}{q}\left[n\right]_q\left[n-2\right]_q^2}{\left[n\right]_q^5}\right.\\
&+\frac{12(\frac{2}{q}-1)\left[n\right]_q\left[n-1\right]_q\left[n-2\right]-(3q^2+5q+6-\frac{3}{q})\left[n-1\right]_q\left[n-2\right]_q^2}{\left[n\right]_q^5}\\
&\left.+\frac{6q^2\left[n-1\right]_q\left[n-2\right]_q\left[n-3\right]_q+(q-4)\left[n\right]_q}{\left[n\right]_q^5}\right)\\
&+x\left(\frac{\frac{12}{q}\left[n\right]_q^3-(\frac{24}{q})\left[n\right]_q^2\left[n-1\right]_q+(6+\frac{12}{q})\left[n\right]_q^2\left[n-2\right]_q+(4-\frac{4}{q})\left[n\right]_q\left[n-2\right]_q^2+\left[n-2\right]_q^3}{\left[n\right]_q^6}\right.\\
&+\frac{(3q^2+4q+3-\frac{3}{q})\left[n-1\right]_q\left[n-2\right]_q^2-4q^2\left[n-1\right]_q\left[n-2\right]_q\left[n-3\right]_q}{\left[n\right]_q^6}\\
&\left.-\frac{(12+\frac{8}{q})\left[n\right]_q\left[n-1\right]_q\left[n-2\right]_q}{\left[n\right]_q^6}\right)\\
&+\frac{(1-\frac{4}{q})\left[n\right]_q^3+\frac{6}{q}\left[n\right]_q^2\left[n-1\right]_q-\frac{\left[n-2\right]_q}{q}\left(\left[n-2\right]_q^2-4\left[n\right]_q\left[n-2\right]_q-6\left[n\right]_q^2\right)}{\left[n\right]_q^7}\\
&+\frac{\frac{8}{q}\left[n\right]_q\left[n-1\right]_q\left[n-2\right]_q-(q^2+3q-\frac{3}{q})\left[n-1\right]_q\left[n-2\right]_q^2+q^3\left[n-1\right]_q\left[n-2\right]_q\left[n-3\right]_q}{\left[n\right]_q^7}
\end{aligned}
$$

Proof. From the linearity of the operator (1.2) we can calculate as below

$$
\begin{aligned}
i)U_n^q((t-x);x) &= U_n^q(t;x)-U_n^q(1;x)\\
&= x(\frac{1}{q})+\frac{q-1}{q\left[n\right]_q}-1\\
&= x(\frac{1-q}{q})+\frac{q-1}{q\left[n\right]_q}\\
ii)U_n^q((t-x)^2;x) &= U_n^q(t^2;x)-2xU_n^q(t;x)+x^2U_n^q(1;x)\\
&= x^2\left(\frac{1}{q^2}-\frac{1}{q^2\left[n\right]_q}\right)+x\left(\frac{2\left[n\right]_q-2\left[n-1\right]_q+\left[n-2\right]_q}{q\left[n\right]_q^2}\right)\\
&+\frac{\left[n-1\right]_q-\left[n-2\right]_q-(q-2)\left[n\right]_q}{q\left[n\right]_q^3}-2x(x(\frac{1}{q})+\frac{q-1}{q\left[n\right]_q})+x^2\\
&= x^2\left((\frac{q-1}{q})^2-\frac{1}{q^2\left[n\right]_q}\right)+x\left(\frac{2(1-q)\left[n\right]_q+(2q^2-2q+1)\left[n-2\right]_q+2q}{q\left[n\right]_q^2}\right)\\
&+\frac{(q^3-2q^2+q-1)\left[n-2\right]_q+(q^2-q-1)}{q\left[n\right]_q^3}
\end{aligned}
$$

as the same way we can simply calculate *iii*), so the proof is complete. □

Remark. *According to the Bohman-Korovkin Theorem (see:*[6]*); for the known generalized Bernstein operators, the uniform convergence of the sequences* $B_n(f;x)$

operators to $f \in C[0,1]$ explained as a specific case of the convergence and that is provided by the following two features;

a) The operator $B_n(f;x)$ is monotone

b) The operator $B_n(f;x)$ convergence uniformly to $f \in C[0,1]$ respectively to $f(x) = 1$, x, x^2.

From Lemma 2.1 and Lemma 2.2 we have seen that since $q \in (0,1)$ then the $\lim\limits_{n\to\infty} [n]_q = \frac{1}{1-q}$ as $n \to \infty$. In this case that implies $U_n^q(r_1;x), U_n^q(r_2;x)$ and $U_n^q((x-t)^2;x)$ can not converge to x, x^2 and 0 as $n \to \infty$. In order to acquire convergence for the operator $U_n^q(f;x)$, it is necessary to use the following technique in place of the prior ones. Getting $q = q_n$ and q_n will be the sequences in $\mathbb{R}$ such that $0 < q_n < 1$, $\lim\limits_{n\to\infty} q_n = 1$, $\lim\limits_{n\to\infty} \frac{1}{[n]_{q_n}} = 0$ as $n \to \infty$.By this way the difficulty can be solved.

3. Uniform and rate of convergence

In this part; we will see in Theorem 3.1 the constructed operator (1.2) converges uniformly and in Theorem 3.2 by the help of moduli of continuity we compute the degree of convergence to see the smoothness of approximation of the function f by the operator (1.2). First we will give some definitions and notations: Moduli of continuity for a bounded and continuous function $f : [\lambda, \beta] \to \mathbb{R}$ is given as

$$\omega(f,\delta) = \sup_{\substack{a,b \in [\lambda,\beta] \\ |b-a| \le \delta}} |f(b) - f(a)|$$

Since $\delta > 0, \omega(f,\delta)$ has some useful properties which can be found in [2]. As we know, as usual the space $C[0,1]$ indicate the real-valued continuous functions on $[0,1]$ this space endowed with the norm for a function f as below

$$\|f\|_{C[0,1]} = \sup_{x\in[0,1]} |f(x)|$$

As we know the operator (1.2) in section introduction defined on the interval $\left[\frac{1}{[n]_q}, \frac{[n-1]_q}{[n]_q}\right]$ thus it can be clearly seen from here $C[\frac{1}{[n]_q}, \frac{[n-1]_q}{[n]_q}] \subset C[0,1]$ so to compute the convergence easily we can use the space $C[0,1]$ instead of $C[\frac{1}{[n]_q}, \frac{[n-1]_q}{[n]_q}]$. Now we will give a theorem for the uniform convergence of the operator (1.2).

Theorem 3.1. *Assume that the q_n denote a sequence such that; for $q_n \in (0,1)$, $\lim\limits_{n\to\infty} q_n = 1$ and $\lim\limits_{n\to\infty} \frac{1}{[n]_{q_n}} = 0$ as $n \to \infty$. Since $x \in [0,1]$, $g \in C[0,1]$ and $n \ge 3$ then,*

$$\lim_{n\to\infty} \|U_n^{q_n}(g;x) - g(x)\|_{C[0,1]} = 0$$

Proof. According to Bohman-Korovkin Theorem in [6] we have to prove the operator (1.2) converges uniformly on $[0,1]$. Let $r_i(v) = v^i$ where $i = 0,1,2$, since $U_n^{q_n}(1;x) = 1$ from Lemma 2.1 we can write

$$\lim_{n\to\infty} \|U_n^{q_n}(1;x) - 1\|_{C[0,1]} = 0$$

for $i = 1$ we calculated in Lemma 2.1 that,

$$U_n^{q_n}(r_1;x) = x(\frac{1}{q_n}) + \frac{q_n - 1}{q_n\,[n]_{q_n}}$$

$$\begin{aligned}\lim_{n\to\infty}\|U_n^{q_n}(r_1;x)-x\|_{C[0,1]} &= \lim_{n\to\infty}\sup_{x\in[0,1]}\left|x(\frac{1}{q_n})+\frac{q_n-1}{q_n\,[n]_{q_n}}-x\right| \\ &= \lim_{n\to\infty}\sup_{x\in[0,1]}\left|x(\frac{1}{q_n}-1)+\frac{q_n-1}{q_n\,[n]_{q_n}}\right|\end{aligned}$$

Since $\lim\limits_{n\to\infty} q_n = 1$ for $0 < q_n < 1$ and $\lim\limits_{n\to\infty} \frac{1}{[n]_{q_n}} = 0$ as $n \to \infty$.

$$\lim_{n\to\infty}\|U_n^{q_n}(r_1;x)-x\|_{C[0,1]} = 0$$

by using similar method for $i = 2$ and applying Lemma 2.1, it is easy to see

$$\lim_{n\to\infty}\left\|U_n^{q_n}(r_2;x)-x^2\right\|_{C[0,1]}$$

$$= \lim_{n\to\infty}\sup_{x\in[0,1]}\left|\begin{array}{c} x^2\left(\frac{1}{q_n^2}-\frac{1}{q_n^2\,[n]_{q_n}}-1\right)+x\left(\frac{[n-2]_{q_n}-2[n-1]_{q_n}+2[n]_{q_n}}{q_n\,[n]_{q_n}^2}\right) \\ +\frac{[n-1]_{q_n}-[n-2]_{q_n}-(q_n-2)[n]_{q_n}}{q_n\,[n]_{q_n}^3}\end{array}\right| \to 0$$

So the operator (1.2) converges uniformly to $g(x)$ on $[0,1]$. □

Theorem 3.2. *Assume that q_n denote a sequence and fulfills for $q_n \in (0,1)$, $\lim\limits_{n\to\infty} q_n = 1$ and $\lim\limits_{n\to\infty} \frac{1}{[n]_{q_n}} = 0$ as $n \to \infty$, then for $x \in [0,1]$, $n \geq 3$ and $h \in C[0,1]$ the follows inequality holds*

$$|U_n^{q_n}(h;x)-h(x)| \leq 2\omega(h;\delta_{n,q_n}(x))$$

where

$$\delta_{n,q_n} = \sqrt{\begin{array}{c}\left[x^2\left((\frac{q_n-1}{q_n})^2-\frac{1}{q_n^2\,[n]_{q_n}}\right)+x\left(\frac{-2(q_n-1)[n]_{q_n}+(2q_n^2-2q_n+1)[n-2]_{q_n}+2q_n}{q_n\,[n]_{q_n}^2}\right)\right. \\ \left.+\frac{(q_n^3-2q_n^2+q_n-1)[n-2]_{q_n}+(q_n^2-q_n-1)}{q_n\,[n]_{q_n}^3}\right]\end{array}}$$

Proof. By using the features of the moduli of continuity in [2], we get the inequality as below

$$|h(t)-h(x)| \leq \left(1+\frac{|t-x|}{\delta_{n,q_n}}\right)\omega(h;\delta_{n,q_n})$$

Since δ_{n,q_n} is a sequence of positive numbers and also by using the linearity and positivity of the operator (1.2) then we may write

$$|U_n^{q_n}(h;x)-h(x| \leq \left(1+\frac{1}{\delta_{n,q_n}}\left(U_n^{q_n}(|t-x|\,;x)\right)\right)\omega(h;\delta_{n,q_n})$$

after this step applying Cauchy-Schwarz inequality and by using Lemma 2.2 we obtain that

$$|U_n^{q_n}(h;x)-h(x| \leq \left(1+\frac{1}{\delta_{n,q_n}}\sqrt{(U_n^{q_n}((t-x)^2;x)}\right)\omega(h;\delta_{n,q_n})$$

$$\leq \left(1+\frac{1}{\delta_{n,q_n}}\sqrt{\begin{array}{c}x^2\left((\frac{q_n-1}{q_n})^2-\frac{1}{q_n^2\,[n]_{q_n}}\right)+x\left(\frac{(2q_n^2-2q_n+1)[n-2]_{q_n}-(q_n-1)2[n]_{q_n}+2q_n}{q_n\,[n]_{q_n}^2}\right) \\ +\frac{(q_n^3-2q_n^2+q_n-1)[n-2]_{q_n}+(q_n^2-q_n-1)}{q_n\,[n]_{q_n}^3}\end{array}}\right)\omega(h;\delta_{n,q_n})$$

by choosing

$$\delta_{n,q_n} = \sqrt{\begin{array}{c} x^2\left((\frac{q_n-1}{q_n})^2 - \frac{1}{q_n^2[n]_{q_n}}\right) + x\left(\frac{(2q_n^2-2q_n+1)[n-2]_{q_n}-(q_n-1)2[n]_{q_n}+2q_n}{q_n[n]_{q_n}^2}\right) \\ +\frac{(q_n^3-2q_n^2+q_n-1)[n-2]_{q_n}+(q_n^2-q_n-1)}{q_n[n]_{q_n}^3} \end{array}}$$

hence, the desired result is obtained as

$$|U_n^{q_n}(h;x) - h(x)| \leq 2\omega(h;\delta_{n,q_n}(x)).$$

□

4. Voronovskaja's Type Asymptotic Theorem

In this part in order to proof the Voronovskaja's type asymptotic theorem, we need first to give a lemma which will help us to prove our major theorem.

Lemma 4.1. *Assume that q_n denote a sequence and fulfills for $q_n \in (0,1)$, $q_n \to 1, q_n^n \to 1$ and $\lim\limits_{n\to\infty} \frac{1}{[n]_{q_n}} = 0$ as $n \to \infty$.Since for $\forall\ x \in \left[\frac{1}{[n]_{q_n}}, \frac{[n-1]_{q_n}}{[n]_{q_n}}\right]$ and $n \geq 3$ then the following limits holds*

$$i) \lim_{n\to\infty} [n]_{q_n} U_n^{q_n}((t-x);x) = 0$$

$$ii) \lim_{n\to\infty} [n]_{q_n} U_n^{q_n}((t-x)^2;x) = x - x^2$$

$$iii) \lim_{n\to\infty} [n]_{q_n}^2 U_n^{q_n}((t-x)^4;x) = 3x^2$$

Proof. By using the conditions of this Lemma and with the help of Lemma 2.2 then we get as follows

$$i) \lim_{n\to\infty} [n]_{q_n} \left(x(\frac{1-q_n}{q_n}) + \frac{q_n-1}{[n]_{q_n q_n}} \right)$$

$$= \lim_{n\to\infty} x\frac{(1-q_n^n)}{q_n} + \lim_{n\to\infty} \frac{q_n-1}{q_n} = 0$$

$$ii) \lim_{n\to\infty} [n]_{q_n} U_n^{q_n}((t-x)^2;x)$$

$$= \lim_{n\to\infty} [n]_{q_n} \left[\begin{array}{c} x^2\left((\frac{q_n-1}{q_n})^2 - \frac{1}{q_n^2[n]_{q_n}}\right) + x\left(\frac{(2q_n^2-2q_n+1)[n-2]_{q_n}-(q_n-1)2[n]_{q_n}+2q_n}{q_n[n]_{q_n}^2}\right) \\ +\frac{(q_n^3-2q_n^2+q_n-1)[n-2]_{q_n}+(q_n^2-q_n-1)}{q_n[n]_{q_n}^3} \end{array} \right]$$

$$= \lim_{n\to\infty} x^2 \left(\frac{1-q_n^n}{1-q_n}(\frac{q_n-1}{q_n})^2 - \frac{1}{q_n^2} \right) + \lim_{n\to\infty} x \left(\frac{(2q_n^2-2q_n+1)\,[n-2]_{q_n} - (q_n-1)2\,[n]_{q_n} + 2q_n}{q_n\,[n]_{q_n}} \right)$$

$$+ \lim_{n\to\infty} \frac{(q_n^3-2q_n^2+q_n-1)\,[n-2]_{q_n} + (q_n^2-q_n-1)}{q_n\,[n]_{q_n}^2}$$

It is easy to see the first limit goes to $-x^2$, second to x and the last one goes to 0, so we obtain

$$ii) \lim_{n\to\infty} [n]_{q_n} U_n^{q_n}((t-x)^2;x) = x - x^2$$

finally by the similar way and with the help of Lemma 2.2, we can easily get desired result

$$iii)\lim_{n\to\infty}[n]_{q_n}^2\,U_n^{q_n}((t-x)^4;x)=3x^2$$

□

Theorem 4.2. *(Voronovskaja's type result) Let* q_n *denote a sequence and fulfills that* $q_n\in(0,1)\,,q_n\to 1,\ q_n^n\to 1$ *and* $\lim\limits_{n\to\infty}\frac{1}{[n]_{q_n}}=0$ *as* $n\to\infty$*,then uniformly for any* $x\in[0,1]$ *and* $h\in C^2[0,1]$ *the follows equality holds*

$$\lim_{n\to\infty}[n]_{q_n}\left(U_n^{q_n}(h;x)-h(x)\right)=\frac{x-x^2}{2}h''(x)$$

Proof. With the help of Taylors' expansion formula and due to acceptance of $x\in[0,1]$ and $h\in C^2[0,1]$ we can write

$$h(t)=h(x)+h'(x)(t-x)+\frac{1}{2}h''(x)(t-x)^2+k(t;x)(t-x)^2 \tag{4.1}$$

$k(t;x)$ in (4.1) is a form of Peano of the rest term, $k(.;x)\in C[0,1]$ and $\lim\limits_{t\to x}k(t;x)=0$.

Applying $U_n^{q_n}(h;x)$ operators to (4.1) then we get,

$$\begin{aligned}[n]_{q_n}\left(U_n^{q_n}(h;x)-h(x)\right)&=[n]_{q_n}U_n^{q_n}((t-x);x)h'(x)+\frac{1}{2}[n]_{q_n}U_n^{q_n}((t-x)^2;x)h''(x)\\&+[n]_{q_n}U_n^{q_n}(k(t;x)(t-x)^2;x)\end{aligned} \tag{4.2}$$

we implement the Cauchy-Schwarz inequality to the last term of equation (4.2), then we obtain

$$[n]_{q_n}U_n^{q_n}(k(t;x)(t-x)^2;x)\le\sqrt{U_n^{q_n}(k^2(t;x);x)}\sqrt{U_n^{q_n}((t-x)^4;x)} \tag{4.3}$$

it is clear to trace that $k^2(x;x)=0$, $k^2(.;x)\in C[0,1]$ and also $U_n^{q_n}((t-x)^4;x)$ is finite

then for $x\in[0,1]$ it follows from Theorem 3.1 that

$$\lim_{n\to\infty}U_n^{q_n}(k^2(t;x);x)=k^2(x;x)=0 \tag{4.4}$$

Combining (4.3), (4.4) and by using Lemma 2.2 so we get immediately

$$\lim_{n\to\infty}[n]_{q_n}U_n^{q_n}(k(t;x)(t-x)^2;x)=0$$

Consequently, we obtain the desired result as below

$$\lim_{n\to\infty}[n]_{q_n}\left(U_n^{q_n}(h;x)-h(x)\right)=\frac{x-x^2}{2}h''(x)$$

□

5. Figures and Numerical Tables

In this section, we give some plots and numerical tables to see the difference approximation and error bounds between the operators U_n^q, B_n^q and U_n^q, U_n.

In the numerical tables to see the which polynomial convergence faster than other so firstly we recall some definitions: Let $a=(a_n)$ and $b=(b_n)$ be sequences with limits a_0 and b_0 and $a \neq (a_n), b \neq (b_n)$ for all n and $0 < M < \infty$(see more details in [7]) then we get the following result:

$$\lim_{n\to\infty} \frac{|a_n - a_0\ |}{|b_n - b_0\ |} = \begin{cases} 0, & (a_n) \text{ convergence faster than } (b_n) \\ M, & \text{convergence}(a_n) \text{ and } (b_n) \text{ are equivalent} \\ \infty, & (b_n) \text{ convergence faster than } (a_n) \end{cases} \tag{5.1}$$

Let $f(x)$, $F_1(x)$ and $F_2(x)$ is respectively given as follows;

$$f(x) = 2\sin(\pi^2 x)\exp(-x)$$

$$F_1(x) = \frac{|U_n^q(f;x) - f(x)|}{|B_n^q(f;x) - f(x)|}, F_2(x) = \frac{|U_n^q(f;x) - f(x)|}{|U_n(f;x) - f(x)|}$$

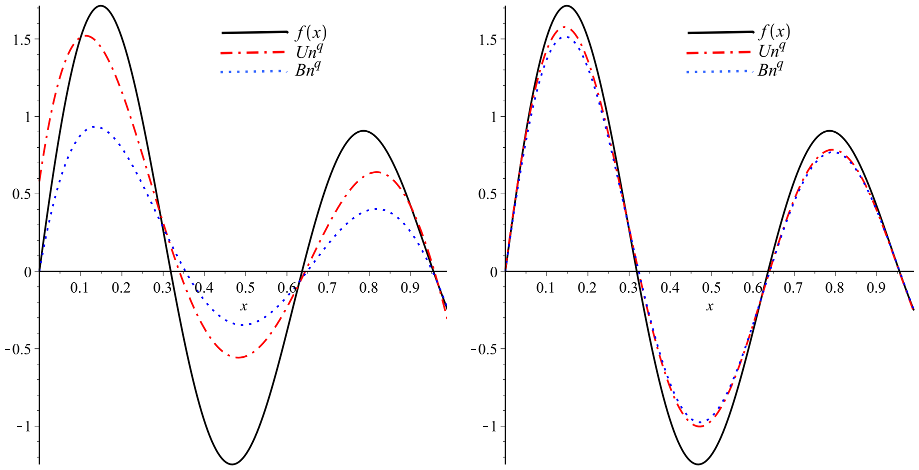
In Table 1 and Table 2 we calculated for different values of x and n, the error bounds by choosing respectively for $q = 0.9999$ and $q = 0.999$. Also we made drawings as Figure 1 and Figure 2 to see the approximation between the abovementioned operators for different n and q values. By considering (5.1), it is clear from Table 1 that to say on some values of x and n; $U_n^q(f;x)$ has better convergence than $B_n^q(f;x)$ and also by referring from Table 2, on some values of x and n; operator $U_n^q(f;x)$ convergence faster than the operator $U_n(f;x)$. As conclusion, by more powerful computers with higher speeds the error intervals can decreases easily.

$F_1(x)$	0.15	0.5	0.8
$n = 10$	0.342258361	0.757069185	0.531807975
$n = 25$	0.717094979	0.872346566	0.786784121
$n = 100$	0.856728405	0.942478376	0.929448984
$n = 250$	0.546141052	0.852543838	0.898573497

TABLE 1. Error bounds at different x and n values for $F_1(x)$.

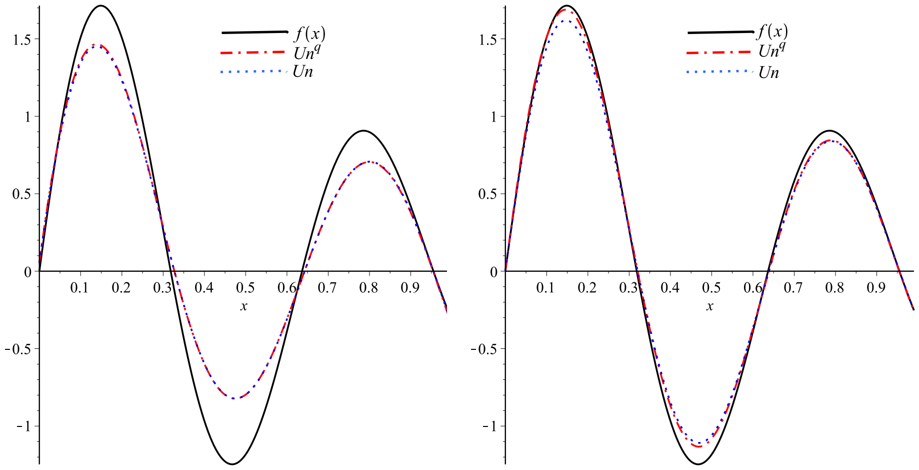
$F_2(x)$	0.15	0.5	0.8
$n = 10$	0.968456394	1.003108428	1.010315718
$n = 25$	0.946809831	1.000093210	1.010540543
$n = 100$	0.281860365	0.828482701	0.925513073
$n = 250$	0.186534697	0.427839722	0.181417734

TABLE 2. Error bounds at different x and n values for $F_2(x)$.



$n = 10$ $n = 50$

FIGURE 1. Approximation between U_n^q and B_n^q to the function $f(x) = 2\sin(\pi^2 x)\exp^{-x}$ for $q = 0.9999$



$n = 25$ $n = 100$

FIGURE 2. Approximation between U_n^q and U_n to the function $f(x) = 2\sin(\pi^2 x)\exp^{-x}$ for $q = 0.999$

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