

**DENYING A SHORT PROOF OF A DETERMINANTAL
 FORMULA FOR GENERALIZED FIBONACCI POLYNOMIALS**

FENG QI

ABSTRACT. In the paper, the author denies the main result and its proof in the paper “M. Anđelić and C. M. da Fonseca, *A short proof for a determinantal formula for generalized Fibonacci numbers*, *Matematiche (Catania)* **74** (2019), no. 2, 363–367; available online at <https://doi.org/10.4418/2019.74.2.9>”.

1. PRELIMINARIES

A tridiagonal determinant is the determinant of a square matrix with nonzero elements only on the diagonal and slots horizontally or vertically adjacent the diagonal. In other words, a determinant $H = |h_{ij}|_{n \times n}$ is called a tridiagonal determinant if $h_{ij} = 0$ for all pairs (i, j) such that $|i - j| > 1$.

The Fibonacci numbers

$$F_n = \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}}, \quad n \in \{0\} \cup \mathbb{N}$$

form a sequence of natural numbers and satisfy $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0$ and $F_1 = F_2 = 1$.

As a generalization of the Fibonacci numbers F_n , the Fibonacci polynomials

$$F_n(s) = \frac{1}{2^n} \frac{(s + \sqrt{4 + s^2})^n - (s - \sqrt{4 + s^2})^n}{\sqrt{4 + s^2}}, \quad n \in \{0\} \cup \mathbb{N}$$

can be generated by

$$\frac{z}{1 - sz - z^2} = \sum_{n=0}^{\infty} F_n(s) z^n = z + sz^2 + (s^2 + 1)z^3 + s(s^2 + 2)z^4 + \dots \quad (1.1)$$

As a generalization of the Fibonacci polynomials $F_n(s)$, generalized Fibonacci polynomials $F_n(s, t)$ can be defined by two initial conditions $F_0(s, t) = 0$ and $F_1(s, t) = 1$ and by the recurrence relation

$$F_n(s, t) = sF_{n-1}(s, t) + tF_{n-2}(s, t), \quad n \geq 2$$

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and can be generalized by

$$\frac{z}{1 - sz - tz^2} = \sum_{n=0}^{\infty} F_n(s, t)z^n = z + sz^2 + (s^2 + t)z^3 + s(s^2 + 2t)z^4 + (s^4 + 3s^2t + t^2)z^5 + \dots \tag{1.2}$$

It is clear that $F_n(s, 1) = F_n(s)$ and $F_n(1, 1) = F_n$ for $n \in \{0\} \cup \mathbb{N}$.

For known results on the Fibonacci numbers and polynomials, please refer to the paper [1, 4, 5], Sections 7 and 8 in [7], and closely related references therein.

2. BACKGROUNDS

By virtue of a general formula in [3, p. 40, Exercise 5)] for the n th derivative of the ratio of two n times differentiable functions, the following determinantal formulas for the Fiboonacci numbers, the Fibonacci polynomials, and generalized Fibonacci polynomials were established in [8, 9].

Applying the general formula in [3, p. 40, Exercise 5)] to the generating function $\frac{z}{1-sz-z^2}$ in (1.1) yields [9, Theorem 1.1] which is recited as follows.

Theorem 1 ([9, Theorem 1.1]). *For $n \in \{0\} \cup \mathbb{N}$, the Fibonacci polynomials $F_n(s)$ can be determinantly expressed as*

$$F_n(s) = \frac{1}{n!} \begin{vmatrix} 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & s & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2 & 2s & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 6 & 3s & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (n-2)s & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & (n-1)(n-2) & (n-1)s & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & n(n-1) & ns \end{vmatrix}. \tag{2.1}$$

Applying the general formula in [3, p. 40, Exercise 5)] to the generating function $\frac{z}{1-sz-tz^2}$ in (1.2) yields [9, Theorem 1.1] which is recited as follows.

Theorem 2 ([8, Theorem 1.1]). *For $n \in \{0\} \cup \mathbb{N}$, the generalized Fibonacci polynomials $F_n(s, t)$ can be determinantly expressed as*

$$F_n(s, t) = \frac{1}{n!} \begin{vmatrix} 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & s & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2t & 2s & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 6t & 3s & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (n-2)s & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & (n-1)(n-2)t & (n-1)s & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & n(n-1)t & ns \end{vmatrix}. \tag{2.2}$$

If taking $t = 1$ in (2.2), then the determinantal formula (2.1) can be derived immediately. If setting $s = t = 1$ in Theorem 2 or letting $s = 1$ in Theorem 1, then we see readily that the Fibonacci numbers F_n for $n \in \{0\} \cup \mathbb{N}$ can be determinantly

expressed as

$$F_n = \frac{1}{n!} \begin{vmatrix} 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 6 & 3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n-2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & (n-1)(n-2) & n-1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & n(n-1) & n \end{vmatrix}.$$

3. DENYING

In [2, Section 2], the authors gave a short proof of the determinantal formula (2.2) in Theorem 2 by the multilinearity of determinants. Concretely speaking, they proved the formula

$$F_n(s, t) = \frac{1}{n!} \begin{vmatrix} s & -1 & 0 & \cdots & 0 & 0 & 0 \\ 2t & 2s & -1 & \cdots & 0 & 0 & 0 \\ 0 & 6t & 3s & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (n-2)s & -1 & 0 \\ 0 & 0 & 0 & \cdots & (n-1)(n-2)t & (n-1)s & -1 \\ 0 & 0 & 0 & \cdots & 0 & n(n-1)t & ns \end{vmatrix} \quad (3.1)$$

for $n \in \mathbb{N}$. The proof of the formula (3.1) can be recited as

$$\begin{aligned} & \begin{vmatrix} s & -1 & 0 & \cdots & 0 & 0 & 0 \\ 2t & 2s & -1 & \cdots & 0 & 0 & 0 \\ 0 & 6t & 3s & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (n-2)s & -1 & 0 \\ 0 & 0 & 0 & \cdots & (n-1)(n-2)t & (n-1)s & -1 \\ 0 & 0 & 0 & \cdots & 0 & n(n-1)t & ns \end{vmatrix} \\ = & \begin{vmatrix} s & -1 & 0 & \cdots & 0 & 0 & 0 \\ 2t & 2s & -2 & \cdots & 0 & 0 & 0 \\ 0 & 3t & 3s & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (n-2)s & -(n-2) & 0 \\ 0 & 0 & 0 & \cdots & (n-1)t & (n-1)s & -(n-1) \\ 0 & 0 & 0 & \cdots & 0 & nt & ns \end{vmatrix} = n! \begin{vmatrix} s & -1 & 0 & \cdots & 0 & 0 & 0 \\ t & s & -1 & \cdots & 0 & 0 & 0 \\ 0 & t & s & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & s & -1 & 0 \\ 0 & 0 & 0 & \cdots & t & s & -1 \\ 0 & 0 & 0 & \cdots & 0 & t & s \end{vmatrix}_{n \times n} \\ & = n! F_n(s, t). \end{aligned} \quad (3.2)$$

We observe that the formula (3.1) is wrong, since, for example, when $n = 3$, the formula (3.1) becomes

$$F_3(s, t) = \frac{1}{3!} \begin{vmatrix} s & -1 & 0 \\ 2t & 2s & -1 \\ 0 & 6t & 3s \end{vmatrix} = s(s^2 + 2t)$$

which, by comparing it with the coefficient of the term containing z^3 in the equation (1.2), is clearly wrong. Comparing it with the coefficient of the term containing z^4 in the equation (1.2) hints that the formula (3.1) perhaps should be corrected as

$$F_{n+1}(s, t) = \frac{1}{n!} \begin{vmatrix} s & -1 & 0 & \cdots & 0 & 0 & 0 \\ 2t & 2s & -1 & \cdots & 0 & 0 & 0 \\ 0 & 6t & 3s & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (n-2)s & -1 & 0 \\ 0 & 0 & 0 & \cdots & (n-1)(n-2)t & (n-1)s & -1 \\ 0 & 0 & 0 & \cdots & 0 & n(n-1)t & ns \end{vmatrix} \quad (3.3)$$

for $n \in \mathbb{N}$.

From the formula (2.2) in Theorem 2, we can deduce

$$F_{n+1}(s, t) = \frac{1}{(n+1)!} \begin{vmatrix} 2s & -1 & 0 & \cdots & 0 & 0 & 0 \\ 6t & 3s & -1 & \cdots & 0 & 0 & 0 \\ 0 & 12t & 4s & \cdots & 0 & 0 & 0 \\ 0 & 0 & 20t & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (n-1)s & -1 & 0 \\ 0 & 0 & 0 & \cdots & n(n-1)t & ns & -1 \\ 0 & 0 & 0 & \cdots & 0 & (n+1)nt & (n+1)s \end{vmatrix} \quad (3.4)$$

for $n \in \mathbb{N}$. See also [8, Remark 4.1]. The formulas (3.3) and (3.4) are seemingly different from each other, or say, they are not apparently same. Consequently, if the formula (3.3) were valid, it would be significant and would be worth being verified.

The first equality in (3.2), that is,

$$\begin{vmatrix} s & -1 & 0 & \cdots & 0 & 0 & 0 \\ 2t & 2s & -1 & \cdots & 0 & 0 & 0 \\ 0 & 6t & 3s & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (n-2)s & -1 & 0 \\ 0 & 0 & 0 & \cdots & (n-1)(n-2)t & (n-1)s & -1 \\ 0 & 0 & 0 & \cdots & 0 & n(n-1)t & ns \end{vmatrix} = \begin{vmatrix} s & -1 & 0 & \cdots & 0 & 0 & 0 \\ 2t & 2s & -2 & \cdots & 0 & 0 & 0 \\ 0 & 3t & 3s & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (n-2)s & -(n-2) & 0 \\ 0 & 0 & 0 & \cdots & (n-1)t & (n-1)s & -(n-1) \\ 0 & 0 & 0 & \cdots & 0 & nt & ns \end{vmatrix},$$

can not be understood. Indeed, when taking $n = 4$, the above equality becomes

$$24s^2(s^2 + 2t) = \begin{vmatrix} s & -1 & 0 & 0 \\ 2t & 2s & -1 & 0 \\ 0 & 6t & 3s & 0 \\ 0 & 0 & 12t & 4s \end{vmatrix} = \begin{vmatrix} s & -1 & 0 & 0 \\ 2t & 2s & -2 & 0 \\ 0 & 3t & 3s & -3 \\ 0 & 0 & 4t & 4s \end{vmatrix} = 24(s^4 + 3s^2t + t^2)$$

which leads to a contradiction. Consequently, the above recited proof of the formula (3.1) in [2, Section 2] is surely wrong.

Is the formula (3.3) correct? When letting $n = 1, 2, 3$, the formula (3.3) becomes

$$F_2(s, t) = \frac{1}{1!} |s|_{1 \times 1} = s, \quad F_3(s, t) = \frac{1}{2!} \begin{vmatrix} s & -1 \\ 2t & 2s \end{vmatrix} = s^2 + t, \quad \text{and}$$

$$F_4(s, t) = \frac{1}{3!} \begin{vmatrix} s & -1 & 0 \\ 2t & 2s & -1 \\ 0 & 6t & 3s \end{vmatrix} = s(s^2 + 2t).$$

These three quantities coincide with coefficients of z^2 , z^3 , and z^4 in the power series expansion (1.2). But, when letting $n = 4$, the formula (3.3) becomes

$$F_5(s, t) = \frac{1}{4!} \begin{vmatrix} s & -1 & 0 & 0 \\ 2t & 2s & -1 & 0 \\ 0 & 6t & 3s & 0 \\ 0 & 0 & 12t & 4s \end{vmatrix} = s^2(s^2 + 2t).$$

We cannot find the coefficient $s^2(s^2 + 2t)$ in all terms of the power series expansion (1.2). This demonstrates that the formula (3.3) is not correct!

In conclusion, the above mentioned formula (3.1) and its proof in [2, Section 2] are all essentially and thoroughly wrong!

Remark 1. This paper is a slightly revised version of the electronic preprint [6].

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COLLEGE OF MATHEMATICS, INNER MONGOLIA UNIVERSITY FOR NATIONALITIES, TONGLIAO 028043, INNER MONGOLIA, CHINA; SCHOOL OF MATHEMATICAL SCIENCES, TIANJIN POLYTECHNIC UNIVERSITY, TIANJIN 300387, CHINA; INSTITUTE OF MATHEMATICS, HENAN POLYTECHNIC UNIVERSITY, JIAOZUO 454010, HENAN, CHINA

E-mail address: qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com

URL: <https://qifeng618.wordpress.com>