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A STUDY OF THE SHOCK WAVE SCHEMES FOR THE MODIFIED BURGERS' EQUATION

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Abstract. This work concerns the study of the Burgers nonlinear equation for a perfect and a weakly viscous fluid. To solve the equation, we adopt the finite difference method combined with explicit and implicit schemes. We add to the original equation a numerical dispersion due to truncation errors (discretization errors). Then, we study the stability and convergence of the solution and make a comparison with some existing results. For illustration, the numerical simulations are given to support the theory.

1. Introduction

Every mathematical model describing a phenomenon can be written by various equations in form ordinary differential equation (ODE) and/or partial differential equation (PDE) which are in reality only approximations of the phenomenon. This mathematical formulation is sometimes considered to be the real representation of the phenomenon, generally named the *original equation*. However, there may exist other better models in the literature which are more convenient for the numerical approximations and the prediction of the quantitative behavior of the differential equations (see [4, 7, 18–20]). The idea is then to replace the proposed *original* ODE or PDE by a new equation named the *modified equation*. This consists in modifying the solution of the former equation without changing the solution of the latter by integrating or embedding a term into the original equation; it should not be interpreted in the sense of small perturbations. The principal interests of such modification are summarized as. (i) To improve a solution method in terms of robustness, stability or performance. (ii) To facilitate the analysis, as well as general understanding, of a specific solution method. (iii) To relate the existing and new solution methods.

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For a better explanation, we refer to Figure 1.

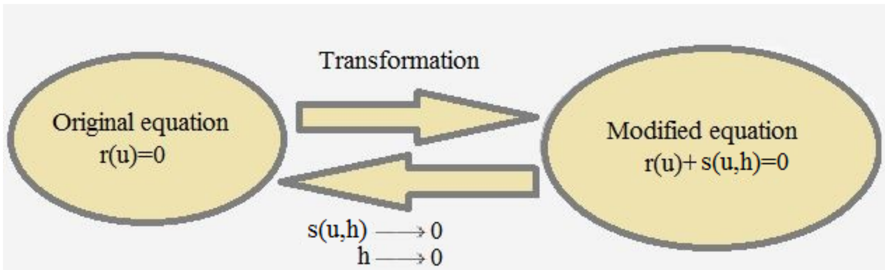

FIGURE 1. Transformation of the original equation into a modified equation is named differently

The named equation $r(u) = 0$ is replaced by a modified equation $r(u)+s(u;h) = 0$. The additional term $s(u;h)$ is known by many names in the literature [4] and depends on some parameters collected in h as well as on u, and although h may be given some interpretations (see [2,4,10,15,20]). It is often a set of scalar variables, the vector $h = [hi]$ is chosen, so that if $h_i \to 0$, $s(u;h_i) \to 0$, then the modified equation is reduced to $r(u) = 0$.

In this work, we are interested in studying a class of the hyperbolic equations that naturally appears in the modeling of certain phenomena such as transport, acoustic waves, gas dynamics, free surface environmental flows and traffic flows (see [3,4,9,17]). More especially, we consider the following Burgers inviscid equation.

For any $x \in]-M;M[$ and $t > 0$, find u such as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \tag{1.1}$$

subject to the initial condition

$$u(x,0) = \phi(x), \tag{1.2}$$

where $\phi(x)$ is a given function at time $t = 0$.
Eq. (1.1) is quasi-linear and models the propagation of waves without dissipation. It often appears as a simplification of the Navier Stokes equations (NSE) and allows us to understand some of the inside behavior of the latter. It can be written in the conservative form

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x}(u) = 0, \text{ with } f(u) = \frac{u^2}{2}, \tag{1.3}$$

which is easily recognizable as the structure of scalar hyperbolic conservation laws. We will often refer to Eq. (1.1) in the form Eq. (1.3) and the development will be valid for a general $f(u)$ in Eq. (1.3).
Another writing form of Eq. (1.1) is given below

$$\frac{\partial u}{\partial t}+A\frac{\partial u}{\partial x}=0, \tag{1.4}$$

where $A=df/du=u$ is the Jacobian matrix and its eigenvalues are all real.

The analytical solution of Eq. (1.1) can be obtained by the characteristic method which leads to a family of curves of PDE. Such a method is no longer valid when the characteristic curves intersect, therefore there are yielding some discontinuities (*shock waves*). The nonlinearity term $u\ \partial u/\partial x$ hides mathematical and numerical complexities (see [5, 10, 17]). The appearance of discontinuities in finite time, even if the initial condition is smooth, give rise to the phenomen of shock waves with important application in physics (see [1, 10, 12, 21]). Major difficulties arise when one tries to approximate the solutions admitting discontinuities. It may happen that the method converges to another weak solution instead of that of the original equation. The solution of Eq. (1.1) is presented in the form of the traveling wave (see Figure 2.)

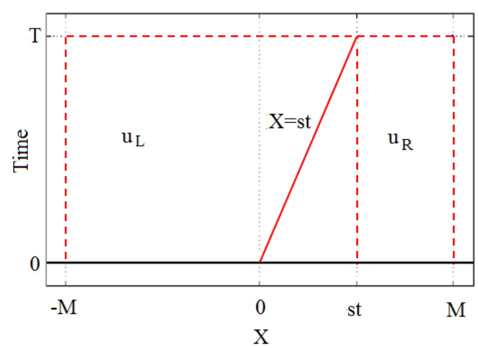

FIGURE 2. The appearance of a shock wave

$$u(x,t)=V(x-x_0-st), \tag{1.5}$$

where $V(y)$ is a step function

$$V(y)=\begin{cases} u_L\ y<0\\ u_R\ y>0\end{cases},$$

with $u_L>u_R$, there is appearance of a shock wave. We give the speed of propagation s of the shock wave

$$s=\frac{u_L+u_R}{2}, \tag{1.6}$$

and for Eq. (1.3) is

$$s=\frac{f(u_L)-f(u_R)}{u_L-u_R}=\frac{jump\ in\ f(u)}{jump\ in\ u}.$$

The aim of this study is to use a modified equation instead of the original equation in order to reduce or eliminate possible oscillations at the vicinity of the shock points. Some comparisons have been made between the obtained results for the modified equation and those available in the literature. To this end, we employ higher schemes as much as possible.

The outline of the paper is as follows. Section 2 displays the numerical schemes for Burgers equation. Section 3 contains the numerical results. Section 4 gives remarks and conclusion.

2. Numerical schemes for the Burgers equation

In this section, we propose linearized numerical schemes for the Burgers equation and give the stability analysis. Afterwards, we built a modified equation.
To this end, we give some schemes for general scalar conservation laws of the Eq. (1.3) and construct numerical solutions in the region $D = [-M < x < M] \times [t > t_0]$, with its boundary ∂D consisting of the lines $x = -M$, $x = M$ and $t = t_0$, is covered with a rectangular mesh of points of coordinates $(x, t) = (x_i, t_j) = (-M + i\Delta x,\ t_0 + j\Delta t)$ for $i = 0, 1, ..., N$ and $j = 0, 1, ..., \overline{N}$.

Denoting by
u_i^j and f_i^j respectively the approximation of u and f at point (x_i, t_j).

2.1. **A linearized centered explicit scheme.**

The finite difference method (FDM) applied to Eq. (1.3) using a simple forward in time and centered in space discretization gives

$$\frac{u_i^{j+1} - u_i^j}{\Delta t} + \frac{f_{i+1}^j - f_{i-1}^j}{2\Delta x} = 0, \ \text{ for } i = 1, ..., N, \tag{2.1}$$

or

$$u_i^{j+1} = u_i^j - \frac{\Delta t}{2\Delta x}\left(f_{i+1}^j - f_{i-1}^j\right). \tag{2.2}$$

The numerical scheme (2.2) is $\theta(\Delta t, \Delta x^2)$ accurate. Introducing now the following relation

$$f_{i+1}^j - f_{i-1}^j = \alpha(u_{i+1}^j - u_{i-1}^j), \tag{2.3}$$

where $\alpha = \underset{i}{sup}\left|f'(u_i^j)\right|$ is the maximum eigenvalue of the matrix A of the Jacobian df/du_i; in our case it is just the single element u.

Then, the linearized explicit scheme can be written as

$$u_i^{j+1} = \alpha\frac{\Delta t}{2\Delta x}u_{i-1}^j + u_i^j - \alpha\frac{\Delta t}{2\Delta x}u_{i+1}^j, \quad \text{for } i = 1, ..., N, \tag{2.4}$$

which can be rewritten in matrix form

$$Au = C, \tag{2.5}$$

where $A = [0;\ 1;\ 0]$ and $C = \left[\frac{\alpha\Delta t}{2\Delta x};\ 1;\ \frac{-\alpha\Delta t}{2\Delta x}\right]$.

Using the Maple program, we obtain the amplification factor of the scheme (2.4) which has a complex number form $1 + i(.)$ (see [10]) and whose absolute value is always greater than unity as shown in the following formula.

$$|G|^2 = 1 + \alpha^2\frac{\Delta t^2}{\Delta x^2}\sin(y)^2, \ \text{ for any } y \neq 0(\pi). \tag{2.6}$$

Then, we deduce that the scheme (2.4) is unconditionally unstable; also, the numerical solution of the Eq. (2.4) does not verify the Eq. (1.3). So, the idea is to modify the Eq. (2.4) by introducing the truncation error as follows

$$ErrT = Eq^h.(1.3) - Eq.(2.4). \tag{2.7}$$

Expanding equation (2.7) using Taylor series about the point (x_i, t_j) and retaining only the first two terms gives

$$ErrT = \frac{1}{2}\left(\frac{\partial^2 u}{\partial t^2}\right)\Delta t + \frac{1}{6}\alpha\left(\frac{\partial^3 u}{\partial x^3}\right)\Delta x^2 + \theta(\alpha\Delta x^2, \Delta t^2) \tag{2.8}$$

We now eliminate the term $\frac{1}{2}\left(\partial^2 u/\partial^2 t\right)$ of the right-hand of Eq. (2.8) by using equation

$$\frac{\partial u}{\partial t} = -\frac{\partial f(u)}{\partial x},$$

and the following development

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= -\frac{\partial}{\partial t}\left(\frac{\partial f(u)}{\partial x}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial f(u)}{\partial t}\right)\\
&= -\frac{\partial}{\partial x}\left(f'(u)\frac{\partial u}{\partial t}\right) = -\frac{\partial}{\partial x}\left(f'(u)\left(\frac{-\partial f}{\partial x}\right)\right)\\
&= \frac{\partial}{\partial x}\left(u\frac{\partial f}{\partial x}\right) = \left(\frac{\partial u}{\partial x}\frac{\partial f}{\partial x} + u\frac{\partial^2 f}{\partial x^2}\right)\\
&= \left(2u\left(\frac{\partial u}{\partial x}\right)^2 + u^2\frac{\partial^2 u}{\partial x^2}\right) = \frac{\partial^2 G}{\partial x^2}(u), \quad \text{with } G(u) = \frac{u^3}{3}.
\end{aligned}$$

Eq. (2.7) then becomes

$$Eq^h.(1.3) - Eq.(2.4) = \frac{1}{2}\left(\frac{\partial^2 G(u)}{\partial x^2}\right) + \theta(\alpha\Delta x^2, \Delta t^2) \tag{2.9}$$

Therefore, the modified equation is written as

$$Eq^h.(1.3) \simeq \frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} + \frac{1}{2}\left(\frac{\partial^2 G(u)}{\partial x^2}\right) = 0. \tag{2.10}$$

The discretization of term $\partial^2 G(u)/\partial x^2$ about the point (x_i, t_j) using the FDM is given by

$$\left(\frac{\partial^2 G(u)}{\partial x^2}\right) \simeq \left(\frac{G(u_{i+1}^j) - 2G(u_i^j) + G(u_{i-1}^j)}{\Delta x^2}\right)$$

Introducing also the following relations

$$\beta(u_{i+1}^j - u_i^j) = G(u_{i+1}^j) - G(u_i^j), \text{ for } i = 0, ..., N-1, \tag{2.11}$$

and

$$\gamma(u_i^j - u_{i-1}^j) = G(u_i^j) - G(u_{i-1}^j), \text{ for } i = 1, ..., N, \tag{2.12}$$

where β and γ are the maximum eigenvalue of the Jacobian matrix dG/du_i which is just the the single element u^2.

Using both the relations (2.3), (2.11) and (2.12), for $i = 1, ..., N$, then the linearized modified scheme is given by

$$\begin{aligned} u_i^{j+1} &= \frac{\Delta t}{2\Delta x}\left(\alpha + \frac{\gamma \Delta t}{\Delta x}\right) u_{i-1}^j + \left(1 - \left(\frac{\Delta t^2}{2\Delta x^2}\right)(\beta+\gamma)\right) u_i^j \\ &\quad + \frac{\Delta t}{2\Delta x}\left(-\alpha + \frac{\beta \Delta t}{2\Delta x}\right) u_{i+1}^j. \end{aligned} \tag{2.13}$$

Or in matrix form

$$A^{(1)} u = C^{(1)}, \tag{2.14}$$

where the matrices $A^{(1)}$ and $C^{(1)}$ are respectively given by

$$A^{(1)} = [0;\ 1;\ 0],$$

$$C^{(1)} = \left[\frac{\Delta t}{2\Delta x}\left(\alpha + \frac{\gamma\Delta t}{\Delta x}\right);\ 1 - \left(\frac{\Delta t^2}{2\Delta x^2}\right)(\beta+\gamma);\ \frac{\Delta t}{2\Delta x}\left(-\alpha + \frac{\beta\Delta t}{2\Delta x}\right)\right].$$

2.2. **A linearized forward explicit scheme.** The discretization of Eq. (1.3) using the forward in time and in space yields

$$\frac{u_i^{j+1} - u_i^j}{\Delta t} - \frac{\left(f_{i+1}^j - f_i^j\right)}{\Delta x}, \quad \text{for } i = 0, ..., N-1. \tag{2.15}$$

Using relation (2.3), the Eq. (2.15) becomes

$$u_i^{j+1} = \left(1 + \frac{\alpha \Delta t}{\Delta x}\right) u_i^j - \frac{\alpha\Delta t}{\Delta x} u_{i+1}^j, \quad \text{for } i = 0, ..., N-1. \tag{2.16}$$

The scheme (2.16) in matrix form is given by

$$A^{(1)} u = C^{(1)}, \tag{2.17}$$

where $A^{(1)}$ and $C^{(1)}$ are two matrices, with

$$A^{(1)} = [0, 1, 0] \text{ and } C^{(1)} = \left[0, 1 + \frac{\alpha\Delta t}{\Delta x}, \frac{-\alpha\Delta t}{\Delta x}\right].$$

Therefore, the amplification factor (see [5], [10]) of Eq. (2.16) is given by

$$|G|^2 = 1 + \frac{4\alpha\Delta t}{\Delta x}\left(1 + \frac{\alpha\Delta t}{\Delta x}\right)\sin^2(\frac{y}{2}). \tag{2.18}$$

We deduce that $|G|^2 > 1$, for any $y \neq 0$ (π) and the explicit forward scheme (2.16) is unstable.
In the same way as before, we get the following truncation error

$$ErrT = Eq^h.\ (1.3) - Eq.(2.15) \ = \frac{1}{2}\left(\frac{\partial^2 u}{\partial t^2}\right) dt + \theta(\Delta x, \Delta t^2). \tag{2.19}$$

We proceed as before by eliminating the term $\frac{1}{2}\left(\partial^2 u/\partial t^2\right)$. We then obtain

$$Eq^h.(1.3) - Eq.(2.16) = \frac{1}{2}\left(\frac{\partial^2 G(u)}{\partial x^2}\right) + \theta(\Delta x, \Delta t^2).$$

The modified equation is therefore

$$Eq^h \simeq \frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} + \frac{1}{2}\left(\frac{\partial^2 G(u)}{\partial x^2}\right) = 0. \tag{2.20}$$

Using relations (2.3), (2.11) and (2.12), we obtain the modified linearized scheme as

$$u_i^{j+1} = \left(-\frac{\beta\Delta t^2}{2\Delta x^2}\right) u_{i+2}^j + \left(\frac{-\alpha\Delta t}{\Delta x} + \frac{\beta\Delta t^2}{2\Delta x^2} + \frac{\gamma\Delta t^2}{2\Delta x^2}\right) u_{i+1}^j + \left(1 + \frac{\alpha\Delta t}{\Delta x} - \frac{\gamma\Delta t^2}{2\Delta x^2}\right) u_i^j, \tag{2.21}$$

or in matrix form

$$A^{(2)}u = C^{(2)},$$

where $A^{(2)}$ and $C^{(2)}$ are respectively two matrices:

$$A^{(2)} = [0;\ 1;\ 0],$$

$$C^{(2)} = \left[\left(1 + \frac{\alpha\Delta t}{\Delta x} - \frac{\gamma\Delta t^2}{2\Delta x^2}\right);\ \left(\frac{-\alpha\Delta t}{\Delta x} + \frac{\beta\Delta t^2}{2\Delta x^2} + \frac{\gamma\Delta t^2}{2\Delta x^2}\right);\ \left(-\frac{\beta\Delta t^2}{2\Delta x^2}\right)\right].$$

2.3. **A linearized implicit scheme.** The MDF applied to Eq. (1.3) by using the forward in time, centered in space discretization implicit scheme yields

$$u_i^{j+1} = u_i^j - \frac{\Delta t}{2\Delta x}\left(f_{i+1}^{j+1} - f_{i-1}^{j+1}\right). \tag{2.22}$$

Using relation (2.3), Eq. (2.22) becomes

$$u_i^{j+1} = u_i^j - \frac{\alpha\Delta t}{2\Delta x}\left(u_{i+1}^{j+1} - u_{i-1}^{j+1}\right). \tag{2.23}$$

The implicit scheme is written as follows

$$\frac{-\alpha\Delta t}{2\Delta x}\, u_{i+1}^{j+1} + u_i^{j+1} + \frac{-\alpha\Delta t}{2\Delta x}\, u_{i+1}^{j+1} = u_i^j, \quad \text{for } i = 1, ..., N. \tag{2.24}$$

Using a maple program, the amplification factor of Eq. (2.24) is given by

$$|G|^2 = \frac{\Delta x^2}{dx^2 + \alpha^2\Delta t^2 \sin(y)^2} < 1,\ \forall y.$$

The latter is exactly the opposite of the amplification factor of the explicit centered scheme (2.4); it is always less than 1, therefore the implicit scheme is unconditionally stable. The truncation error of the scheme (2.24) is

$$ErrT = \left(\alpha\left(\frac{\partial^2 u(t,x)}{\partial x\partial t}\right) + \frac{1}{2}\frac{\partial^2 u}{\partial t^2}\right)\Delta t + \theta(\alpha\Delta x^2, dt^2).$$

We proceed as previously by eliminating respectively the terms $\partial^2 u(t,x)/\partial x\partial t$ and $1/2(\partial^2 u/\partial t^2)$, and using the relations (2.11) and (2.12), we then obtain

$$\frac{-\alpha\Delta t}{2\Delta x}u_{i-1}^{j+1} + u_i^{j+1} - \frac{\alpha\Delta t}{2\Delta x}u_{i+1}^{j+1} = \gamma u_{i-1}^j + (1-\beta-\gamma)u_i^j + \beta u_{i+1}^j, \tag{2.25}$$

which in matrix form gives

$$A^{(4)}u^{j+1} = C^{(4)}u_i^j, \text{ with } u^j = (u_1^j, ..., u_N^j)^t,$$

where

$$A^{(4)} = \left[\frac{-\alpha\Delta t}{2\Delta x};\ 1;\ -\frac{\alpha\Delta t}{2\Delta x}\right]$$
$$C^{(4)} = [\gamma;\ (1-\beta-\gamma);\ \beta]$$

2.4. **Lax-Friedrichs scheme.** The Lax-Friedrichs method is considered to be a typical first-order method as the application to a nonlinear equation and the dissipative character of the result. The Lax-Friedrichs scheme of Eq. (1.3) is given by

$$\frac{u_i^{j+1} - \frac{1}{2}(u_{i-1}^j + u_{i+1}^j)}{\Delta t} + \frac{f_{i+1}^j - f_{i-1}^j}{2\Delta x} = 0,$$

and its amplification factor (see [5], [10]) is

$$G = \cos(k\Delta x) - i\frac{\Delta t}{\Delta x}A\sin(k\Delta x),$$

where the matrix A is the Jacobian matrix df/du_i which is just the single element u. The stability requires that the Courant-Friedrichs-Lewy (CFL) condition must be verified, i.e.,

$$\left|\frac{\Delta t}{\Delta x}\alpha\right| \leq 1,$$

Using relation (2.3), the linearized scheme is written as

$$u_i^{j+1} = \left(\frac{1}{2} + \frac{\alpha\Delta t}{2\Delta x}\right)u_{i-1}^j + \left(\frac{1}{2} - \frac{\alpha\Delta t}{2\Delta x}\right)u_{i+1}^j. \tag{2.26}$$

This scheme is an explicit two-level scheme in first-order time. It is stable if the CFL condition is verified.
The amplification factor in [10] is given by

$$|G|^2 = \frac{\alpha^2\Delta t^2}{\Delta x^2} + cos(y)^2\left(1 - \frac{\alpha^2\Delta t^2}{\Delta x^2}\right). \tag{2.27}$$

The module of (2.27) is less than 1 under the CFL condition.
Through the Maple program, we can obtain the following truncation error

$$ErrT = \frac{1}{2}\left(\frac{\partial^2 u}{\partial t^2}\right)\Delta t - \frac{1}{2\Delta t}\left(\frac{\partial^2 u}{\partial x^2}\right)\Delta x^2 + \theta(\alpha\Delta x^2, \Delta t^2). \tag{2.28}$$

In the same way as before, we have

$$ErrT = \frac{1}{2}\left(\frac{\partial^2 G(u)}{\partial x^2}\right)\Delta t + \theta(\alpha\Delta x, \Delta t^2),\ \text{with } G(u) = \frac{u^3}{3}. \tag{2.29}$$

Then, the modified Lax-Friedrichs equation is given by

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} - \frac{1}{2}\left(\frac{\partial^2 G(u)}{\partial x^2}\right)\Delta t = 0. \tag{2.30}$$

For $i = 1, ..., N$, the modified Lax-Friedichs scheme is

$$u_i^{j+1} = \left(\frac{1}{2} + \frac{\alpha\Delta t}{2\Delta x} + \frac{\gamma\Delta t^2}{2\Delta x^2}\right)u_{i-1}^j + \frac{\Delta t^2(\beta-\gamma)}{2\Delta x^2}u_i^j + \left(\frac{1}{2} - \frac{\alpha\Delta t}{2\Delta x} - \frac{\beta\Delta t^2}{2\Delta x^2}\right)u_{i+1}^j \tag{2.31}$$

3. Experimental simulations

To show the reduction of the oscillations around shock points, we draw some graphs for the linearized and modified schemes by taking different initial conditions:

1) $\phi_1(x) = \begin{cases} 1, & x \leq 0, \\ 0, & x > 0. \end{cases}$ (with this condition, the problem is named Riemann problem)

2) $\phi_2(x) = \begin{cases} 0.5, & x \leq 0, \\ 1, & x > 0. \end{cases}$

3) $\phi_3(x) = \exp(-2 * (x-1)^2)$, (named, Gaussian condition).

4) $\phi_4(x) = \begin{cases} 1, & x < 0 \\ 1-x, & 0 \leq x < 1 \\ 0, & x > 1 \end{cases}$

For some values of α, we fix the values β and γ and calculate respectively the norms $||A||_\infty$, $||A||_1$ and $||A||_2$ as shown in the tables below

The values			Centered explicit scheme			Forward scheme		
α	β	γ	$\lVert A\rVert_\infty$	$\lVert A\rVert_1$	$\lVert A\rVert_2$	$\lVert A\rVert_\infty$	$\lVert A\rVert_1$	$\lVert A\rVert_2$
0.5	0.06	0.3	1.0398	1.0398	1.0000	1.3050	1.6101	1.3361
0.01	0.06	0.3	1.0004	1.0004	0.9999	0.9111	1.0388	0.9177
0.001	0.06	0.3	1	1	0.9999	0.9038	1.0388	0.9114

We remark that, in the table above for the different values of α, the norms $||.||_\infty$and $||.||_1$ have exactly the same values for the explicit centered scheme. For the forward scheme, when α decreases the norms $||.||_\infty$ and $||.||_2$ are lower than 1, *i.e.*, there is more stability.

In the following case, we fix α ($\alpha = 0.1$) and we vary β and γ.

The values			Centered explicit scheme			Forward scheme		
α	β	γ	$\lVert A\rVert_\infty$	$\lVert A\rVert_1$	$\lVert A\rVert_2$	$\lVert A\rVert_\infty$	$\lVert A\rVert_1$	$\lVert A\rVert_2$
0.1	0.06	0.03	1.0076	1.0076	0.9999	0.983	1.3878	1.0242
0.1	0.6	0.3	1.0041	1.0041	0.9999	0.983	1.0388	0.9843
0.1	3	0.6	1	1	0.9999	0.983	1.0388	0.9155

We remark that, in the table above for the different values of β, the norms $||.||_\infty$and $||.||_1$ have exactly the same values for the explicit centered scheme. For the forward scheme when β increases, the norms $||.||_\infty$ is lower than 1.

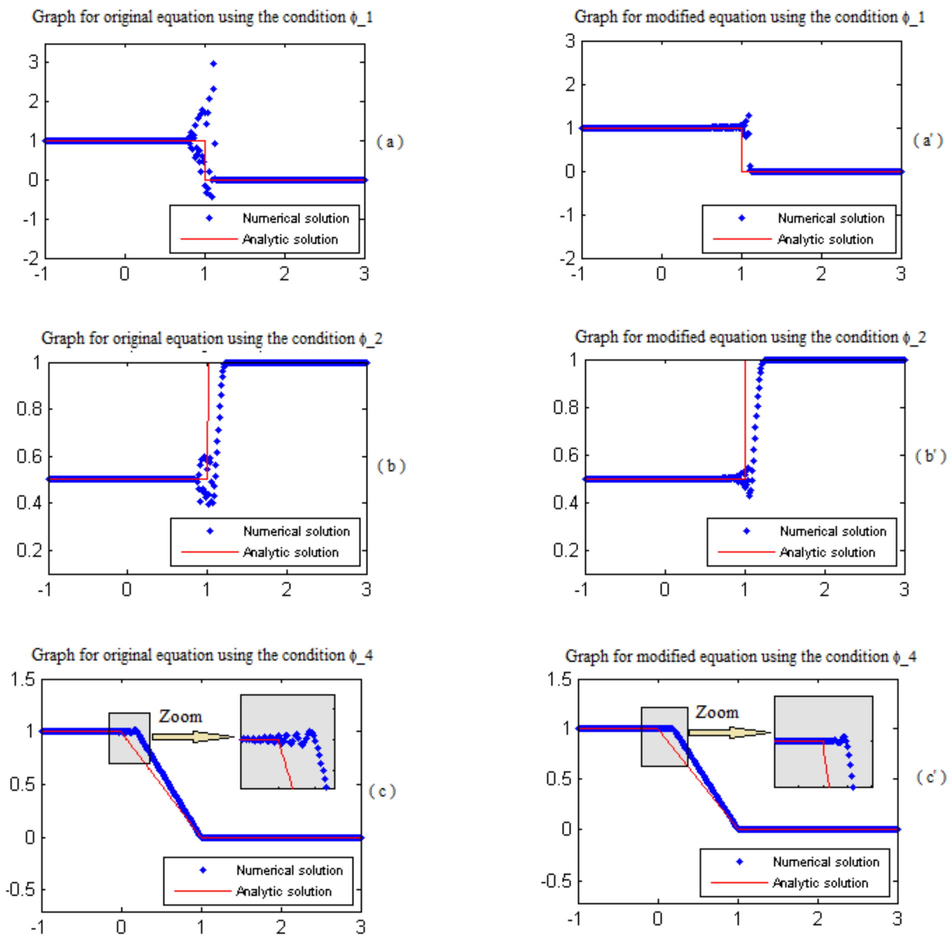

FIGURE 3. Graphs for explicit centered scheme (2.4) and the modified scheme (2.13)

Using the initial conditions 1, 2 and 4, the graphs (a'), (b') and (c ') show that there is a reduction in the number of shock points around of the discontinuity points compared to those of the respective graphs (a), (b) and (c), of the original equation.

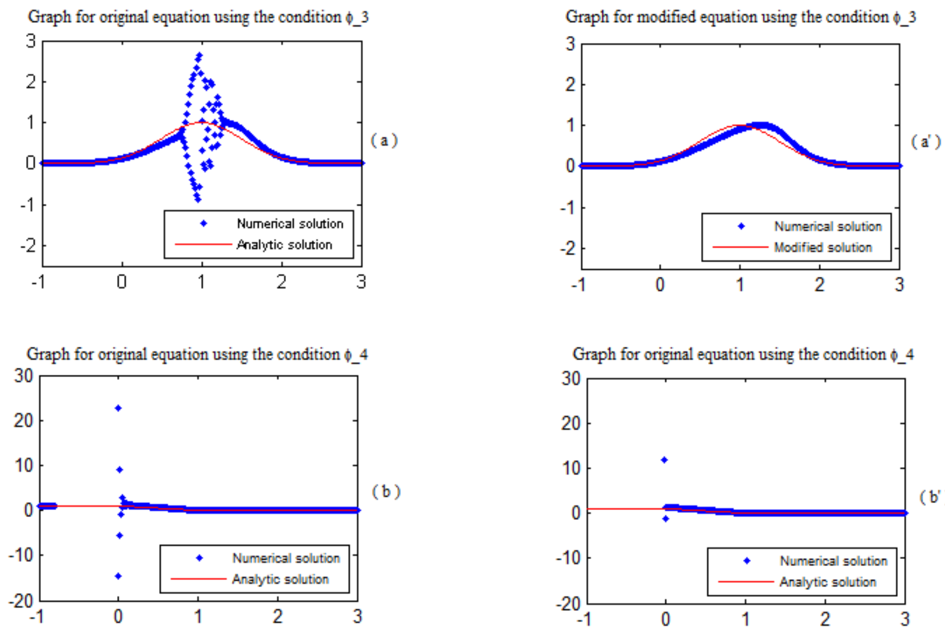

FIGURE 4. Graphs for forward scheme (2.15) and the modified scheme (2.20)

Using the initial condition 3, the graph (a') shows that there is elimination of shock points around the discontinuity points compared to those of the graph (a) of the original equation. The graph (b') shows that there is a reduction in the number of shock points around the points of discontinuity compared to those of the graph (b) of the original equation.

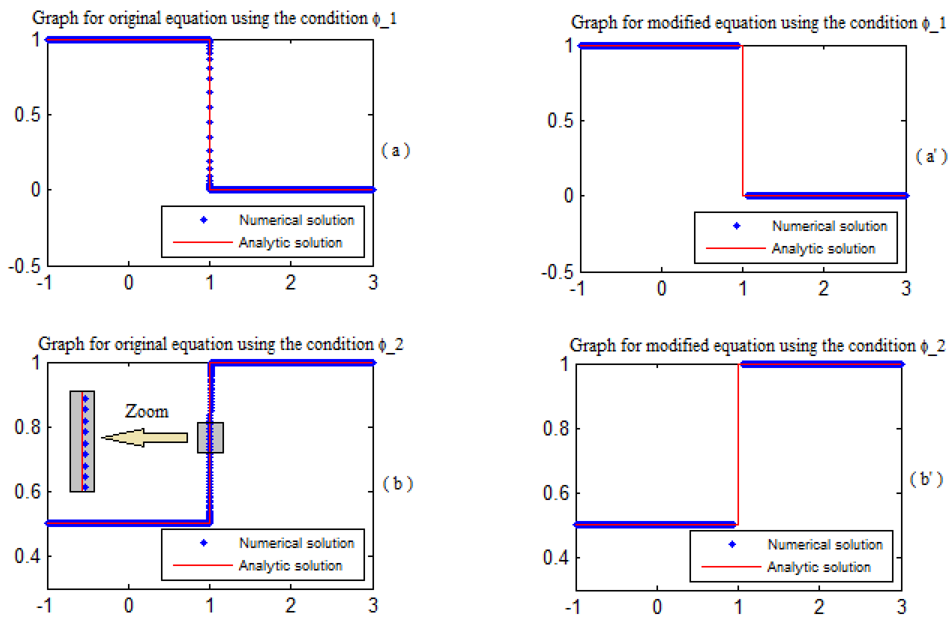

FIGURE 5. Graphs for Lax-Friedrichs scheme (2.26) and the modified scheme (2.31)

Using the initial condition 1 and 2, the graphs (a') and (b') show that there is no improvement around the discontinuity points compared to those of the respective graphs (a) and (b) of the original equation.

4. Remarks and conclusions

The mathematical theory of weak solutions for hyperbolic equations is a relatively recent development. The example of Burgers' equation may be a simple analogue of Euler equations for the flow of inviscid fluid and is a prototype for conservation equations that can develop discontinuities (shocks wave). In this study, we were interested in the effect of the nonlinear term which possesses some proprieties making Burgers equation a proper model for testing numerical algorithms in flows where severe gradients or shocks are anticipated (see [1,21]). A weak solution of Eq. (1.3) is a genuine solution except along a surface in (x, t) space, across which the function u may be discontinuous. Comparing the analytical solution and numerical solutions, the modified equation can give more accurate results and reduce the oscillations at points of discontinuities. The right-moving discontinuity is correctly positioned and is sharply defined. Even though the method uses central differences, some asymmetry will occur, since the wave is moving. For future work, we intend to apply this approach to Burgers-Fischer and Burgers-Huxley $2D$ equations.

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