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FIXED POINTS FOR CYCLICALLY SET-VALUED MAPPINGS AND APPLICATIONS FOR VARIATIONAL RELATIONS PROBLEMS

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Abstract. Fixed point theorems for cyclic maps were investigated with different conditions on space and the contraction of mappings. In this paper, we shall give some results above the fixed point theorem for cyclic set valued mappings. Also we applied our results to prove the existence solution for variational relations problems.

1. Introduction and preliminaries

In past five decades, issues related to fixed point theory increasingly occupied a central role in the study of nonlinear phenomena. In fact, one of the most important reasons to expanding of this theory is its wide-ranging applications in physics([6],[39]), game theory([36],[38]), equilibrium point in the economic model of supply and demand([26],[34]), optimization([13],[14]), control theory([8],[28]), biology([4],[12]), medical science([17]) and computer science([1],[3]). It is well known that a fundamental result of this theory is the Banach contraction principle ([11]). A number of generalizations of that theorem have appeared in [18], [23], [42], [43].

Nadler [35] proved an extension of Banach fixed point theorem for set-valued contraction map. Further Smithson [44] proved a fixed point theorem for set-valued contractive map. A number of generalizations of that theorem have appeared in [15], [20], [21], [33], [37], [41] , [45]. To familiarize the applications of the fixed point theorems for set-valued mappings we note that the several classes of problems like variational Inclusion Problem ([25]), equilibrium problems([5]), optimization problems ([32]) and differential inclusions ([7]), can be gather under the more general model of variational relations problems(VRP). This approach was proposed by Luc [32], and it was continued in numerous papers(see [9], [10],[27],[31],[29]). Inoan [27] proved existence of solution for (VRP) by some classical set-valued fixed point theorems.

The key feature of Bannach's fixed point and its generalizations is the mapping's

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contraction. That is, in the many of articles, for the generalizations of Banach's fixed point theorem (in the case of sel-maps and set-valued mappings) it has always been attempted to develope the contraction property. For example in [30] authors introduced the mappings of the type $f : A_i \to A_{i+1}, i = 1, 2, , p+1$, with $A_{p+1} = A_p$ and generalized Banach fixed point theorem and contractive fixed point theorems for those mappings. Later some authors ([19]) named those mappings cyclic maps. In the cyclic mappings case, contraction or contractive assumptions are restricted to pairs $(x, y) \in A_i \times A_{i+1}$. Then these mappings are extension of Banach fixed point theorem. Some other fixed point theorems for cyclic self mappings are proved in [2],[16],[19] , [40]. The most important feature in this type of generalization is that the contraction property is confined to subsets of space rather than the entire space.

In this paper we extend some fixed point theorem of cyclic mappings to set-valued mappings that we called them cyclic set-valued maps. As an application of our results, we study the existence of solutions for a general variational problem by using the fixed point results that proved in this paper.

1.1. **Some basic notations and definitions.** Here are some basic topics for the study of this article. Let (X, d) be a metric space and $A \subseteq X$ .We denote the family of all nonempty closed and bounded subsets of A by $CB(A)$. For B and C, two nonempty closed subsets of X, Hausdorff metric is defined as following:

$$H(B, C) = \max\{\sup_{x\in C} d(x, B), \sup_{y\in B} d(y, C)\}$$

An orbit ([44]) of the set-valued map $T : X \to 2^X$ at the point $x \in X$ is a sequence $\{x_n \in Tx_{n-1}\}_{n\in\mathbb{N}}$ where $x_0 = x$. we shall use $O(x)$ as a sequence and as a set as the situation demands. An orbit $O(x)$ is called a regular iff

$$d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1})$$

and

$$d(x_{n+1}, x_{n+2}) \leq H(Tx_n, Tx_{n+1})$$

Let $\{A_i\}_{i=1}^p$ be nonempty subsets of X. $T : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ said to be is a cyclic ([30]) if for each $x \in A_i$ $(1 \leq i \leq p)$ $Tx \in A_{i+1}(A_{p+1} = A_1)$. A cyclic map $T : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ said to be is a cyclic contraction if there exists $k \in [0, 1)$ such that for each $x \in A_i$ and $y \in A_{i+1}(1 \leq i \leq p,\ A_{p+1} = A_1)$ we have

$$d(Tx, Ty) \leq kd(x, y).$$

Also a cyclic map $T : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ said to be is a cyclic contractive ([30]) if for each $x \in A_i$ and $y \in A_{i+1}$ with $x \neq y(1 \leq i \leq p,\ A_{p+1} = A_1)$ we have

$$d(Tx, Ty) < d(x, y).$$

Definition 1. *Let $\{A_i\}_{i=1}^p$ be nonempty subsets of X. We call $T : \cup_{i=1}^p A_i \to \cup_{i=1}^p CB(A_i)$ to be cyclic set-valued map, if for each $x \in A_i(1 \leq i \leq p)$ we have $Tx \subseteq A_{i+1}(A_{p+1} = A_1)$.*

Definition 2. *Let $\{A_i\}_{i=1}^p$ be nonempty subsets of X. We call a cyclic set-valued map $T : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p CB(A_i)$ to be cyclic set-valued contraction, if there exists $k \in [0,1)$ such that for each $x \in A_{i+1}$ and $y \in A_i (1 \leq i \leq p,\ A_{p+1} = A_1)$*

$$H(Tx, Ty) \leq kd(x, y).$$

Definition 3. *Let $\{A_i\}_i^p$ be nonempty subsets of X. We call a cyclic set-valued map $T : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p CB(A_i)$ to be cyclic set-valued contractive, if for each $x \in A_i$ and $y \in A_{i+1}$ and $x \neq y (1 \leq i \leq p,\ A_{p+1} = A_1)$*

$$H(Tx, Ty) < d(x, y).$$

Let $Y \subseteq X$ and $T : Y \to 2^Y$ be set-valued map with nonempty values and $R(x, y)$ is a relation linking $x, y \in Y$. Variational relation problem (VRP) is defined as following:

"Find $\bar{x} \in Y$ such that $\bar{x} \in T(\bar{x})$ and $R(\bar{x}, y)$ holds for every $y \in T(\bar{x})$."

which is a particular case of the one formulated in [32]. Now for every $x \in Y$, consider the set valued mappings $\Gamma : Y \to 2^Y$ as following

$$\Gamma(x) = \{z \in Tx \quad | \quad R(z, w) \quad holds \quad for \quad every \quad w \in Tx\}.$$

If $\bar{x}$ is a fixed point of set-valued mapping Γ ($\bar{x} \in \Gamma\bar{x}$) then $\bar{x}$ is a solution of (VRP).

2. Fixed Points of Cyclic Set-Valued Mappings

Lemma 2.1. *Let (X, d) is the complete metric space and $B, C \subseteq X$. Then for $\epsilon > 0$ and $x \in B$ there exists $y \in C$ such that $d(x, y) \leq H(B, C) + \epsilon$.*

Proof. Let $x \in B$. Using the definition of the Hausdorff metric, we have:

$$d(x, C) \leq \sup_{z \in B} d(z, C) \leq H(B, C).$$

By using the definition of the infimum for $\epsilon > 0$, there exists $y \in C$ such that

$$d(x, y) \leq H(B, C) + \epsilon.$$

□

Theorem 2.2. *Let (X, d) is the complete metric space and*
(i) $\{A_i\}_{i=1}^p$ are nonempty subsets of X that at least one of which is closed;
(ii) $T : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p CB(A_i)$ is cyclic set-valued contraction;
Then T has at least a fixed point.

Proof. Let A_1 is closed and $x_0 \in A_1$. There exist $x_1 \in Tx_0$ such that

$$d(x_1, Tx_1) \leq H(Tx_0, Tx_1).$$

Using the lemma 2.1, for $\epsilon = k$, we may select $x_2 \in Tx_1$ such that:

$$d(x_1, x_2) \leq H(Tx_0, Tx_1) + k.$$

Similarly, for $\epsilon = k^2$, we may select $x_3 \in Tx_2$ such that:

$$d(x_2, x_3) \leq H(Tx_1, Tx_2) + k^2.$$

Repeating this process, for each $r \in \mathbb{N}$ and $\epsilon = k^r$, we may select a $x_{r+1} \in Tx_r$ such that:

$$d(x_r, x_{r+1}) \leq H(Tx_{r-1}, Tx_r) + k^r.$$

Then we have;

$$\begin{aligned} d(x_r, x_{r+1}) &\le H(Tx_{r-1}, Tx_r) + k^r. \\ &\le kd(x_{r-1}, dx_r) + k^r \\ &\le k[H(Tx_{r-2}, Tx_{r-1}) + k^{r-1}] + k^r. \\ &\le k^2 d(x_{r-2}, x_{r-1}) + 2k^r \\ &\le \cdots \\ &\le k^r d(x_0, x_1) + rk^r. \end{aligned}$$

Therefore

$$\sum_{r=0}^{\infty} d(x_r, x_{r+1}) \le d(x_0, x_1)(\sum_{r=0}^{\infty} k^r) + \sum_{r=0}^{\infty} rk^r < \infty.$$

Hence $\{x_n\}$ is a cauchy sequence, so there exist $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Assume that $\{x_{kp+i}\}_{k=0}^{\infty} = \{x_n\} \cap A_i$(for $1 \le i \le p$). So for each $1 \le i \le p$ we have $\lim_{k\to\infty} x_{kp+i} = x$. Specially $\lim_{k\to\infty} x_{kp+1} = x$. A_1 is closed then $x \in A_1$. By (ii) and definition of Hausdorff metric we get;

$$0 \le d(x_{kp+3}, Tx) \le H(Tx_{kp+2}, Tx) \le d(x_{kp+2}, x).$$

For $k \to \infty$ we get $d(x, Tx) = 0$. Tx is closed then $x \in Tx$. □

All assumptions of theorem 2.2 satisfy in the example 1 for $A_1 = A$, $A_2 = B$ and $p = 2$.

Example 1. *Let $A = [0, 1]$ and $B = (-1, 0]$. For every $x \in A$ define $Tx = [-\frac{x}{2}, 0]$ and for every $y \in B$ define $Ty = [0, -\frac{y}{2}]$. For each $x \in A$ and $y \in B$ we have*

$$H(Tx, Ty) = H([-\frac{x}{2}, 0], [0, -\frac{y}{2}]) = \max\{\sup_{z\in Tx} d(z, Ty), \sup_{z\in Ty} d(z, Tx)\}$$

$$= \max\{\sup_{z\in[-\frac{x}{2},0]} d(z, [0, -\frac{y}{2}]), \sup_{z\in[0,-\frac{y}{2}]} d(z, [-\frac{x}{2}, 0])\}.$$

$$= \max\{|\frac{x}{2}|, |\frac{y}{2}|\} \le ||\frac{x}{2}| + |\frac{y}{2}|| = |\frac{x}{2} - \frac{y}{2}| = \frac{1}{2}|x - y|$$

Hence $T : A \cup B \longrightarrow CB(A) \cup CB(B)$ is cyclic set-valued contraction.

Theorem 2.3. *Let (X, d) is the complete metric space and*
(i) $\{A_i\}_{i=1}^{p}$ are nonempty closed subsets of X, at least one of which is compact;
(ii) $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} CB(A_i)$ is cyclic set-valued contractive;
Then T has at least a fixed point.

Proof. Let A_1 is compact and

$$d = dist(A_1, A_p) := \inf\{d(x, y) : x \in A_1, y \in A_p\}.$$

By compactness there exist $x_1 \in A_1$ and a sequence $\{u_n\} \subseteq A_p$ such that $\lim_{n\to\infty} d(x_1, u_n) = d$. Assume $d > 0$. Then for each $n \in \mathbb{N}$

$$H(Tx_1, Tu_n) < d(x_1, u_n).$$

For each $n \in \mathbb{N}$, assume that $x_{2,n} \in Tx_1 \subseteq A_2$ and

$$d(x_{2,n}, Tu_n) \le H(Tx_1, Tu_n) < d(x_1, u_n).$$

Hence there exist $u_{n,1} \in Tu_n \subseteq A_1$ such that

$$d(x_{2,n}, u_{n,1}) < d(x_1, u_n).$$

Similarly for every $n \in \mathbb{N}$ we may select $\{u_{n,i}\}_{i=1}^{p+1}$, $\{x_{i+1,n}\}_{i=1}^{p+1}$ such that

$$d(x_{p+2,n}, u_{n,p+1}) < d(x_{p+1,n}, u_{n,p}) < \cdots < d(x_{2,n}, u_{n,1}) < d(x_1, u_n). \tag{2.1}$$

Since $\{u_{n,p+1}\}_{n=1}^{\infty} \subset A_1$ so there exist convergence subsequence $\{v_n\}_{n=1}^{\infty}$ of $\{u_{n,p+1}\}_{n=1}^{\infty}$ and $z \in A_1$ such that $v_n \to z$. By (2.1), for each $n \in \mathbb{N}$, we have

$$0 \leq d(x_{p+2,n}, u_{n,p+1}) < d(x_1, u_n). \tag{2.2}$$

Since $d(x_1, u_n)$ is convergence then it is bounded. Thus we may deduce that the nonnegative real sequence $\{d(x_{p+2,n}, v_n)\}_{n=1}^{\infty} \subseteq \{d(x_{p+2,n}, u_{n,p+1})\}_{n=1}^{\infty}$ is bounded and so it has a convergence subsequence $\{d(x_{p+2,m}, v_m)\}_{m=1}^{\infty}$. By (2.2) and triangle inequality for each $m \in \mathbb{N}$ we get:

$$d(x_{p+2,m}, z) \leq d(v_m, z) + d(x_{p+2,m}, v_m) < d(v_m, z) + d(x_1, v_m).$$

Hence $\{d(x_{p+2,m}, z)\}_{m=1}^{\infty}$ is convergence and since $v_n \to z$, $d(x_1, v_n) \to d$ we get

$$0 \leq \lim_{m \to \infty} d(x_{p+2,m}, z) \leq d.$$

We have $\{x_{p+2,m}\}_{m=1}^{\infty} \subset A_2$. So we may select $t \in A_2$ such that

$$0 \leq d(t, z) \leq d.$$

Since $z \in A_1$ and $t \in A_2$ we have

$$H(Tt, Tz) < d(t, z).$$

However this implies

$$d(t_{p-1}, z_{p-1}) < d,$$

since $z_{p-1} \in A_p$, $t_{p-1} \in A_1$ we get a contradiction. Then we have $d = 0$ and $A_1 \cap A_p \neq \emptyset$. If $x \in A_1 \cap A_p$ we have $Tx \subseteq A_1$ and $Tx \subseteq A_2$. Thus $A_1 \cap A_2 \neq \emptyset$. Now we consider sets

$$A'_1 = A_1 \cap A_2, A'_2 = A_2 \cap A_3, \cdots, A'_p = A_p \cap A_1.$$

In view of cyclic property these sets are all nonempty (and closed) and A'_1 is compact. Thus the assumptions $(i) - (ii)$ of the theorem is satisfy for family $\{A'_i\}_{i=1}^{p}$ and T. By repeating the argument just given we conclude that

$$A'_1 \cap A'_p \neq \emptyset.$$

This is turn implies $A_1 \cap A_2 \cap A_3 \neq \emptyset$. Continuing step-by-step we conclude that

$$A := \cap_{i=1}^{p} A_i \neq \emptyset.$$

If $x \in A$ then for $1 \leq i \leq p$ and $x \in A_i$ we have $Tx \subseteq A_i$. Hence $T : A \to CB(A)$ is contractive set-valued map. Let $y_0 \in A$ and there exists $y_1 \in Ty_0$ such that

$$d(y_1, Ty_1) \leq H(Ty_0, Ty_1) < d(y_0, y_1).$$

Thus there exists $y_2 \in Ty_1$ such that

$$d(y_1, y_2) < d(y_0, y_1).$$

and

$$d(y_1, y_2) < H(Ty_0, Ty_1)$$

By repeating the argument just given we may select $y_{n+1} \in Ty_n$ such that

$$d(y_{n+1}, y_{n+2}) < d(y_n, y_{n+1}),$$

$$d(y_{n+1}, y_{n+2}) < H(Ty_n, Ty_{n+1})$$

Thus $O(y) = \{y_n\}_{n=1}^{\infty}$ is regular orbit and $O(y) \subseteq A$. Since A is compact, $\{y_n\}_{n=1}^{\infty}$ has a convergence subsequence and by the theorem (1) of [44] T has a fixed point. □

Example 2. *Let* $(\mathbb{R}^2, ||.||)$ *is Euclidean metric space and*

$$A_1 = \{(0, \frac{1}{2n})|n \in \mathbb{N}\}, \quad A_2 = \{(0, \frac{1}{2n+1})|n \in \mathbb{N}\} \cup \{(0,0)\},$$

$$B_1 = \{(\frac{-1}{n}, \frac{-1}{n})|n \in \mathbb{N}\}, \quad B_2 = \{(\frac{1}{n}, \frac{-1}{n})|n \in \mathbb{N}\} \cup \{(0,0)\}.$$

Set $A = A_1 \cup A_2$, $B = B_1 \cup B_2$ *and define* $T : A \cup B \longrightarrow A \cup B$ *such that :*

$$T(0, \frac{1}{2n}) = \frac{1}{2}(\frac{-1}{2n}, \frac{-1}{2n}), \quad T(0, \frac{1}{2n+1}) = \frac{1}{2}(\frac{1}{2n+1}, \frac{-1}{2n+1}), \quad T(0,0) = (0,0)$$

and

$$T(\frac{-1}{m}, \frac{-1}{m}) = (0, \frac{1}{2m}), \quad T(\frac{1}{m}, \frac{-1}{m}) = (0, \frac{1}{2m+1}), \quad T(0,0) = (0,0).$$

It is easy to seen that T *is cyclic map. Now we prove that,* $T : A \cup B \longrightarrow A \cup B$ *is contractive cyclic map.*
For any $x \in A_1$ *and* $y \in B_1$ *we have:*

$$||T(0, \frac{1}{2n}) - T(\frac{-1}{m}, \frac{-1}{m})|| = \sqrt{(\frac{1}{4n})^2 + (\frac{1}{4n} + \frac{1}{2m})^2},$$

$$||(0, \frac{1}{2n}) - (\frac{-1}{m}, \frac{-1}{m})|| = \sqrt{(\frac{1}{m})^2 + (\frac{1}{2n} + \frac{1}{m})^2}.$$

Hence

$$||T(0, \frac{1}{2n}) - T(\frac{-1}{m}, \frac{-1}{m})|| < ||(0, \frac{1}{2n}) - (\frac{-1}{m}, \frac{-1}{m})||$$

iff

$$\sqrt{(\frac{1}{4n})^2 + (\frac{1}{4n} + \frac{1}{2m})^2} < \sqrt{(\frac{1}{m})^2 + (\frac{1}{2n} + \frac{1}{m})^2}.$$

Last inequality satisfies for each $m, n \in \mathbb{N}$. *For any* $x \in A_1$ *and* $y \in B_2$ *such that* $y \neq (0,0)$ *we have:*

$$||T(0, \frac{1}{2n}) - T(\frac{1}{m}, \frac{-1}{m})|| = \sqrt{(\frac{1}{4n})^2 + (\frac{1}{4n} + \frac{1}{2m+1})^2},$$

$$||(0, \frac{1}{2n}) - (\frac{1}{m}, \frac{-1}{m})|| = \sqrt{(\frac{1}{m})^2 + (\frac{1}{2n} + \frac{1}{m})^2}.$$

Hence

$$||T(0, \frac{1}{2n}) - T(\frac{1}{m}, \frac{-1}{m})|| < ||(0, \frac{1}{2n}) - (\frac{1}{m}, \frac{-1}{m})||$$

iff

$$\sqrt{(\frac{1}{4n})^2 + (\frac{1}{4n} + \frac{1}{2m+1})^2} < \sqrt{(\frac{1}{m})^2 + (\frac{1}{2n} + \frac{1}{m})^2}.$$

Last inequality satisfies for each $m, n \in \mathbb{N}$. *For any* $x \in A_2$ *such that* $x \neq (0,0)$ *and* $y \in B_1$ *we have:*

$$||T(0, \frac{1}{2n+1}) - T(\frac{-1}{m}, \frac{-1}{m})|| = \sqrt{(\frac{1}{2(2n+1)})^2 + (\frac{1}{2(2n+1)} + \frac{1}{2m})^2},$$

$$||(0,\frac{1}{2n+1})-(\frac{-1}{m},\frac{-1}{m})||=\sqrt{(\frac{1}{m})^2+(\frac{1}{2n+1}+\frac{1}{m})^2}.$$

Hence

$$||T(0,\frac{1}{2n+1})-T(\frac{-1}{m},\frac{-1}{m})||<||(0,\frac{1}{2n+1})-(\frac{-1}{m},\frac{-1}{m})||$$

iff

$$\frac{1}{4(2n+1)^2}+\frac{1}{4(2n+1)^2}+\frac{1}{4m^2}+\frac{1}{2m(2n+1)}<$$
$$\frac{1}{m^2}+\frac{1}{(2n+1)^2}+\frac{1}{m^2}+\frac{1}{m(2n+1)}.$$

Last inequality satisfies for each $m,n\in\mathbb{N}$. For any $x\in A_2$ and $y\in B_2$ such that $x,y\neq(0,0)$ we have:

$$||T(0,\frac{1}{2n+1})-T(\frac{1}{m},\frac{-1}{m})||=\sqrt{\frac{1}{4(2n+1)^2}+(\frac{1}{2(2n+1)}+\frac{1}{2m+1})^2},$$

$$||(0,\frac{1}{2n+1})-(\frac{1}{m},\frac{-1}{m})||=\sqrt{(\frac{1}{m})^2+(\frac{1}{2n+1}+\frac{1}{m})^2}.$$

Hence

$$||T(0,\frac{1}{2n+1})-T(\frac{1}{m},\frac{-1}{m})||<||(0,\frac{1}{2n+1})-(\frac{1}{m},\frac{-1}{m})||$$

iff

$$\frac{1}{4(2n+1)^2}+\frac{1}{4(2n+1)^2}+\frac{1}{(2m+1)^2}+\frac{1}{(2m+1)(2n+1)}<\frac{1}{m^2}+\frac{1}{(2n+1)^2}+\frac{1}{m^2}+\frac{2}{m(2n+1)}.$$

Last inequality satisfies for each $m,n\in\mathbb{N}$. For any $x\in A_1$ and $(0,0)\in B_2$ we have:

$$||T(0,\frac{1}{2n})-T(0,0)||=\frac{\sqrt{2}}{4n},\quad ||(0,\frac{1}{2n})-(0,0)||=\frac{1}{2n}.$$

Hence

$$||T(0,\frac{1}{2n})-T(0,0)||<||(0,\frac{1}{2n})-(0,0)||.$$

Last inequality satisfies for each $n\in\mathbb{N}$. For any $x\in A_2$ ($x\neq(0,0)$) and $(0,0)\in B_2$ we have:

$$||T(0,\frac{1}{2n+1})-T(0,0)||=\frac{1}{2}\frac{\sqrt{2}}{2n+1},\quad ||(0,\frac{1}{2n+1})-(0,0)||=\frac{1}{2n+1}.$$

Hence

$$||T(0,\frac{1}{2n+1})-T(0,0)||<||(0,\frac{1}{2n+1})-(0,0)||\Leftrightarrow\frac{1}{2}\frac{\sqrt{2}}{2n+1}<\frac{1}{2n+1}.$$

Last inequality satisfies for each $n\in\mathbb{N}$. For any $y\in B$ and $(0,0)\in A_2$ we have:

$$||T(\frac{-1}{m},\frac{-1}{m})-T(0,0)||=\frac{1}{2m},\quad ||(\frac{-1}{m},\frac{-1}{m})-(0,0)||=\frac{\sqrt{2}}{m}.$$

Hence

$$||T(\frac{-1}{m},\frac{-1}{m})-T(0,0)||<||(\frac{-1}{m},\frac{-1}{m})-(0,0)||\Leftrightarrow\frac{1}{2m}<\frac{\sqrt{2}}{m}.$$

Last inequality satisfies for each $m\in\mathbb{N}$. For any $y\in B_2$ ($y\neq(0,0)$) and $(0,0)\in A_2$ we have:

$$||T(\frac{1}{m},\frac{-1}{m})-T(0,0)||=\frac{1}{2m+1},\quad ||(\frac{1}{m},\frac{-1}{m})-(0,0)||=\frac{\sqrt{2}}{m}.$$

Hence

$$||T(\frac{1}{m},\frac{-1}{m})-T(0,0)||<||(\frac{1}{m},\frac{-1}{m})-(0,0)||\Leftrightarrow\frac{1}{2m+1}<\frac{\sqrt{2}}{m}$$

Last inequality satisfies for each $m\in\mathbb{N}$. So for any $x\in A, y\in B(x\neq y)$ we have

$$||Tx-Ty||<||x-y||.$$

Hence $T: A\cup B\longrightarrow A\cup B$ is contractive cyclic map. But T is not contractive map. To prove this, let $n\in\mathbb{N}$. We have:

$$||T(0,\frac{1}{2n})-T(0,\frac{1}{2n+1})||=\frac{1}{2}\sqrt{(\frac{1}{2n}+\frac{1}{2n+1})^2+(\frac{1}{2n}-\frac{1}{2n+1})^2},$$

$$||(0,\frac{1}{2n})-(0,\frac{1}{2n+1})||=\sqrt{(\frac{1}{2n}-\frac{1}{2n+1})^2}.$$

Hence

$$||T(0,\frac{1}{2n})-T(0,\frac{1}{2n+1})||>||(0,\frac{1}{2n})-(0,\frac{1}{2n+1})||$$

iff

$$\frac{1}{4}(\frac{1}{4n^2}+\frac{1}{(2n+1)^2}+\frac{1}{n(2n+1)}+\frac{1}{4n^2}+\frac{1}{(2n+1)^2}-\frac{1}{n(2n+1)})>$$
$$\frac{1}{4n^2}+\frac{1}{(2n+1)^2}-\frac{1}{n(2n+1)}.$$

By multiplying both sides of the last inequality to $2n^2(2n+1)^2$ we have:

$$8n(2n+1)>(2n+1)^2+4n^2$$

Last inequality satisfies for each $n\in\mathbb{N}$.

Corollary 2.4. *It is easy to check that the map $T: A\cup B\longrightarrow CB(A)\cup CB(B)$, such that A and B is the sets defined in example 2 and*

$$T(0,\frac{1}{2n})=\{\frac{1}{2}(\frac{-1}{2n},\frac{-1}{2n})\},\quad T(0,\frac{1}{2n+1})=\{\frac{1}{2}(\frac{1}{2n+1},\frac{-1}{2n+1})\},$$
$$T(0,0)=\{(0,0)\},$$
$$T(\frac{-1}{m},\frac{-1}{m})=\{(0,\frac{1}{2m})\},\quad T(\frac{1}{m},\frac{-1}{m})=\{(0,\frac{1}{2m+1})\},\quad T(0,0)=\{(0,0)\}.$$

is cyclic set-valued contractive map but it is not contractive set-valued map.

3. Existence results for variational relations problems

Theorem 3.1. *Let (X,d) is the complete metric space and*
(C1) $\{A_i\}_{i=1}^p$ are nonempty subsets of X, at least one of which is closed;
(C2) for every $x\in\cup_{i=1}^p A_i$, Γx is nonempty,
(C3) There exists $q\in(0,1]$ such that for every $x_1\in A_i, x_2\in A_{i+1}$ if $z_1\in\Gamma x_1$ then there exists $z_2\in\Gamma x_2$ such that

$$d(z_1,z_2)\leq qH(Tx,Ty),$$

(C4) The Relation R is closed in the first variable; that is, for any $y\in\cup_{i=1}^p A_i$ fixed, if $\{z_n\}\subset\cup_{i=1}^p A_i$ is a sequence with $z_n\to z$ and $R(z_n,y)$ holds for any $n\in\mathbb{N}$, then $R(z,y)$ holds too,
(C5) $T:\cup_{i=1}^p A_i\to\cup_{i=1}^p CB(A_i)$ be cyclic set-valued contraction.
Then the variational relation problem (VRP) has at least a solution.

Proof. By (C5) , (C2) and (C4) we can follows that for each fixed $x \in X$, Γx is nonempty and closed and $\Gamma : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} CB(A_i)$ is cyclic set-valued mappings. Let $x_1 \in A_i, x_2 \in A_{i+1}$ and $z_1 \in \Gamma x_1$. Then according (C3) there exists $z_2 \in \Gamma x_2$ such that:

$$d(z_1, z_2) \leq qH(Tx_1, Tx_2).$$

In the same way, starting from any $z_2 \in \Gamma x_2$ there exists $z_1 \in \Gamma x_1$ for which the previous inequality holds. Since T is set-valued contraction, It follow that:

$$H(\Gamma x_1, \Gamma x_2) \leq qH(Tx_1, Tx_2) \leq qkd(x_1, x_2).$$

Hence Γ is cyclic set-valued contraction. Thus Γ satisfies in theorem 2.2 and admits a fixed point. Therefore the (VRP) admits a solution. □

Example 3. *Let $A = [0, 1]$ and $B = (-1, 0]$. For any $x \in A$ define $Tx = [-\frac{x}{2}, 0]$ and for any $y \in B$ define $Ty = [0, -\frac{y}{2}]$. By example 1 it follows that $T : A \cup B \longrightarrow CB(A) \cup CB(B)$ is cyclic set-valued contraction. Now consider the problem:*

"Find $\bar{x}$ such that $\bar{x} \in T\bar{x}$ and $3|y| \leq 4|\bar{x}|$ for any $y \in T\bar{x}$. (1)"

For any $x \in [0, 1)$ and $y \in (-1, 0]$, we have $\Gamma x = [-\frac{1}{2}x, -\frac{3}{8}x]$ and $\Gamma y = [\frac{3}{8}|y|, \frac{1}{2}|y|]$. Then if $x \in [0, 1)$ and $y \in (-1, 0]$ we get

$$H(\Gamma x, \Gamma y) = \max\{\frac{1}{2}|x - y|, \frac{3}{8}|x - y|\} = \frac{1}{2}|x - y|.$$

Hence $\Gamma : A \cup B \longrightarrow CB(A) \cup CB(B)$ is cyclic set-valued contraction. Also relation $R(z, w) \iff 3|w| \leq 4|z|$ satisfy (C4) of theorem 3.1. Thus (VRP) (1) admits a solution.

4. Conclusion

The results in this paper,

1. extend the work of Eldered and Veeramani ([19]) from a single valued maps to set-valued maps.

1. extend the work of Kirk, Srinivasan and Veeramani ([30]) from a single valued maps to set-valued maps.

3. show that the category of cyclic contractive mappings is not equal to the category of contractive mappings by example 2.

Remark. *One of the closest development to our results is the investigation of the fixed point theorem to cyclic φ-contractions([2]) in the case of set-valued mappings. Other useful development is the investigation of the best proximity point theorem for cyclic contraction and contractive set-valued mappings. Furthermore it seems that, it is interesting question that, "is the category of cyclic contractive mappings is not equal to the category of contractive mappings in the metric like spaces same as the generalized metric space([3]) or $G-$metric space([22])"?*

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