Journal of Mathematical Analysis
ISSN: 2217-3412, URL: www.ilirias.com/jma
Volume 11 Issue 1 (2020), Pages 17-26.

THE ANALYTIC SOLUTION OF SEQUENTIAL SPACE-TIME FRACTIONAL DIFFUSION EQUATION INCLUDING PERIODIC BOUNDARY CONDITIONS

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Abstract. In this paper, the analytic solution of sequential space-time fractional differential equation with periodic boundary conditions in one dimension is established. It is constructed in the form of a Fourier series by taking the eigenfunctions of Sturm-Liouville problem with fractional derivative in Caputo sense into account.

1. Introduction

As partial differential equations (PDEs) of fractional order plays an important role in modelling for the numerous processes and systems in various scientifc research areas such as applied mathematics, physics chemistry etc., the interest of this topic is increasing enourmously. Since the fractional derivative is non-local, the model with fractional derivative for physical problems turns out to be the best choice to analyze the behaviour of the complex non linear processes. That is why it attracts increasing number of researchers. The derivative in the sense of Caputo is one of the most common one since mathematical models with Caputo derivatives gives better results compare to the analysis of ones including other fractional derivatives. In literature increasing number of studies can be found supporting this conclusion [1],[2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [15], [16], [17], [18], [19], [22], [23]. Moreover the Caputo derivative of constant is zero which is not hold by many fractional derivatives. The solutions of fractional PDEs and ordinary differential equations (ODEs) are determined in terms of Mittag-Leffler function. The diffusion problem including fractional derivative in Caputo sense has been studied by Sevindir and Demir [21]. This study can be regarded as an extension of it.

2. Preliminary Results

In this section, we recall fundamental definition and well known results about fractional derivative in Caputo sense.

Definition 1. The q^{th} order fractional derivative of $u(t)$ in Caputo sense is defined as

2000 *Mathematics Subject Classification.* 26A33, 65M70.
Key words and phrases. Caputo fractional derivative; Space-fractional diffusion equation; Mittag-Leffler function; Periodic boundary conditions; Spectral method.

Submitted September 12, 2019. Published November 3, 2019.
Communicated by Cemil Tunc.

$$D^q u(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^{t} (t-s)^{n-q-1} u^{(n)}(s) ds, t \in [t_0, t_0+T] \tag{2.1}$$

where $u^{(n)}(t) = \frac{d^n u}{dt^n}$, $n-1 < q < n$. Note that Caputo fractional derivative is equal to integer order derivative when the order of the derivative is integer [14], [20].

Definition 2. The q^{th} order Caputo fractional derivative for $0 < q < 1$ is defined in the following form [14], [20]:

$$D^q u(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} u'(s) ds, t \in [t_0, t_0+T] \tag{2.2}$$

The Mittag–Leffler function with two-parameters which is taken into account in eigenvalue problem, is given by

$$E_{\alpha,\beta}\left(\lambda(t-t_0)^{\alpha}\right) = \sum_{k=0}^{\infty} \frac{\left(\lambda(t-t_0)^{\alpha}\right)^k}{\Gamma(\alpha k+\beta)}, \alpha, \beta > 0 \tag{2.3}$$

including constant λ. Especially, for $t_0 = 0$, $\alpha = \beta = q$ we have

$$E_{\alpha,\beta}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma(qk+q)}, q > 0. \tag{2.4}$$

Mittag–Leffler function coincides with exponential function i.e., $E_{1,1}(\lambda t) = e^{\lambda t}$ for $q = 1$. For details see [14], [20].

The following significant functions are defined in terms of the Mittag–Leffler function with two parameters as

$$\sin_{\mathrm{q}}(\mu t^q) = \frac{E_{q,1}(i\mu t^q) - E_{q,1}(-i\mu t^q)}{2i} = \sum_{k=0}^{\infty} \frac{(-1)^k (\mu t^q)^{2k+1}}{\Gamma((2k+1)q+1)} \tag{2.5}$$

and

$$\cos_{\mathrm{q}}(\mu t^q) = \frac{E_{q,1}(i\mu t^q) + E_{q,1}(-i\mu t^q)}{2} = \sum_{k=0}^{\infty} \frac{(-1)^k (\mu t^q)^{2k}}{\Gamma(2kq+1)} \tag{2.6}$$

Note that for $q = 1$ these functions are usual trigonometric functions $\sin(\mu t)$ and $\cos(\mu t)$.

The main goal of this study is to establish the analytic solution of following sequential space-time fractional differential equations with periodic boundary and initial condition.

$$D_t^{\alpha} u(x,t) = \gamma^2 D_x^{2\beta} u(x,t), \tag{2.7}$$

$$\begin{cases} u(-l,t) = u(l,t) \\ D_x^{\beta} u(-l,t) = D_x^{\beta} u(l,t) \end{cases} \tag{2.8}$$

$$u(x,0) = f(x) \tag{2.9}$$

where $0 < \alpha < 1,\ 1 < 2\beta < 2, -l \le x \le l,\ 0 \le t \le T, \gamma \in \mathbb{R}$.

2.1. **Inner Product.** Let the vector space V be defined as span of the functions $\sin_\beta\left(\mu(\frac{x}{b-a})^\beta\right)$ and $\cos_\beta\left(\mu(\frac{x}{b-a})^\beta\right)$ for fixed β where $0<\beta\leq 1$, $\mu\in\mathbb{R}$ on the interval $I=[a,b]$, i.e., $V=span\{\sin_\beta\left(\mu(\frac{x}{b-a})^\beta\right), \cos_\beta\left(\mu(\frac{x}{b-a})^\beta\right)\}$. As a result the linear transformation $T:V\to span\{\sin\left(\frac{\mu x}{b-a}\right), \cos(\frac{\mu x}{b-a})\}$ becomes one-to-one and onto. Hence its inverse transformation T^{-1} exists. The mapping $<\bullet,\ \bullet>: V\times V\to\mathbb{R}$ is defined as

$$< u\left(x;\beta\right),\ v\left(x;\beta\right) > = T^{-1}(\int Tu\left(x;\beta\right).Tv\left(x;\beta\right)dx)|_{x=a}^{b} \tag{2.10}$$

where $Tu\left(x;\beta\right)=u(x;1)$ and $Tv\left(x;\beta\right)=v(x;1)$ [21].

3. Main Results

By means of separation of variables method, The generalized solution of above problem is constructed in analytical form. Thus a solution of problem (2.7)-(2.9) have the following form:

$$u\left(x,t;\alpha,\beta\right)=X(x;\beta)\ T(t;\alpha,\beta) \tag{3.1}$$

where $-l\leq x\leq l,\ 0\leq t\leq T$.

Plugging (3.1) into (2.7) and arranging it, we have

$$\frac{D_t^\alpha\left(T\left(t;\alpha,\beta\right)\right)}{T\left(t;\alpha,\beta\right)}=\gamma^2\frac{D_x^{2\beta}\left(X\left(x;\beta\right)\right)}{X\left(x;\beta\right)}=-\lambda^2(\beta) \tag{3.2}$$

Note that the value of λ varies based on β. Equation (3.2) produces two fractional differential equations with respect to time and space. The first fractional differential equation is obtained by taking the equation on the right hand side of Eq. (3.2). Hence with boundary conditions (2.8), we have the following problem

$$D_x^{2\beta}\left(X\left(x;\beta\right)\right)+\lambda^2\left(\beta\right)X\left(x;\beta\right)=0 \tag{3.3}$$

$$\begin{cases} X\left(-l\right)=X\left(l\right) \\ X'\left(-l\right)=X'\left(l\right) \end{cases} \tag{3.4}$$

The solution of eigenvalue problem (3.3)-(3.4) is accomplished by making use of the Mittag-Lefffer function of the following form:

$$X\left(x;\beta\right)=E_{\beta,1}(rx^\beta) \tag{3.5}$$

Hence the characteristic equation is computed in the following form:

$$r^2+\lambda^2(\beta)=0 \tag{3.6}$$

Case 1. If $\lambda(\beta)=0$, the characteristic equation have two coincident roots $r_1=r_2$, leading to the general solution of the eigenvalue problem (3.3)-(3.4) having the following form:

$$X\left(x;\beta\right)=k_1\frac{x^\beta}{\beta}+k_2 \tag{3.7}$$

$$D_x^\beta X\left(x,\beta\right)=k_1\Gamma(\beta) \tag{3.8}$$

The first boundary condition yields

$$X(-l) = -k_1\frac{l^\beta}{\beta} + k_2 = k_1\frac{l^\beta}{\beta} + k_2 = X(l) \Rightarrow k_1 = 0 \tag{3.9}$$

which leads to the following solution

$$X(x;\beta) = k_2 \tag{3.10}$$

Similarly second boundary condition leads to

$$D_x^\beta X(-l) = 0 = D_x^\beta X(l) \tag{3.11}$$

The representation of the solution is established as

$$X_0(x) = k_2 \tag{3.12}$$

Case 2. If $\lambda(\beta) > 0$, the Eq. (3.6) have two distinct real roots $r_1,\ r_2$ yielding the general solution of the problem (3.3)-(3.4) in the following form:

$$X(x;\beta) = c_1 E_{\beta,1}\left(r_1 x^\beta\right) + c_2 E_{\beta,1}(r_2 x^\beta) \tag{3.13}$$

By making use of the first boundary condition, we have

$$X(-l) = c_1 E_{\beta,1}\left(r_1(-l)^\beta\right) + c_2 E_{\beta,1}\left(r_2(-l)^\beta\right) = c_1 E_{\beta,1}\left(r_1 l^\beta\right) + c_2 E_{\beta,1}\left(r_2 l^\beta\right) = X(l) \tag{3.14}$$

$$c_1\left(E_{\beta,1}\left(r_1(-l)^\beta\right) - E_{\beta,1}\left(r_1 l^\beta\right)\right) + c_2\left(E_{\beta,1}\left(r_2(-l)^\beta\right) - E_{\beta,1}\left(r_2 l^\beta\right)\right) = 0 \tag{3.15}$$

Since $E_{\beta,1}\left(r_1(-l)^\beta\right) - E_{\beta,1}\left(r_1 l^\beta\right)$ and $E_{\beta,1}\left(r_2(-l)^\beta\right) - E_{\beta,1}\left(r_2 l^\beta\right)$ are linearly independent the equation (3.15) is satisfied if and only if $c_1 = 0 = c_2$ which implies that $X(x;\beta) = 0$ which implies that there is not any solution for $\lambda(\beta) > 0$.

Case 3. If $\lambda(\beta) < 0$, the characteristic equation have two complex roots yielding the general solution of the problem (3.3)-(3.4) in the following form:

$$X(x;\beta) = c_1 \cos_\beta\left(\lambda(\beta) x^\beta\right) + c_2 sin_\beta(\lambda(\beta) x^\beta) \tag{3.16}$$

By making use of the first boundary condition we have

$$X(-l) = c_1 \cos_\beta\left(\lambda(\beta) l^\beta\right) + c_2 sin_\beta(-\lambda(\beta) l^\beta) = c_1 \cos_\beta\left(\lambda(\beta) l^\beta\right) + c_2 sin_\beta(\lambda(\beta) l^\beta) = X(l) \tag{3.17}$$

which implies that

$$\Rightarrow 2c_2 sin_\beta\left(\lambda(\beta) l^\beta\right) = 0 \Rightarrow c_2 = 0 \tag{3.18}$$

Hence the solution becomes

$$X(x;\beta) = c_1 \cos_\beta(\lambda(\beta) x^\beta) \tag{3.19}$$

$$D_x^\beta X(x,\beta) = -c_1 \lambda(\beta) sin_\beta\left(\lambda(\beta) x^\beta\right) \tag{3.20}$$

Similarly last boundary condition leads to

$$D_x^{\beta} X(-l) = -c_1 \lambda(\beta)\, sin_{\beta}\left(-\lambda(\beta)\, l^{\beta}\right) = -c_1 \lambda(\beta)\, sin_{\beta}\left(\lambda(\beta)\, l^{\beta}\right) = D_x^{\beta} X(l) \tag{3.21}$$

$$\Rightarrow 2c_1 \lambda(\beta)\, sin_{\beta}\left(\lambda(\beta)\, l^{\beta}\right) = 0 \tag{3.22}$$

which implies that

$$sin_{\beta}\left(\lambda(\beta)\, l^{\beta}\right) = 0 \tag{3.23}$$

which yields the following eigenvalues

$$\lambda_n(\beta) = \frac{w_n(\beta)}{l^{\beta}}, \lambda_1(\beta) < \lambda_2(\beta) < \lambda_3(\beta) < \dots \tag{3.24}$$

where $w_n(\beta)$ satisfy the equation $sin_{\beta}(w_n(\beta)) = 0$.

As a result the solution is obtained as follows:

$$X_n(x;\beta) = \cos_{\beta}\left(w_n(\beta)\left(\frac{x}{l}\right)^{\beta}\right), n = 1, 2, 3, \dots \tag{3.25}$$

The second equation in (3.2) for eigenvalue $\lambda_n(\beta)$ yields the fractional differential equation below:

$$\frac{D_t^{\alpha}\left(T(t;\alpha,\beta)\right)}{T(t;\alpha,\beta)} = -\gamma^2 \lambda_n^2(\beta) \tag{3.26}$$

which yields the following solution

$$T_n(t;\alpha,\beta) = E_{\alpha,1}\left(-\gamma^2 \frac{w_n^2(\beta)}{l^{2\beta}} t^{\alpha}\right) n = 0, 1, 2, 3, \dots \tag{3.27}$$

The solution for every eigenvalue $\lambda_n(\beta)$ is constructed as

$$u_n(x,t;\alpha,\beta) = X_n(x;\beta)\, T_n(t;\alpha,\beta) = E_{\alpha,1}\left(-\gamma^2 \frac{w_n^2(\beta)}{l^{2\beta}} t^{\alpha}\right) \cos_{\beta}\left(w_n(\beta)\left(\frac{x}{l}\right)^{\beta}\right), n = 0, 1, 2, 3, \dots \tag{3.28}$$

which leads to the following general solution

$$u(x,t;\alpha,\beta) = A_0 + \sum_{n=1}^{\infty} A_n \cos_{\beta}\left(w_n(\beta)\left(\frac{x}{l}\right)^{\beta}\right)\ E_{\alpha,1}\left(-\gamma^2 \frac{w_n^2(\beta)}{l^{2\beta}} t^{\alpha}\right) \tag{3.29}$$

Note that it satifies boundary condition and fractional differential equation.

The coefficients of general solution are established by taking the following initial condition into account:

$$u(x,0) = f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos_{\beta}\left(w_n(\beta)\left(\frac{x}{l}\right)^{\beta}\right) \tag{3.30}$$

The coefficients A_n for $n = 0, 1, 2, 3, \dots$ are determined by the help of inner product (2.10) as follows:

$$\left\langle f\left(x\right),\ \cos_\beta\left(w_k\left(\beta\right)\left(\frac{x}{l}\right)^\beta\right)\right\rangle=\left\langle A_0,\ \cos_\beta\left(w_k\left(\beta\right)\left(\frac{x}{l}\right)^\beta\right)\right\rangle+$$
$$+\sum_{n=1}^{\infty}A_n\left\langle\cos_\beta\left(w_n\left(\beta\right)\left(\frac{x}{l}\right)^\beta\right),\ \cos_\beta\left(w_k\left(\beta\right)\left(\frac{x}{l}\right)^\beta\right)\right\rangle \tag{3.31}$$

$$T^{-1}\left(\int T\left[\cos_\beta\left(w_k\left(\beta\right)\left(\frac{x}{l}\right)^\beta\right)\ f\left(x\right)\right]dx\right)\Bigg|_{x=-l}^{x=l}=A_0\,T^{-1}\left(\int T\left[\cos_\beta\left(w_k\left(\beta\right)\left(\frac{x}{l}\right)^\beta\right)\right]dx\right)\Bigg|_{x=-l}^{x=l}+$$
$$+\sum_{n=1}^{\infty}A_n\,T^{-1}\left(\int T\left[\cos_\beta\left(w_n\left(\beta\right)\left(\frac{x}{l}\right)^\beta\right)\ \cos_\beta\left(w_k\left(\beta\right)\left(\frac{x}{l}\right)^\beta\right)\right]dx\right)\Bigg|_{x=-l}^{x=l} \tag{3.32}$$

$$T^{-1}\left(\int\cos\left(\frac{k\pi x}{l}\right)\ f(x)dx\right)\Bigg|_{x=-l}^{x=l}=A_0\,T^{-1}\left(\int\cos\left(\frac{k\pi x}{l}\right)\ dx\right)\Bigg|_{x=-l}^{x=l}$$
$$+\sum_{n=1}^{\infty}A_n\,T^{-1}\left(\int\cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{k\pi x}{l}\right)\ dx\right)\Bigg|_{x=-l}^{x=l} \tag{3.33}$$

$$A_0=\frac{1}{2l}T^{-1}\left(\int f(x)dx\right)\Bigg|_{x=-l}^{x=l} \tag{3.34}$$

$$A_n=\frac{1}{l}T^{-1}\left(\int f(x)\cos\left(\frac{k\pi x}{l}\right)\ dx\right)\Bigg|_{x=-l}^{x=l} \tag{3.35}$$

4. Illustrative Example

In this section, we first consider the following initial periodic boundary value problem:

$$u_t(x,t)=u_{xx}(x,t)$$
$$\begin{cases} u\left(-1,t\right)=u\left(1,t\right) \\ u_x\left(-1t,t\right)=u_x\left(1,t\right)\end{cases}$$
$$u\left(x,0\right)=\cos(\pi x) \tag{4.1}$$

which has the solution in the following form:

$$u\left(x,t\right)=\cos(\pi x)\ e^{-\pi^2 t}. \tag{4.2}$$

where $-1\leq x\leq 1$, $0\leq t\leq T$.

Now let the following problem called fractional heat-like problem be taken into consideration:

$$D_t^\alpha u\left(x,t\right)=D_x^{2\beta}u\left(x,t\right) \tag{4.3}$$

$$\begin{cases} u\left(-1,t\right)=u\left(1,t\right) \\ D_x^\beta u\left(-1t,t\right)=D_x^\beta u\left(1,t\right)\end{cases} \tag{4.4}$$

$$u\left(x,0\right)=\cos(\pi x) \tag{4.5}$$

where $0<\alpha<1,\ 1<2\beta<2,\ -1\leq x\leq 1,\quad 0\leq t\leq T$.

The separation of the variables method yields the following equations:

$$\frac{D_t^{\alpha}\left(T\left(t;\alpha,\beta\right)\right)}{T\left(t;\alpha,\beta\right)}=\frac{D_x^{2\beta}\left(X\left(x;\beta\right)\right)}{X\left(x;\beta\right)}=-\lambda^2(\beta) \tag{4.6}$$

Note that the value of λ varies based on β. Equation (3.2) produces two fractional differential equations with respect to time and space. The first fractional differential equation is obtained by taking the equation on the right hand side of Eq. (4.6). Hence with boundary conditions (4.4), we have the following problem

$$D_x^{2\beta}\left(X\left(x;\beta\right)\right)+\lambda^2\left(\beta\right)X\left(x;\beta\right)=0 \tag{4.7}$$

$$\begin{cases} X\left(-1\right)=X\left(1\right) \\ D_x^{\beta}X\left(-1\right)=D_x^{\beta}X\left(1\right) \end{cases} \tag{4.8}$$

The representation of the solution for the eigenvalue problem (4.7)-(4.8) is obtained as

$$X_n\left(x;\beta\right)=\cos_{\beta}\left(w_n\left(\beta\right)x^{\beta}\right),n=0,1,2,3,\ldots \tag{4.9}$$

The second eqution in (4.6) for every eigenvalue $\lambda_n(\beta)$ yields the following equation:

$$\frac{D_t^{\alpha}\left(T\left(t;\alpha,\beta\right)\right)}{T\left(t;\alpha,\beta\right)}=-\lambda_n^2\left(\beta\right) \tag{4.10}$$

which has the following solution

$$T_n\left(t;\alpha,\beta\right)=E_{\alpha,1}\left(-w_n^2\left(\beta\right)t^{\alpha}\right)\ n=0,1,2,3,\ldots \tag{4.11}$$

For each eigenvalue $\lambda_n(\beta)$, we obtain the following solution:

$$u_n\left(x,t;\alpha,\beta\right)=E_{\alpha,1}\left(-w_n^2\left(\beta\right)t^{\alpha}\right)\cos_{\beta}\left(w_n\left(\beta\right)x^{\beta}\right)\ n=0,1,2,3,\ldots \tag{4.12}$$

and hence we have the following sum:

$$u\left(x,t;\alpha,\beta\right)=A_0+\sum_{n=1}^{\infty}A_n\cos_{\beta}\left(w_n\left(\beta\right)x^{\beta}\right)\ \ E_{\alpha,1}\left(-w_n^2\left(\beta\right)t^{\alpha}\right) \tag{4.13}$$

Note that the general solution (4.13) satisfy both boundary conditions (4.4) and the fractional equation (4.3). By making use of the inner product defined in (2.10), we determine the coefficients A_n in such a way that the general solution (4.13) satisfes the initial condition (4.5). Plugging $t=0$ in to the general solution (4.13) and making equal to the initial condition (4.5) we have

$$u\left(x,0\right)=A_0+\sum_{n=1}^{\infty}A_n\cos_{\beta}\left(w_n\left(\beta\right)x^{\beta}\right) \tag{4.14}$$

The coefficients A_n for $n=0,1,2,3,\ldots$ are determined by the help of the inner product as follows:

$$\begin{aligned} A_0 = \frac{1}{2}T^{-1}\left(\int \cos(\pi x)\ dx\right)\Big|_{x=-1}^{x=1} &= \frac{1}{2}T^{-1}\left(\frac{1}{\pi}\sin(\pi x)\ \right)\Big|_{x=-1}^{x=1} \\ &= \frac{1}{2}\,\frac{sin_\beta\left(w_1\left(\beta\right)x^\beta\right)}{w_1(\beta)}\Bigg|_{x=-1}^{x=1} \\ &= \frac{1}{2w_1(\beta)}\left[sin_\beta\left(w_1\left(\beta\right)\right) - sin_\beta\left(-w_1\left(\beta\right)\right)\right] \\ &= 0 \end{aligned} \tag{4.15}$$

$$A_n = T^{-1}\left(\int \cos(\pi x)\ \cos\left(n\pi x\right)\ dx\right)\Big|_{x=-1}^{x=1} \tag{4.16}$$

For $n \neq 1$ $A_n = 0$ dır. $n = 1$ we get

$$\begin{aligned} A_1 = T^{-1}\left(\int \cos^2(\pi x)\ dx\right)\Big|_{x=-1}^{x=1} &= T^{-1}\left(\int\left(\frac{1}{2} + \frac{\cos(2\pi x)}{2}\right)dx\right)\Big|_{x=-1}^{x=1} \\ &= T^{-1}\left(\frac{x}{2} + \frac{\sin(2\pi x)}{4\pi}\right)\Big|_{x=-1}^{x=1} \\ &= \frac{x^\beta}{2} + \frac{sin_\beta\left(w_2\left(\beta\right)x^\beta\right)}{w_4(\beta)}\Bigg|_{x=-1}^{x=1} \\ &= \frac{1}{2} - \left(\frac{(-1)^\beta}{2}\right) \\ &= 1 \end{aligned} \tag{4.17}$$

Thus

$$u\left(x, t; \alpha, \beta\right) = cos_\beta\left(w_1\left(\beta\right)x^\beta\right)E_{\alpha,1}\left(-w_1^2\left(\beta\right)t^\alpha\right) \tag{4.18}$$

It is important to note that plugging $\alpha = \beta = 1$ in to the solution (4.18) gives the solution (4.2) which confirm the accuracy of the method we apply.

5. Conclusion

In this research, the analytic solution of sequential space-time fractional differential equation with periodic boundary conditions in one dimension is constructed. By making use of seperation of variables the solution is formed in the form of a Fourier series with respect to the eigenfunctions of a corresponding Sturm-Liouville eigenvalue problem including fractional derivative in Caputo sense.

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