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# SOME REMARKS ON FUNCTIONS WITH NON-DECREASING INCREMENTS

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Abstract. The main purpose of this work is to establish relationship of functions with non-decreasing increments with other functions of high importance. Especially, we pay attention to the role of representation and connection among functions with non-decreasing increments and arithmetic integral mean, Wright convex functions, convex functions, $\nabla-$convex functions, Jensen $m-$connvex functions, $m-$convex functions, $m-\nabla-$convex functions, $k-$monotonic functions, absolutely monotonic functions, completely monotonic functions, Laplace Transform and exponentially convex functions, by using the method of finite difference operator as different cases of $\Delta_h^m f(y)$. We also consider function with non-decreasing increments of order three and obtain the generalisation of the Levinson's-type inequality and Jensen-Mercer's-type inequality by using Jensen-Boas inequality and also deduce some results.

## 1. Introduction and Preliminaries

This distinct topic of function with non-decreasing increments (FWNDI) was introduced by H. D. Brunk in 1964 (see [5]). Book [21] discussed in detail about the properties of FWNDI. In past few decades FWNDI has gained popularity in several branches of Mathematics; that is a reason for increased interest in FWNDI by using finite difference operator. There are interesting topics in Numerical Method, Differential Equation, Physics, Biology and Engineering that play an important role where we use finite difference operators (see [17]). There are many applications of these operators in the area like Networking, Probabilistic, Fractal and Random Media, Fractionally Dependent Component, Applications to Mechanics, Controls Theory, Transport Phenomena, Fractional Equations and Chaos, and Future ideas as documented in the book [27]. Finally, these operators are naturally connected to different inequalities; various general inequalities for FWNDI, see present contribution in that field [10].

We can see the applications of functions with non-decreasing increments in the ultramodular function which is the especial case of FWNDI but the only difference is that the range of ultramodular function is $[0,1]$ while the range of FWNDI is

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$\mathbb{R}$. In Statistics, ultramodular functions play an important role in modelling stochastic orders and positive dependence among random vectors and ultramodular function also use in the field of economics. It is well known that the class of absolutely monotonic function is subclass of completely monotonic function and this is subclass of FWNDI if differentiability exists, further that absolutely monotonic function is used in copulas and there are many applications of copulas in different fields such as; Probability Theory, Quantitative Finance, Civil Engineering, Reliability Engineering, Warranty Data Analysis, Medicine, Solar Irradiance Variability, Hydrology Research, Climate and Weather Research. All these applications of FWNDI are extracted from [19].

Current article has an aim to collect the established facts about the FWNDI, together with some other important functions and notions, that can help out for finding whether a given funtion is FWNDI or not. In the second section, we would like to establish the connection among functions with non-decreasing increments and arithmetic integral mean, $W-$convex functions, convex functions, $\nabla-$convex functions, Jensen $m-$convex functions, $m-$convex functions, $m-\nabla-$convex functions, $k-$monotonic functions, absolutely monotonic functions, completely monotonic functions, Laplace transform and exponentially convex functions, by using the method of finite difference operator as different cases of $\Delta_h^m f(y)$ with examples. In third section, we would like to obtain the generalisation of the Levinson's-type inequality and Jensen-Mercer's-type inequality by using Jensen-Boas inequality for function with non-decreasing increments of order three and also deduce some results.

Let us recall few important definitions and significant results extracted from ([1], [12], [18], [25] and [28]). Throughout the article we would use the following notations $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_* = [0, \infty)$ and $\mathbb{R}_+ = (0, \infty)$ and also $I$ an interval in $\mathbb{R}$, $\mathbf{I}$ and $[\mathbf{a}, \mathbf{b}]$ both are intervals in $\mathbb{R}^k$.

H. D. Brunk introduced an interesting class of multivariate real valued functions known as functions with non-decreasing increments. Let us introduce some notations to recall the definition of FWNDI as follows:
Let $\mathbb{R}^k$ represent $k-$dimensional vector lattice of elements $\mathbf{y} = (y_1, y_2, y_3, \dots, y_k)$, $y_i$ be real, with partial ordering " $\leq''$ on $\mathbb{R}^k$ is here stated as $(y_1, y_2, y_3, \dots, y_k) \leq (z_1, z_2, z_3, \dots, z_k) \iff y_1 \leq z_1, \dots, y_k \leq z_k$, that is $y_i \leq z_i$ ; $i \in \{1, 2, \dots, k\}$. We begin with a fundamental definition of FWNDI as follow:

**Definition 1.1.** *A function $f : \mathbf{I} \to \mathbb{R}$, where $\mathbf{I} \subset \mathbb{R}^k$ and $k$ is a fixed positive integer, is said to have non-decreasing increments if following inequality*

$$f(\mathbf{b} + \mathbf{h}) - f(\mathbf{b}) \geq f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}),$$

*holds, where* $\mathbf{0} \leq \mathbf{h} \in \mathbb{R}^k, \mathbf{a} \leq \mathbf{b}; \mathbf{a}, \mathbf{b} + \mathbf{h} \in \mathbf{I}$

Some especial examples and properties of FWNDI was given by Brunk in paper [5] and also see [13] for more discussion.

**Examples of FWNDI:**

- (*i*) The simplest example of a FWNDI is a constant function.
- (*ii*) Lines in the form of $\mathbf{y} = \mathbf{a}t + \mathbf{b}$, where $(0, \dots, 0) \leq \mathbf{a} \in \mathbb{R}^k, \mathbf{b} \in \mathbb{R}^k$ whose direction cosines are nonnegative also belong to the family of functions with non-decreasing increments.
- (*iii*) An important continuous function $\vartheta : \mathbb{R}^2 \to \mathbb{R}$ stated as $\vartheta(y, z) = yz$ is a function with non-decreasing increments.

(*iv*) A continuous function $\nu : [0,\infty)^k \to \mathbb{R}$ stated as $\nu(\mathbf{y}) = \prod_{i=1}^{k} y_i$ is another useful function with non-decreasing increments.

(*v*) $F(\mathbf{y}+\mathbf{z}) = F(\mathbf{y}) + F(\mathbf{z})$ is the Cauchy functional equation which is an interesting and widely used example of such functions.

**Properties of FWNDI:**
FWNDI possesses the following properties:

(*i*) A FWNDI need not be continuous.

(*ii*) If function $f : \mathbf{I} \to \mathbb{R}$ has 1st order partial derivatives $\forall\, \mathbf{y} \in \mathbf{I}$, then $f$ has non-decreasing increments if and only if every of those partial derivatives is non-decreasing in every arguments.

(*iii*) If function $f : \mathbf{I} \to \mathbb{R}$ has 2nd order partial derivatives $\forall\, \mathbf{y} \in \mathbf{I}$, then $f$ has non-decreasing increments if and only if every of those partial derivatives is non-negative.

(*iv*) A function $\upsilon : [0,1] \to \mathbb{R}$ is convex, stated as $\upsilon(t) = f(t\mathbf{a}+\mathbf{b})$, if $f$ FWNDI is continuous in $\mathbf{b} \le \mathbf{y} \le \mathbf{b}+\mathbf{a}$; $\mathbf{0} \le \mathbf{a} \in \mathbb{R}^k$.

We define here a especial type of function which belongs to the class of FWNDI and themselves connect/contain the class of other functions that are already provided in the starting of the recent heading of this article, by using several cases of $\Delta_h^m f(y)$.

Here we recall the definition of finite difference of the function of order $m$.

**Definition 1.2.** [25] *The **finite difference of the function of order** $m$ on $I = [a,b] \in \mathbb{R}$, where $m$ is non-negative integers, is defined by*

$$\Delta_h^0 f(y) = f(y)$$
$$\Delta_h^m f(y) = \Delta_h^{m-1} f(y+h) - \Delta_h^{m-1} f(y)$$

*where $h \neq 0$ and $y + ih \in I$ for $i \in \{0, 1, \cdots, m\}$. Then it can easy to write the statement as*

$$\Delta_h^m f(y) = \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} f(y+ih).$$

**Definition 1.3.** *The **mth order divided difference** of a function $f : I \to \mathbb{R}$, at distinct elements $y_i, y_{i+1}, \ldots, y_{i+m} \in I = [a,b] \subset \mathbb{R}$, where $i \in \mathbb{N}$ is stated as:*

$$\begin{aligned} [y_j; f] &= f(y_j), \quad j \in \{i, i+1, \ldots, i+m\} \\ [y_i, \ldots, y_{i+m}; f] &= \frac{[y_{i+1}, \ldots, y_{i+m}; f] - [y_i, \ldots, y_{i+m-1}; f]}{y_{i+m} - y_i}. \end{aligned}$$

Further can be written as

$$[y_i, \ldots, y_{i+m}; f] = \sum_{k=0}^{m} \frac{f(y_{i+k})}{\prod_{j=i, j\neq i+k}^{i+m} (y_{i+k} - y_j)}.$$

We present some remarks about the relationship among finite difference, divided difference and derivative of the function.

**Remark.** *Some important remarks are following:*

(i) *Let us denote $[y_i, \ldots, y_{i+m}; f]$ by $\Delta_{(m)} f(y_i)$. The value $[y_i, \ldots, y_{i+m}; f]$ is independent of elements order $y_i, y_{i+1}, \ldots, y_{i+m}$.*

(ii) *We can extend this above definition by including case in which few elements or all elements coincide by supposing that $y_i \leq \cdots \leq y_{i+m}$ (see [25]) and letting*

$$[y_i, \ldots, y_{i+m}; f] = \frac{f^{(m)}(y_i)}{m!},$$

*provided that $f^{(m)}(y_i)$ exists.*

(iii) *The following identity is valid (see [25, 26]):*

$$\Delta^m_{\mathbf{h}} f(\mathbf{y}) = m! h^m \Delta_m f(\mathbf{y})$$

*provided that $y_i$ are equally spaced.*

Using finite difference operator, we would like to state alternative form of functions with non-decreasing increments as;

$$\Delta_{\mathbf{h_1}}\Delta_{\mathbf{h_2}} f(\mathbf{a}) \geq 0 \tag{1.1}$$

Since
$$\begin{aligned} \Delta_{\mathbf{h_1}}(\Delta_{\mathbf{h_2}} f(\mathbf{a})) &= \Delta_{\mathbf{h_1}}(f(\mathbf{a}+\mathbf{h_2}) - f(\mathbf{a})) \geq 0 \\ &= f(\mathbf{a}+\mathbf{h_2}+\mathbf{h_1}) - f(\mathbf{a}+\mathbf{h_2}) - f(\mathbf{a}+\mathbf{h_1}) + f(\mathbf{a}) \geq 0 \end{aligned}$$

Setting $\mathbf{h} = \mathbf{h_1}, \mathbf{b} = \mathbf{a} + \mathbf{h_2}$ in above statement then

$$f(\mathbf{b}+\mathbf{h}) - f(\mathbf{b}) - f(\mathbf{a}+\mathbf{h}) + f(\mathbf{a}) \geq 0$$

If taking $\mathbf{h_1} = \mathbf{h_2} = \mathbf{h}$, then we can obtain especial case of (1.1)

$$\Delta_{\mathbf{h}} f(\mathbf{b}) - \Delta_{\mathbf{h}} f(\mathbf{a}) \geq 0$$
$$\Delta_{\mathbf{h}}(f(\mathbf{b}) - f(\mathbf{a})) \geq 0$$
$$\Delta^2_{\mathbf{h}} f(\mathbf{a}) \geq 0, \text{ where } \mathbf{a} \leq \mathbf{b}$$

We know that $\Delta_{\mathbf{h_1}} f(\mathbf{y}) = f(\mathbf{y}+\mathbf{h_1}) - f(\mathbf{y})$ and further,

$$\Delta_{\mathbf{h_1}}\Delta_{\mathbf{h_2}} \cdots \Delta_{\mathbf{h_m}} f(\mathbf{y}) = \Delta_{\mathbf{h_1}}(\Delta_{\mathbf{h_2}} \cdots \Delta_{\mathbf{h_m}} f(\mathbf{y})) \text{ for } m \geq 2$$

where $\mathbf{y}, \mathbf{y}+\mathbf{h_1}+\cdots+\mathbf{h_m} \in \mathbf{I}, \mathbf{0} \leq \mathbf{h_i} \in \mathbb{R}^k$ for $i \in \{1, 2, \ldots, m\}$.

Similarly we can extend this definition for mth order as

**Definition 1.4.** *A function $f\colon \mathbf{I} \to \mathbb{R}$ is called **function with non-decreasing increments of order** $m$ if holds*

$$\Delta_{\mathbf{h_1}}\Delta_{\mathbf{h_2}} \cdots \Delta_{\mathbf{h_m}} f(\mathbf{y}) \geq 0. \tag{1.2}$$

*whenever $\mathbf{y}, \mathbf{y}+\mathbf{h_1}+\cdots+\mathbf{h_m} \in \mathbf{I}, \mathbf{0} \leq \mathbf{h_i} \in \mathbb{R}^k$ for $i \in \{1, 2, \ldots, m\}$. Then the especial case is given by*

$$\Delta^m_{\mathbf{h}} f(\mathbf{y}) \geq 0 \tag{1.3}$$

*where $f$ is called FWNDI of order $m$ with equally spaced $\mathbf{h}$.*

## 2. Relationship Among Functions With Non-decreasing Increments and Many Others

We would like to use the above definition to establish relationship among functions with non-decreasing increments and many other functions, the detailed list of other functions already mention at initial in the introductory section. In the current section we would like to recall some important definitions which are extracted from the articles [1, 2, 4, 7, 6, 8, 9, 10, 11, 12, 18, 20, 21, 22, 24, 25, 28] and these will relate to functions with non-decreasing increments, by using the finite difference operator as different cases of $\Delta_h^m f(y) \geq 0$. In this connection we will use the relationships of finite difference, derivative and some other differences (see [6, 8, 25, 26]).

2.1. **Arithmetic integral mean vs FWNDI.** Let us have A is an arithmetic integral mean of a function $f$ on an interval $[0, a]$ (see in [10, 16]).

**Definition 2.2.** *A function A (non-decreasing function) is said to be a* ***Arithmetic integral mean*** *on an interval* $[0, a]$*, such that*

$$A(t) = \frac{1}{t}\int_0^t f(y)dy$$

*provided* $f : [0, a] \to \mathbb{R}, a > 0$*, is a non-negative and non-decreasing function.*

Now, we will recall extension of above-mentioned result to functions with non-decreasing increments of higher order, extracted from [10].

**Theorem 2.1.** *Let the function* $f : [\mathbf{a}, \mathbf{b}] \to \mathbb{R}$ *be continuous and with non-decreasing increments of order* $m$*. Then the function* $A$*, defined as*

$$A(\mathbf{t}) = \left(\prod_{i=1}^{k}(t_i - a_i)\right)^{-1} \int_{a_1}^{t_1} \cdots \int_{a_k}^{t_k} f(\mathbf{v})\mathbf{dv},$$

*is a function with non-decreasing increments of order* $m$ *on* $[\mathbf{a}, \mathbf{b}]$*, where* $\mathbf{v} = (v_1, \ldots ,v_k)$ *and* $\mathbf{dv} = dv_1 \cdots dv_k$*.*

The **Alternative form of Arithmetic integral mean of order m** on an interval $[\mathbf{a}, \mathbf{b}]$, using equation (1.2) it is defined by

$$\Delta_{\mathbf{h_1}}\Delta_{\mathbf{h_2}} \cdots \Delta_{\mathbf{h_m}} A(\mathbf{t}) \geq 0, \qquad \mathbf{h_i} \geq \mathbf{0}.$$

**Remark.** *If take* $\mathbf{h_1} = \mathbf{h_2} = \cdots = \mathbf{h_m}$ *in above inequality, then obtain especial case as* $\Delta_{\mathbf{h}}^m A(\mathbf{t}) \geq 0$*.*

**Example 2.3.** *Let a function* $f : [0, a] \to \mathbb{R}_+$ *which is stated as*

$$f(y) = e^{y-c^2}$$

*Since* $\Delta_h^m f(y) \geq 0$*, then from the above remark also holds* $\Delta_h^m A(t) \geq 0$ *when* $h$ *is very small, therefore we can say that* $A$ *is Arithmetic integral mean of order* $m$ *on the interval* $[0, a]$ *for every* $a > 0, c \in \mathbb{R}$*.*

Now first of all we would like to mention here the 1st dimensional case of function with non-decreasing increments which is Wright–convex function and we will give the equivalent form.

2.4. **Wright–convex function vs FWNDI.**

**Definition 2.5.** *A function* $f : [a, b] \to \mathbb{R}$ *is called* ***Wright–convex*** *function, if following inequality is valid* $\forall\ y \leq z; x \geq 0; y, z + x \in [a, b]$*.*

$$f(z + x) - f(z) \geq f(y + x) - f(y)$$

*It can also be written as* $\Delta_x^2 f(y) \geq 0 \qquad y \leq z;\ y, z + x \in [a, b]$
*This is computed same as function with non-decreasing increments, i.e.,*

$$\Delta_h^2 f(y) \geq 0 \tag{2.1}$$

*If there exists $f''$, then $f$ is **Wright–convex** function iff*

$$h^2 f''(y) \geq 0. \tag{2.2}$$

*And the **equivalent form of** (2.2) on $I \subset \mathbb{R}$, using equation (1.3) it is also defined by same (2.1), when $h$ is very small.*

**Example 2.6.** *Let a function $f : [a, b] \to \mathbb{R}$ which is stated as*

$$f(y) = y(ay - b)$$

*Since $h^2 f''(y) \geq 0$, i.e. $\Delta_h^2 f(y) \geq 0$, therefore the function $f$ is Wright–convex on the interval $[a, b] \subset \mathbb{R}$ for every $a \geq 0, b \in \mathbb{R}$.*

**Remark.** *Wright-convex function is an especial case of FWNDI for $k = 1$.*

Now we would like to state generalised convex function may be seen in [12, 25].

2.7. **m–convex function vs FWNDI.**

**Definition 2.8.** *A function $f : I \to \mathbb{R}$, is known as **m–convex** , if the inequality $\Delta_{(m)} f(y) \geq 0$ holds $\forall (m+1)$ different points $y_0, y_1, \ldots, y_m \in I$.*
***Especial case of convex function of order m**, using equation (1.3) and Remark 1 it is defined by*

$$\frac{\Delta_h^m f(y)}{m! h^m} \geq 0, \qquad y \in I, h > 0. \tag{2.3}$$

*If there exists $f^{(m)}$, then function is **m–convex** or **mth order convex** iff*

$$\frac{f^{(m)}(y)}{m!} \geq 0. \tag{2.4}$$

*And the **equivalent form of** (2.4) on $I \subset \mathbb{R}$, using equation (1.3) it is also defined by same (2.3), when $h$ is very small.*

**Example 2.9.** *Let a function $f : I \to \mathbb{R}$ which is stated as*

$$f(y) = \frac{y^m}{m!}$$

*Since $\dfrac{f^{(m)}(y)}{m!} \geq 0$, then $\dfrac{\Delta_h^m f(y)}{m! h^m} \geq 0$, therefore the function $f$ is m–convex on the interval $I$ for $q > 0$, $m \in \{0, 1, 2, \cdots\}$.*

2.10. **m–∇–convex function vs FWNDI.**

**Definition 2.11.** *A function $f : I \to \mathbb{R}$, is known as **m–∇–convex**, if $\forall (m+1)$ different points $y_0, y_1, \ldots, y_m \in I$ we have $\nabla_{(m)} f(y) = (-1)^m \Delta_{(m)} f(y) \geq 0$.*
***Especial case of ∇–convex funtion of order m**, using equation (1.3) and Remark 1 it is defined by*

$$\frac{(-1)^m \Delta_h^m f(y)}{m! h^m} \geq 0, \qquad y \in I, h > 0. \tag{2.5}$$

*If there exists $f^{(m)}$, then function is **m–∇–convex** iff*

$$\frac{(-1)^m f^{(m)}(y)}{m!} \geq 0. \tag{2.6}$$

*And the **equivalent form of** (2.6) on $I \subset \mathbb{R}$, using equation (1.3) it is also defined by same (2.5), when $h$ is very small.*

**Example 2.12.** *Let a function $f : I \to \mathbb{R}$ which is stated as*

$$f(y) = \frac{e^{-ry}}{r^m}$$

*Since* $\frac{(-1)^m f^{(m)}(y)}{m!} \geq 0$, *i.e.* $\frac{(-1)^m \Delta_h^m f(y)}{m!h^m} \geq 0$, *therefore the function $f$ is $m$–$\nabla-$convex on the interval $I \subset \mathbb{R}_*$ for $r \in \mathbb{R} - \{0\}$, $m \in \{0, 1, 2, \cdots\}$.*

**Remark.** *Convex function and $\nabla-$convex function are the especial cases of $m$–convex function and $m$–$\nabla-$convex respectively, if we put $m = 2$.*

2.13. **Jensen m–convex function vs FWNDI.**

**Definition 2.14.** *Any function $f : I \to \mathbb{R}$ is known as **Jensen m–convex** or **J–convex of order m** if holds*

$$\Delta_h^m f(y) \geq 0, \qquad \forall h > 0 \text{ and } y \in I. \tag{2.7}$$

*This is done by finite difference operator and if there exists $f^{(m)}$, then $f$ is **J–convex of order m** iff*

$$h^m f^{(m)}(y) \geq 0. \tag{2.8}$$

*And the **equivalent form of** (2.8) on $I \subset \mathbb{R}$, using equation (1.3) it is also defined by same* (2.7)*, when $h$ is very small.*

**Example 2.15.** *Let a function $f : I \to \mathbb{R}$ which is stated as*

$$f(y) = \frac{e^{qy}}{q^m}$$

*Since $h^m f^{(m)}(y) \geq 0$, then $\Delta_h^m f(y) \geq 0$, therefore the function $f$ is J–convex of order $m$ on the interval $I \subset \mathbb{R}_*$ for $q \in \mathbb{R} - \{0\}$, $m \in \{0, 1, 2, \cdots\}$.*

**Remark.** *Jensen convex function is the especial case of Jensen m–convex function, if we put $m = 2$.*

Now we recall definitions of convex function and $\nabla-$convex function and their connection with FWNDI using equation (1.3).

2.16. **Convex function vs FWNDI.**

**Definition 2.17.** *Any continuous function $f : I \to \mathbb{R}$ is known as **convex**, if there exists non-negative second order divided difference, such that*

$$\Delta_2 f(y) \geq 0, \qquad y \in I.$$

***Especial case of convex** function on $I \subset \mathbb{R}$, using equation (1.3) and Remark 1 it is defined by*

$$\frac{\Delta_h^2 f(y)}{2!h^2} \geq 0, \qquad y \in I, h > 0. \tag{2.9}$$

*If there exists $f^{''}$, then $f$ is **convex** iff*

$$\frac{f^{''}(y)}{2!} \geq 0. \tag{2.10}$$

*And the **equivalent form of** (2.10) on $I \subset \mathbb{R}$, using equation (1.3) it is also defined by same* (2.9)*, when $h$ is very small.*

**Example 2.18.** *Let a function $f : I \to \mathbb{R}_*$ which is stated as*

$$f(y) = my^2 + m^2$$

*Since* $\frac{f^{''}(y)}{2!} \geq 0$, *i.e.* $\frac{\Delta_h^2 f(y)}{2!h^2} \geq 0$, *therefore the function $f$ is convex on the interval $I$ for $m \in \{0, 1, 2, \cdots\}$.*

**Remark.** *If sets, "C" convex function, "W" Wright-convex function and "J" Jensen convex function then $C \subset W \subset J$. Moreover, each inclusion is proper (see "Kenyon, 1956 and Klee, 1956) (also see [21, 25]").*

2.19. **$\nabla-$convex function vs FWNDI.**

**Definition 2.20.** *Any continuous function $f\colon I \to \mathbb{R}$ is known as $\nabla-$**convex**, there is existence of $\nabla^2 f(y) = (-1)^2 \Delta_2 f(y)$ and satisfy*

$$\nabla^2 f(y) \geq 0, \qquad y \in I.$$

***Especial case of** $\nabla-$**convex** function on $I \subset \mathbb{R}$, using equation (1.3) and Remark 1 it is defined by*

$$\frac{(-1)^2 \Delta_h^2 f(y)}{2!h^2} \geq 0, \qquad y \in I, h > 0, \tag{2.11}$$

*If there exists $f^{''}$, then $f$ is $\nabla-$**convex** iff*

$$\frac{(-1)^2 f^{''}(y)}{2!} \geq 0. \tag{2.12}$$

*And the **equivalent form of** (2.12) on $I \subset \mathbb{R}$, using equation (1.3) it is also defined by same (2.11), when $h$ is very small.*

**Example 2.21.** *Let a function $f : \mathbb{R}_+ \to \mathbb{R}$ which is stated as*

$$f(y) = \ln \frac{1}{y}$$

*Since* $\frac{(-1)^2 f^{''}(y)}{2!} \geq 0$, *i.e.* $\frac{(-1)^2 \Delta_h^2 f(y)}{2!h^2} \geq 0$, *therefore the function $f$ is $\nabla-$convex on the interval $\mathbb{R}_+$.*

2.22. **Completely monotonic function vs FWNDI.**

**Definition 2.23.** *A continuous function $f$ is known as **completely monotonic**, if there is existence of derivatives of all orders on $I \subset \mathbb{R}$, and if satisfy*

$$(-1)^i f^{(i)}(y) \geq 0, \qquad i \in \{0, 1, \ldots\} \quad ; \quad y \in I.$$

***Equivalent form of completely monotonic** function on $I \subset \mathbb{R}$, using equation (1.3) it is defined by*

$$\frac{(-1)^i \Delta_h^i f(y)}{h^i} \geq 0, \qquad i \in \{0, 1, \ldots\} \quad ; \quad y \in I, h > 0.$$

*when $h$ is very small.*

**Example 2.24.** *Some examples of completely monotonic functions are following:*

(i) $f(y) = \frac{\alpha}{y^{1-\alpha}}, \quad 0 \leq \alpha \leq 1,\ y > 0$

(ii) $f(y) = \frac{1}{(y + \alpha^2)^\beta}, \quad \alpha \geq 0,\ \beta \geq 0,\ y > 0$

(iii) $f(y) = -\ln(1 - 1/y), \quad \forall y \in \mathbb{R}_+$

(iv) $f(y) = e^{1/y}, \quad \forall y \in \mathbb{R}_+$

Completely monotonic function is generalised form of absolutely monotonic function and k–monotonic function, now we will give connection of absolutely monotonic function and k–monotonic function with FWNDI using equation (1.3).

2.25. **Absolutely monotonic function vs FWNDI.**

**Definition 2.26.** *A continuous function $f$ is known as **absolutely monotonic**, if there is existence of derivatives of all orders on $I \subset \mathbb{R}$, and if satisfy*

$$f^{(i)}(y) \geq 0, \qquad i \in \{0, 1, \ldots\} \quad ; \quad y \in I.$$

***Equivalent form of absolutely monotonic** function on $I \subset \mathbb{R}$, using equation (1.3) it is defined by*

$$\frac{\Delta_h^i f(y)}{h^i} \geq 0, \qquad i \in \{0, 1, 2, \ldots\} \quad ; \quad y \in I, h > 0.$$

*when $h$ is very small.*

**Example 2.27.** *Let a function $f : [-1, 1] \to \mathbb{R}$ which is stated as*

$$f(y) = sin^{-1} y$$

*Since $f^{(i)}(y) \geq 0$, then $\dfrac{\Delta_h^i f(y)}{h^i} \geq 0$, therefore the function $f$ is absolutely monotonic in the interval $[0, 1)$.*

2.28. **k–monotonic function vs FWNDI.**

**Definition 2.29.** *A function is known as **k–monotonic** on $I \subset \mathbb{R}$, if all its derivatives $f^{(i)}(y)$ exist and satisfy*

$$(-1)^i f^{(i)}(y) \geq 0, \qquad i \in \{0, 1, \ldots, k\} \quad , \text{where } k \text{ is fixed} \quad ; \quad y \in I.$$

***Equivalent form of k–monotonic** function on interval $I \subset \mathbb{R}$, using equation (1.3) it is defined by*

$$\frac{(-1)^i \Delta_h^i f(y)}{h^i} \geq 0, \qquad i \in \{0, 1, \ldots, k\} \quad ; \quad y \in I, h > 0.$$

*when $h$ is very small.*

**Example 2.30.** *Let a function $f : I \to \mathbb{R}$ which is stated as*

$$f(y) = y^{-r}$$

*Since $(-1)^i f^{(i)}(y) \geq 0$, then $\dfrac{(-1)^i \Delta_h^i f(y)}{h^i} \geq 0$, therefore the function $f$ is k–monotonic on the interval $I \subset \mathbb{R}_*$ for $r \geq 0$.*

2.31. **Laplace transform of $f$ vs FWNDI.**

**Definition 2.32.** *Let $f$ be a function that satisfies $|f(y)| \leq Me^{ay}$ and piecewise continuous, where $a$ and $M$ are real constants. Then the **Laplace transform** of $f(y)$ is stated as*

$$F(s) = L\{f(y)\} = \int_0^\infty f(y) e^{-sy} dy, \qquad s > a.$$

*Similarly, the Laplace transformation of Borel measure $\varphi(t)$ on $\mathbb{R}_*$ is stated as*

$$L\{\varphi(t)\} = \int_0^\infty e^{-yt} d\varphi(t).$$

*At origin the Laplace transformation is continuous iff $\varphi$ is finite.*

Bernstein [4] proved that $f$ on an interval $\mathbb{R}_*$ is completely monotonic , iff there is existence of increasing function $\varphi(t)$ on $\mathbb{R}_*$.

$$f(y) = \int_0^\infty e^{-yt} d\varphi(t).$$

**Remark.** *Equivalent form of above statement can also be present as a function $f$ on the interval $\mathbb{R}_*$ satisfied the condition $\dfrac{(-1)^i \Delta_h^i f(y)}{h^i} \geq 0$ (**completely monotonic**) where $h$ is very small and $y \in I \subset \mathbb{R}_*, i \in \{0, 1, \dots\}$ iff there exists*

$$f(y) = \int_0^\infty e^{-yt} d\varphi(t),$$

*where $\varphi(t)$ is an increasing function on $\mathbb{R}_*$.*

2.33. **Exponentially convex function vs FWNDI.**

**Definition 2.34.** *Any continuous function $\omega : I \to \mathbb{R}$ on open interval $I$ is called **exponentially convex**, if*

$$\sum_{i,j=1}^{m} \rho_i \rho_j \, \omega \left(y_i + y_j\right) \geq 0,$$

$\forall\, m \in \mathbb{N}$ *and* $\forall\ \rho_i, \rho_j \in \mathbb{R}$*; such that* $y_i + y_j \in I$ *and* $i, j \in \{1, \dots, m\}$.

**Theorem 2.2.** *The function $\omega : I \to \mathbb{R}$ on an interval $I$ is exponentially convex iff*

$$\omega(y) = \int_{-\infty}^{\infty} e^{ty} d\varphi(t), \qquad y \in I$$

*for some non-decreasing function $\varphi(t) : \mathbb{R} \to \mathbb{R}$*

*Proof.* See [2], p. 211. □

Little less obvious examples can be deduced by applying above integral representation and some results from **Laplace transform** are following.

**Example 2.35.** *The folowing are examples of Exponentially convex functions as well as Laplace transform of $f$ through above theorem on $\mathbb{R}_+$ include:*

(i) $f(y) = y^{-\alpha}$ , $\forall \alpha > 0$
(ii) $f(y) = e^{-\alpha\sqrt{y}}$, $\forall \alpha > 0$
(iii) $f(y) = e^{-y\sqrt{s}}$, $\forall s > 0$
(iv) $f(y) = e^{-ty}$, $\forall t > 0$

**Remark.** *The above functions are also satisfying condition $\dfrac{(-1)^i \Delta_h^i f(y)}{h^i} \geq 0$ (**completely monotonic**) by using equation (1.3) in these examples when $h$ is very small.*

## 3. Functions with Non-decreasing Increments of Order three

Before we give the generalisation of the Levinson's-type inequality and Jensen-Mercer's-type inequality by using Jensen-Boas inequality for function with non-decreasing increments of order three, we must present following proposition about the inequality of Jensen-Steffensen-type for a FWNDI which is extracted from [23]. Here in current section, bounded variation stands for bv.

**Proposition 3.1.** *Let* $\mathbf{Y} : [a,b] \to \mathbf{I}$ *be a non-decreasing continuous map and let* $B \in bv[a,b]$ *such that*

$$B(a) \leq B(y) \leq B(b), \quad B(a) < B(b).$$

*If* $f : \mathbf{I} \to \mathbb{R}$ *is a continuous function with non-decreasing increments, then*

$$f\left(\frac{\int_a^b \mathbf{Y}(t)\, dB(t)}{\int_a^b dB(t)}\right) \leq \frac{\int_a^b f\left(\mathbf{Y}(t)\right)\, dB(t)}{\int_a^b dB(t)}$$

*holds, where* $\int_a^b \mathbf{Y}\, dB = \left(\int_a^b Y_1\, dB, \ldots, \int_a^b Y_k\, dB\right)$.

We also give generalised form of above proposition using Jensen-Boas inequality for function with non-decreasing increments, extracted from [15] as follows.

**Theorem 3.2.** *Let* $\mathbf{Y} : [a,b] \to \mathbf{I}$ *be a continuous and monotonic (either non-increasing or non-decreasing) map in each of the* $l$ *intervals* $(b_{i-1}, b_i)$. *Let* $B : [a,b] \to \mathbb{R}$ *is continuous or of bounded variation satisfying*

$$B(a) \leq B(a_1) \leq B(b_1) \leq B(a_2) \leq \cdots \leq B(b_{l-1}) \leq B(a_l) \leq B(b) \tag{3.1}$$

*for all* $a_i \in (b_{i-1}, b_i)$ $(b_0 = a,\ b_l = b)$, *and* $B(b) > B(a)$. *If* $f$ *is continuous function having non-decreasing increments in each of the* $l$ *intervals* $(b_{i-1}, b_i)$, *then we have the following inequality*

$$f\left(\frac{\int_a^b \mathbf{Y}(t) dB(t)}{\int_a^b dB(t)}\right) \leq \frac{\int_a^b f(\mathbf{Y}(t)) dB(t)}{\int_a^b dB(t)}. \tag{3.2}$$

**Remark.** (i) *If* $\dfrac{\int_a^b (\mathbf{Y}(t))\, dB(t)}{\int_a^b dB(t)} \in \mathbf{I}$ *and* $\forall\ y \in a_i \in (b_{i-1}, b_i)$ $(b_0 = a,\ b_l = b)$ *we have either* $B(y) \geq B(b)$ *or* $B(y) \leq B(a)$, *then the inequality in* (3.2) *holds for reverse direction.*

(ii) *If putting* $l = 1$, *Theorem* 3.2 *gives a especial case as Proposition* 3.1.

Now we are able to give our main theorems which is a generalisations of the Levinson's inequality and Jensen-Mercer's-type inequality using Jensen-Boas inequality in the following next sub-sections.

3.1. **On Inequalities of Levinson-type.** Levinson's inequality is a generalisation of an inequality of Ky Fan i.e. in other word we say the Ky Fan inequality is the especial case of Levinson's inequality (see "Beckenbach and Bellman, 1961, 1965, p. 5; Mitrinovic, 1970, p. 363; and Hardy, Littlewood, and Polya, 1934, 1952, p. 281-82) and for generalisations of Levinson's inequality, (see Gavrea and Ivan 1981, Gavrea 1985, and Gavrea and Gurzau 1987", also see [3]).

The following theorem is a generalisation of the Levinson's-type inequality using Jensen-Boas inequality.

**Theorem 3.3.** *Let $B \in bv[a,b]$ such that* (3.1) *holds and let* $\mathbf{Y} : [a,b] \to [\mathbf{0}, \mathbf{d}]$, $(\mathbf{d} > \mathbf{0})$ *be a continuous and monotonic (either non-increasing or non-decreasing) map in each of the $l$ intervals $(b_{i-1}, b_i)$. If $f$ is a continuous function with non-decreasing increments of order three on $\mathbf{J} = [\mathbf{0}, 2\mathbf{d}] \subset \mathbb{R}^k$, then the inequality*

$$\frac{\int_a^b f(\mathbf{Y}(t))\, dB(t)}{\int_a^b dB(t)} - f\left(\frac{\int_a^b \mathbf{Y}(t)\, dB(t)}{\int_a^b dB(t)}\right) \leq \frac{\int_a^b f(2\mathbf{d} - \mathbf{Y}(t))\, dB(t)}{\int_a^b dB(t)} - f\left(\frac{\int_a^b (2\mathbf{d} - \mathbf{Y}(t))\, dB(t)}{\int_a^b dB(t)}\right) \tag{3.3}$$

*holds.*

*Proof.* If $f$ is a function with non-decreasing increments of order three on $\mathbf{J}$, then the following inequality holds

$$\Delta_{\mathbf{h_1}} \Delta_{\mathbf{h_2}} \Delta_{\mathbf{h_3}} f(\mathbf{y}) \geq 0 \quad \text{for} \quad \mathbf{y}, \mathbf{y} + \mathbf{h_1} + \mathbf{h_2} + \mathbf{h_3} \in \mathbf{J}, \quad \mathbf{h_1}, \mathbf{h_2}, \mathbf{h_3} \in \mathbb{R}^k_*,$$

i.e.,

$$\Delta_{\mathbf{h_1}} \Delta_{\mathbf{h_2}} \left(f(\mathbf{y} + \mathbf{h_3}) - f(\mathbf{y})\right) \geq 0. \tag{3.4}$$

If $\mathbf{y} \in \mathbf{J}$ and $\mathbf{h_3} = 2\mathbf{d} - 2\mathbf{y}$, we have

$$\Delta_{\mathbf{h_1}} \Delta_{\mathbf{h_2}} \left(f(2\mathbf{d} - \mathbf{y}) - f(\mathbf{y})\right) \geq 0,$$

i.e., the function $\mathbf{y} \mapsto f(2\mathbf{d} - \mathbf{y}) - f(\mathbf{y})$ is a function with non-decreasing increments of order two, i.e., it is a function with non-decreasing increments. Now, using Theorem 3.2, we obtain Theorem 3.3. ☐

**Remark.**
- (i) *If* $\dfrac{\int_a^b (2\mathbf{d} - \mathbf{Y}(t))\, dB(t)}{\int_a^b dB(t)} \in \mathbf{J}$ *and* $\forall\ y \in a_i \in (b_{i-1}, b_i)$ $(b_0 = a,$ $b_l = b)$ *we have either $B(y) \geq B(b)$ or $B(y) \leq B(a)$, then the inequality in* (3.3) *holds for reverse direction.*
- (ii) *If putting $l = 1$, Theorem* 3.3 *gives a especial case as Theorem 3.1 of* [13].
- (iii) *We can obtain discrete version of Theorem* 3.3*, using technique as given in Corollary 1 of* [14].

**Corollary 3.4.** *Let* $\mathbf{Y}$ *satisfies the assumptions of Theorem* 3.3. *Then the inequalities*

$$0 \leq \left(\int_a^b dB(t)\right)^{k-1} \int_a^b \prod_{i=1}^k Y_i(t)\, dB(t) - \prod_{i=1}^k \int_a^b Y_i(t)\, dB(t)$$
$$\leq \left(\int_a^b dB(t)\right)^{k-1} \int_a^b \prod_{i=1}^k (2d_i - Y_i(t))\, dB(t) - \prod_{i=1}^k \int_a^b (2d_i - Y_i(t))\, dB(t)$$

*hold, where all components of* $\mathbf{Y}$ *are nonnegative.*

*Proof.* The function $f(\mathbf{y}) = y_1 \cdots y_k$ is a function with non-decreasing increments of orders two and three for $\mathbf{y} \in \mathbb{R}^k_*$. So, using Theorems 3.2 and 3.3, we obtain Corollary 3.4. ☐

**Remark.** *If putting $l = 1$, Corollary* 3.4 *gives a especial case as Corollary 3.3 (i) of* [13].

**Theorem 3.5.** *Let $B \in bv[a,b]$ such that* (3.1) *holds and let $f$ be a continuous function with non-decreasing increments of order three on $[\mathbf{c},\mathbf{d}] \subset \mathbb{R}^k$. Let $\mathbf{0} < \mathbf{a} < \mathbf{d}-\mathbf{c}$. If $\mathbf{Y} : [a,b] \to [\mathbf{c}, \mathbf{d}-\mathbf{a}]$ is a continuous and monotonic (either non-increasing or non-decreasing) map in each of the $l$ intervals $(b_{i-1}, b_i)$, then the inequality*

$$\frac{\int_a^b f(\mathbf{Y}(t))\,dB(t)}{\int_a^b dB(t)} - f\left(\frac{\int_a^b \mathbf{Y}(t)\,dB(t)}{\int_a^b dB(t)}\right) \leq \frac{\int_a^b f(\mathbf{a}+\mathbf{Y}(t))\,dB(t)}{\int_a^b dB(t)} - f\left(\frac{\int_a^b (\mathbf{a}+\mathbf{Y}(t))\,dB(t)}{\int_a^b dB(t)}\right) \tag{3.5}$$

*holds.*

*Proof.* Using (3.4) for $\mathbf{h_3} = \mathbf{a} = \text{constant} \in \mathbb{R}^k$, we have that $\mathbf{y} \mapsto f(\mathbf{a}+\mathbf{y}) - f(\mathbf{y})$ is a function with non-decreasing increments, so from Theorem 3.2, we obtain Theorem 3.5. □

**Remark.**
(i) *If* $\dfrac{\int_a^b (\mathbf{a}-\mathbf{Y}(t))\,dB(t)}{\int_a^b dB(t)} \in \mathbf{J}$ *and* $\forall\, y \in a_i \in (b_{i-1}, b_i)$ $(b_0 = a,$ $b_l = b)$ *we have either $B(y) \geq B(b)$ or $B(y) \leq B(a)$, then the inequality in* (3.5) *holds for reverse direction.*
(ii) *If putting $l = 1$, Theorem* 3.5 *gives a especial case as Theorem 3.2 of* [13].
(iii) *We can obtain discrete version of Theorem* 3.5*, using technique as given in Corollary 1 of* [14].

**Remark.** *Proposition 3.1 and Theorem 3.2 both are especial cases of the Theorems 3.3 and 3.5.*

**Corollary 3.6.** *Let $\mathbf{Y}$ satisfies the assumptions of Theorem* 3.5*, then the inequalities*

$$0 \leq \left(\int_a^b dB(t)\right)^{k-1} \int_a^b \prod_{i=1}^k Y_i(t)\,dB(t) - \prod_{i=1}^k \int_a^b Y_i(t)\,dB(t)$$
$$\leq \left(\int_a^b dB(t)\right)^{k-1} \int_a^b \prod_{i=1}^k (a_i + Y_i(t))\,dB(t) - \prod_{i=1}^k \int_a^b (a_i + Y_i(t))\,dB(t)$$

*hold, where all components of $\mathbf{Y}$ are nonnegative.*

*Proof.* The function $f(\mathbf{y}) = y_1 \cdots y_k$ is a function with non-decreasing increments of orders two and three for $\mathbf{y} \in \mathbb{R}^k_*$. So, using Theorems 3.2, and 3.5, we obtain Corollary 3.6. □

**Remark.** *If putting $l = 1$, Corollary* 3.6 *gives a especial case as Corollary 3.3 (ii) of* [13].

3.2. **On Inequalities of Jensen-Mercer-type.** The next theorem is the generalisation of Jensen-Mercer inequality using Jensen-Boas inequality, for this purpose using Lemma 14 from [15] for proving the theorems below:

**Theorem 3.7.** *Let $B \in bv[a,b]$ such that* (3.1) *valid and $\mathbf{Y} : [a,b] \to [\mathbf{0}, \mathbf{d}]$, $(\mathbf{d} > \mathbf{0})$ be a monotonic (either non-incresing or non-decresing) and continuous map in* *every of $l$ intervals $(b_{i-1}, b_i)$. If $f$ is a continuous FWNDI of third order in the interval $\mathbf{J} = [\mathbf{0}, 2\mathbf{d}] \subset \mathbb{R}^k$ and $L = \int_a^b dB(t) > 0$, then inequality*

$$\frac{1}{L}\int_a^b f\left(\mathbf{Y}(t)\right)\, dB(t) - f\left(\frac{1}{L}\int_a^b \mathbf{Y}(t)\, dB(t)\right) \leq \frac{1}{L}\int_a^b f\left(2\mathbf{d} - \mathbf{Y}(t)\right)\, dB(t) - f\left(\frac{1}{L}\int_a^b (2\mathbf{d} - \mathbf{Y}(t))\, dB(t)\right) \tag{3.6}$$

*holds.*

*Proof.* Using (3.4) for $\mathbf{h_3} = 2\mathbf{d} - 2\mathbf{y}$, since $\mathbf{y} \mapsto f(2\mathbf{d} - \mathbf{y}) - f(\mathbf{y})$ is a FWNDI of second order, i.e., it is a FWNDI. Now, applying Lemma 14 of [15] and get desired Theorem 3.7. □

**Remark.** (i) *If $\frac{1}{L}\int_a^b (2\mathbf{d} - \mathbf{Y}(t))\, dB(t) \in \mathbf{J}$ and $\forall\ y \in a_i \in (b_{i-1}, b_i)$ $(b_0 = a$, $b_l = b)$ we have either $B(y) \geq B(b)$ or $B(y) \leq B(a)$, then the inequality in* (3.6) *holds for reverse direction.*

(ii) *We can obtain discrete version of Theorem* 3.7*, using technique as given in Corollary 1 of* [14].

**Theorem 3.8.** *Let $B \in bv[a, b]$ such that* (3.1) *valid and $f$ be a continuous FWNDI of third order in the interval $[\mathbf{c}, \mathbf{d}] \subset \mathbb{R}^k$. Let $\mathbf{0} < \mathbf{a} < \mathbf{d} - \mathbf{c}$. If $\mathbf{Y} : [a, b] \to [\mathbf{c}, \mathbf{d} - \mathbf{a}]$ is a monotonic (either non-incresing or non-decresing) and continuous map in every of $l$ intervals $(b_{i-1}, b_i)$ and $L = \int_a^b dB(t) > 0$, then the inequality*

$$\frac{1}{L}\int_a^b f\left(\mathbf{Y}(t)\right)\, dB(t) - f\left(\frac{1}{L}\int_a^b \mathbf{Y}(t)\, dB(t)\right) \leq \frac{1}{L}\int_a^b f\left(\mathbf{a} - \mathbf{Y}(t)\right)\, dB(t) - f\left(\frac{1}{L}\int_a^b (\mathbf{a} - \mathbf{Y}(t))\, dB(t)\right) \tag{3.7}$$

*holds.*

*Proof.* Using (3.4) for $\mathbf{h_3} = \mathbf{a} = \text{constant} \in \mathbb{R}^k$, since $\mathbf{y} \mapsto f(\mathbf{a} + \mathbf{y}) - f(\mathbf{y})$ is a FWNDI, now applying Lemma 14 of [15] and get desired Theorem 3.8. □

**Remark.** (i) *If $\frac{1}{L}\int_a^b (\mathbf{a} - \mathbf{Y}(t))\, dB(t) \in \mathbf{J}$ and $\forall\ y \in a_i \in (b_{i-1}, b_i)$ $(b_0 = a$, $b_l = b)$ we have either $B(y) \geq B(b)$ or $B(y) \leq B(a)$, then the inequality in* (3.7) *holds for reverse direction.*

(ii) *We can obtain discrete version of Theorem* 3.8*, using technique as given in Corollary 1 of* [14].

**Remark.** *Lemma 14 of* [15] *is an especial case of the Theorems 3.7 and 3.8.*

## 4. Conclusion

At the end of this article we can conclude that functions with non-decreasing increments have closed connection with arithmetic integral mean, Wright convex functions, convex functions, $\nabla-$convex functions, Jensen $m-$connvex functions,

$m-$convex functions, $m-\nabla-$convex functions, $k-$monotonic functions, absolutely monotonic functions, completely monotonic functions, Laplace transform and exponentially convex functions, and we established these connections via finite difference operator, i.e. all these functions are especial and/or one dimensional cases of functions with non-decreasing increments where $\mathbf{h_1} = \mathbf{h_2} = \cdots = \mathbf{h_m}$. In this article, we have also given the generalisation of the Levinson's-type inequality and Jensen-Mercer's-type inequality by using Jensen-Boas inequality for function with non-decreasing increments of order three.

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