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ON CHARACTER AMENABILITY AND APPROXIMATE CHARACTER AMENABILITY OF BANACH ALGEBRAS

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Abstract. Some characterizations of character amenability and approximate character amenability of Banach algebras are investigated. In this respect, the new concept character amenability modulo an ideal and approximate character amenability modulo an ideal of Banach algebras are introduced. As an application of the obtained results, character amenability and approximate character amenability of semigroup algebras are characterized. On general semigroups such as E-inversive E-semigroup and eventually regular semigroup, it is shown that the character amenability and approximate character amenability of semigroup algebra $l^1(S)$ is equivalent to amenability of semigroup S. Some examples are included.

1. introduction

Let A be a Banach algebra and $\sigma(A)$ be the spectrum of A .i.e. the space of all non-zero multiplicative linear functionals $\varphi : A \to \mathbb{C}$. For $\varphi \in \sigma(A) \cup \{0\}$, the set of all Banach $A-$bimodule X with the right module action $x.a = \varphi(a)x$ is denoted by $\mathcal{M}^A_\varphi$ and the set of all Banach $A-$bimodule X with the left modulo action $a.x = \varphi(a)x$ is denoted by ${}_\varphi\mathcal{M}^A (a \in A, x \in X)$. A Banach algebra A is called left φ-amenable if every continuous derivation $D : A \to X^*$ is inner for all $X \in \mathcal{M}^A_\varphi$ and A is called left character amenable if it is left φ-amenable for every $\varphi \in \sigma(A) \cup \{0\}$. Similarly, right character amenability is defined by considering $X \in {}_\varphi\mathcal{M}^A$. A is called character amenable if it is both left and right character amenable. The concept of character amenability of Banach algebras was introduced by Monfared [14] and then improved by Kaniuth, Lau and Pym in [9, 10]. As such character amenability is weaker than the classical amenability introduced by Johnson in [7] so all amenable Banach algebras are character amenable. The concept of amenability modulo an ideal of Banach algebras was introduced by the second author and M. Amini in [1]. As an application of this concept, it was shown that amenability of semigroup S is equivalent to amenability of semigroup algebra $l^1(S)$ modulo I_σ, where I_σ is a the closed ideal corresponding to the least group congruence σ

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on S. This paper is devoted to discussion of character amenability and approximate character amenability of a large class of Banach algebras as quotient Banach algebras. In this regard, by introducing the new concepts character amenability and approximate character amenability modulo an ideal of Banach algebras, we present some characterizations of character amenability and approximate character amenability of Banach algebras. Using the presented characterizations, we show that a semigroup S is amenable if and only if the semigroup algebra $l^1(S)$ is character amenable (approximate character amenable) for a large class of semigroups as E-inversive E-semigroup and eventually regular semigroup. Finally we end this paper by giving some interesting examples.

2. Character amenability modulo an ideal

Let A be a Banach algebra and X be a Banach A-bimodule. By a derivation D, we mean a bounded linear mapping $D: A \to X$ with $D(ab) = a.D(b) + D(a)b$, $(a, b \in A)$ and by an inner derivation, we mean a derivation D which there exists $x \in X$ such that $D(a) = ad_x(a) = a.x - x.a$. Let A be a Banach algebra and I be a closed ideal of A, by $I^{\perp}$ we mean the set of all function $f \in A^*$ such that $\langle x, f\rangle = 0$ for all $x \in I$.

Definition 2.1. Let A be a Banach algebra, I be a closed ideal of A and $\varphi \in (\sigma(A) \cup \{0\}) \cap I^{\perp}$. A is left $\varphi-$amenable modulo I if for all $X \in \mathcal{M}_{\varphi}^{A}$ for which $I.X = 0$, every continuous derivation $D : A \to X^*$ is inner on the set theoretic $A \backslash I$. A is left character amenable modulo I if it is left $\varphi-$amenable modulo I for every $\varphi \in (\sigma(A) \cup \{0\}) \cap I^{\perp}$. Right character amenability modulo I is defined analogously by considering ${}_{\varphi}\mathcal{M}^{A}$. A is character amenable modulo I if it is both left and right character amenable modulo I.

Let A be Banach algebra and I be closed ideal of A. Then $\frac{A}{I}$ can be made as an A-bimodule where the module actions are defined by

$$(a + I).b = ab + I\ ,\ b.(a + I) = ba + I\ (a, b \in A).$$

By AI (resp. IA and I^2), we mean the closed ideal generated by $\{a.i : a \in A, i \in I\}$ (resp. $\{i.a : i \in I, a \in A\}$ and $\{i.j : i, j \in I\}$).

Blanket Assumption. All over this paper we fix AI, IA, and I^2 as above unless they are otherwise specified.

Lemma 2.2. *Let I be a closed ideal of Banach algebra A and $\varphi \in (\sigma(A) \cup \{0\}) \cap I^{\perp}$, then $\psi(a + I^2) = \varphi(a)$ (resp. $\psi(a + AI) = \varphi(a)$ and $\psi(a + IA) = \varphi(a)$) is the induction character of φ on $(\sigma(\frac{A}{I^2}) \cup \{0\})$ (resp. $(\sigma(\frac{A}{AI}) \cup \{0\})$ and $(\sigma(\frac{A}{IA}) \cup \{0\})$.*

Proof. Suppose that $a + I^2 = b + I^2$, then $a - b \in I^2 \subseteq I$. Since $\varphi \in I^{\perp}$, $\varphi(a - b) = 0$. Thus $\psi(a + I^2) = \varphi(a)$ is a well-defined character on the $\sigma(\frac{A}{I^2})$. The proof of the other parts are similar. $\square$

According to Lemma 2.2, for the rest of this paper we denote the induction character of $\varphi \in (\sigma(A) \cup \{0\}) \cap I^{\perp}$ on $(\sigma(\frac{A}{I^2}) \cup \{0\})$, $(\sigma(\frac{A}{AI}) \cup \{0\})$ or $(\sigma(\frac{A}{IA}) \cup \{0\})$ by ψ.

Theorem 2.3. *Let A be a Banach algebra and I be a closed ideal of A. Then the following assertions hold.*

i) If $\dfrac{A}{IA}$ is left $\psi-$amenable and $I^2 = IA$, then A is left $\varphi-$amenable modulo I, *ii) If* $\dfrac{A}{AI}$ *is right* $\psi-$*amenable and* $I^2 = AI$*, then* A *is right* $\varphi-$*amenable modulo* I.

Proof. i) Let $\varphi \in (\sigma(A) \cup \{0\}) \cap I^{\perp}$, $X \in \mathcal{M}_{\varphi}^{A}$, $I.X = 0$ and $D : A \to X^*$ be a bounded derivation. For each $x \in X$ and $a \in A$, define $x.(a+IA) = \psi(a+IA)x = \varphi(a)x$ and $(a+IA).x = a.x$. If $(a+IA) = (b+IA)$, then $(a-b) \in IA \subseteq I$. As $I.X = 0$, $(a-b).x = 0$ so $ax = bx$. Also, since $\varphi \in I^{\perp}$ so $\varphi(a-b) = 0$. This implies that $\varphi(a)x = \varphi(b)x$. Hence the above module actions are well-defined and X will have a Banach $\frac{A}{IA}-$bimodule structure. Then $X \in \mathcal{M}_{\psi}^{\frac{A}{IA}}$. Define $\tilde{D}(a+IA) = D(a)$. If $a+IA = b+IA$ then $a-b \in IA = I^2$. So there exist $i, j \in I$ such that $a-b = ij$. We have $D(a-b) = D(ij) = D(i).j + i.D(j) = \varphi(j)D(i) + i.D(j) = 0$. Then $\tilde{D}$ is a well-defined bounded derivation on $\frac{A}{IA}$. Left $\psi-$amenabily of $\frac{A}{IA}$ implies that there exists $\eta \in X^*$ such that $\tilde{D}(a+IA) = ad_{\eta}(a+IA)$. Thus for every $a \in A\backslash I$,

$$\begin{aligned} D(a) &= ad_{\eta}(a+IA) \\ &= (a+IA).\eta + \eta.(a+IA) \\ &= \psi(a+IA)\eta + \eta.(a+IA) \\ &= \varphi(a)\eta + \eta.a \\ &= ad_{\eta}(a). \end{aligned}$$

ii) The proof of (ii) is similar. $\square$

Let A be a Banach algebra, $\varphi \in \sigma(A) \cup \{0\}$ and $\pi : A\hat{\otimes}A \to A$ be the diagonal operator. Recall that a net $(m_{\alpha})_{\alpha} \subseteq A\hat{\otimes}A$ is called a left $\varphi-$approximate diagonal for A if for every $a \in A$,

$$\begin{aligned} m_{\alpha}.a - \varphi(a)m_{\alpha} &\to 0, \\ \langle \varphi \otimes \varphi, m_{\alpha} \rangle = \varphi(\pi(m_{\alpha})) &\to 1, \end{aligned}$$

and $M \in (A\hat{\otimes}A)^{**}$ is called left $\varphi-$virtual diagonal for A if for every $a \in A$,

$$\begin{aligned} M.a &= \varphi(a)M, \\ \langle M, \varphi \otimes \varphi \rangle &= \langle \pi^*(M), \varphi \rangle = 1. \end{aligned}$$

Theorem 2.4. *Let* A *be a Banach algebra and* I *be a closed ideal of* A*. If* A *is left* $\varphi-$*amenable modulo* I *and* $IA = I^2$*, then* $\frac{A}{IA}$ *has a left* $\psi-$*virtual diagonal.*

Proof. Consider the $A-$bimodule Banach algebra $\frac{A}{IA}\hat{\otimes}\frac{A}{IA}$ where the module actions are defined by $a.(\bar{b} \otimes \bar{c}) = \psi(\bar{a})(\bar{b} \otimes \bar{c}) = \varphi(a)(\bar{b} \otimes \bar{c})$ and $(\bar{b} \otimes \bar{c}).a = (\bar{b} \otimes \bar{c}\bar{a})$, $(a \in A, \bar{b}, \bar{c} \in \frac{A}{IA})$. So $\frac{A}{IA}\hat{\otimes}\frac{A}{IA} \in {}_{\psi}\mathcal{M}^{A}$. Then $(\frac{A}{IA}\hat{\otimes}\frac{A}{IA})^*$ will have $A-$bimodule structure where the right module action is $f.a = \psi(\bar{a}).f = \varphi(a)f$, $(a \in A, f \in (\frac{A}{IA}\hat{\otimes}\frac{A}{IA})^*)$. Also $(\frac{A}{IA}\hat{\otimes}\frac{A}{IA})^{**}$ will have $A-$bimodule structure with usual right module action and left module action defined by

$$a.\Phi = \psi(\bar{a}).\Phi, \ \ (a \in A, \Phi \in (\frac{A}{IA}\hat{\otimes}\frac{A}{IA})^{**}).$$

As ψ is a multiplicative on A,

$$\begin{aligned}\langle a.(\psi\otimes\psi),\bar{b}\otimes\bar{c}\rangle &= \langle\psi\otimes\psi,(\bar{b}\otimes\bar{c}).a\rangle\\ &= \langle\psi\otimes\psi,\bar{b}\otimes\overline{ca}\rangle\\ &= \psi(\bar{b})\psi(\overline{ca})\\ &= \psi(\bar{b})\psi(\bar{c})\psi(\bar{a})\\ &= \psi(\bar{a})\langle\psi\otimes\psi,\bar{b}\otimes\bar{c}\rangle,\end{aligned}$$

thus $a.(\psi\otimes\psi)=\psi(\bar{a})(\psi\otimes\psi)$. Obviously $\mathbb{C}.(\psi\otimes\psi)$ is a $1-$dimensional submodule of $(\frac{A}{IA}\hat{\otimes}\frac{A}{IA})^*$ generated by $\psi\otimes\psi$. Consider the quotient Banach $A-$bimodule $\mathcal{X}=\dfrac{(\frac{A}{IA}\hat{\otimes}\frac{A}{IA})^*}{\mathbb{C}.(\psi\otimes\psi)}$ and set $\mathcal{P}:(\frac{A}{IA}\hat{\otimes}\frac{A}{IA})^*\to\mathcal{X}$ as canonical $A-$bimodule quotient map. It is clear that

$$\mathcal{X}^*=(\frac{(\frac{A}{IA}\hat{\otimes}\frac{A}{IA})^*}{\mathbb{C}.(\psi\otimes\psi)})^*\cong(\mathbb{C}.(\psi\otimes\psi))^{\perp}\subseteq(\frac{A}{IA}\hat{\otimes}\frac{A}{IA})^{**},$$

thus $\mathcal{X}^*$ can be identified as a close subspace of $(\frac{A}{IA}\hat{\otimes}\frac{A}{IA})^{**}$. Set $\mathcal{P}^*:\mathcal{X}^*\to(\frac{A}{IA}\hat{\otimes}\frac{A}{IA})^{**}$ as a canonical adjoint of $\mathcal{P}$. Since $\psi\otimes\psi\in\mathcal{X}^*\subseteq(\frac{A}{IA}\hat{\otimes}\frac{A}{IA})^{**}$, by the Hahn-Banach theorem, there exists $\Phi_0\in(\frac{A}{IA}\hat{\otimes}\frac{A}{IA})^{**}$ such that $\langle\Phi_0,\psi\otimes\psi\rangle=1$. Put $ad_{\Phi_0}:A\to(\frac{A}{IA}\hat{\otimes}\frac{A}{IA})^{**}$, then

$$\begin{aligned}ad_{\Phi_0}(a) &= a.\Phi_0-\Phi_0.a\\ &= \psi(\bar{a})\Phi_0-\Phi_0.a\\ &= \psi(\bar{a})\Phi_0-\Phi_0.\bar{a}.\end{aligned}$$

It follows that

$$\begin{aligned}\langle ad_{\Phi_0}(a),\psi\otimes\psi\rangle &= \langle\psi(\bar{a})\Phi_0-\Phi_0.\bar{a},\psi\otimes\psi\rangle\\ &= \langle\Phi_0,\psi(\bar{a})(\psi\otimes\psi)\rangle-\langle\Phi_0,\bar{a}.(\psi\otimes\psi)\rangle\\ &= \langle\Phi_0,\psi(\bar{a})(\psi\otimes\psi)\rangle-\langle\Phi_0,\psi(\bar{a})(\psi\otimes\psi)\rangle\\ &= 0,\end{aligned}$$

thus $ad_{\Phi_0}(a)\in(\mathbb{C}.(\psi\otimes\psi))^{\perp}\simeq\mathcal{X}^*$. On the other hand, $\mathcal{X}$ is an $A-$bimodule Banach algebra with canonical left module action and right module action defined as $x.a=x.\psi(\bar{a})=x.\varphi(a)$, for $x\in\mathcal{X},a\in A$. There exists $g\in(\frac{A}{IA}\hat{\otimes}\frac{A}{IA})^*$ such that $x=g+\mathbb{C}.(\psi\otimes\psi)$ and

$$\begin{aligned}x.a &= (g+\mathbb{C}.(\psi\otimes\psi)).a\\ &= \psi(\bar{a})g+\mathbb{C}.(\psi\otimes\psi),\end{aligned}$$

thus $\mathcal{X}\in\mathcal{M}_{\varphi}^{A}$. Also $I.\mathcal{X}=0$, because for every $i\in I$,

$$\begin{aligned}i.(\bar{a}\otimes\bar{b}) &= i.((a+IA)\otimes(b+IA))\\ &= (i.a+IA)\otimes(b+IA)\\ &= 0\otimes\bar{b}=0.\end{aligned}$$

Put $D:A\to\mathcal{X}^*$ by $D(a)=ad_{\Phi_0}(a)$. Left $\varphi-$amenability modulo I of A implies that there exists $\Phi_1\in\mathcal{X}^*$ such that $D(a)=ad_{\Phi_1}(a)=ad_{\Phi_0}(a)$ (for all $a\in A\setminus I$).

Set $M = \Phi_0 - \Phi_1$. We claim that M is $\psi-$virtual diagonal for $\frac{A}{IA}$. We have

$$\begin{aligned}\langle M, \psi \otimes \psi \rangle &= \langle \Phi_0 - \Phi_1, \psi \otimes \psi \rangle \\ &= \langle \Phi_0, \psi \otimes \psi \rangle - \langle \Phi_1, \psi \otimes \psi \rangle \\ &= 1 - 0 = 1.\end{aligned}$$

So for $\bar{a} = a + IA \in \frac{A}{IA}$, we have

$$\begin{aligned}\bar{a}.\Phi_0 - \Phi_0.\bar{a} &= \psi(\bar{a})\Phi_0 - \Phi_0.\bar{a} \\ &= ad_{\Phi_0}(\bar{a}) \\ &= ad_{\Phi_1}(\bar{a}) \\ &= \bar{a}.\Phi_1 - \Phi_1.\bar{a} \\ &= \psi(\bar{a})\Phi_1 - \Phi_1.\bar{a}.\end{aligned}$$

Hence $\psi(\bar{a})\Phi_0 - \Phi_0.\bar{a} = \psi(\bar{a})\Phi_1 - \Phi_1.\bar{a}$, and $(\Phi_0 - \Phi_1).\bar{a} = \psi(\bar{a})(\Phi_0 - \Phi_1)$. Thus $M.\bar{a} = \psi(\bar{a})M$ and M is a $\psi-$virtual diagonal for $\frac{A}{IA}$. □

We recall that for a Banach algebra A, $\varphi \in \sigma(A) \cup \{0\}$ and $\eta \in A^{**}$, η is a φ-TLI (φ-topologically left invariant) if $\langle \eta, a.f \rangle = \varphi(a)\langle \eta, f \rangle$ and η is called φ-TRI (φ-topologically right invariant) if $\langle \eta, f.a \rangle = \varphi(a)\langle \eta, f \rangle$, $(a \in A, f \in A^*)$. It is shown that if A is a Banach algebra and $\varphi \in \sigma(A)$, then A has a left $\varphi-$virtual diagonal if and only if A has a bounded left $\varphi-$approximate diagonal, if and only if A is left $\varphi-$amenable, if and only if there exists a $\varphi-$TLI $\eta \in A^{**}$ such that $\eta(\varphi) = 1$, if and only if there exists a bounded net (u_α) in A such that $u_\alpha.a - \varphi(a).u_\alpha \to 0$ for all $a \in A$ and $\varphi(u_\alpha) = 1$ for all α [11, Theorem 2.3]. Using Theorems 2.3 and 2.4, we have the following result.

Theorem 2.5. *Let A be a Banach algebra, I be a closed ideal of A, $\varphi \in \sigma(A) \cap I^{\perp}$ and ψ be the induction character of φ on $\frac{A}{IA}$. Then the following assertions are equivalent.*

i) A is left $\varphi-$amenable modulo I.
ii) $\frac{A}{IA}$ has a left $\psi-$virtual diagonal.
iii) $\frac{A}{IA}$ is left $\psi-$amenable.
iv) $\frac{A}{IA}$ has a bounded left $\psi-$approximate diagonal.
v) There exists a $\psi-TLI$ $\eta \in (\frac{A}{IA})^{**}$ such that $\eta(\psi) = 1$.
vi) There exists a bounded net $(\bar{u}_\alpha)_\alpha \subseteq \frac{A}{IA}$ such that $\bar{u}_\alpha.\bar{a} - \psi(\bar{a}).\bar{u}_\alpha \to 0$ for all $\bar{a} \in \frac{A}{IA}$ and $\psi(\bar{u}_\alpha) = 1$ for all α.

Similar condition are equivalent for right $\varphi-$amenability modulo I.

We recall the following well-known result.

Proposition 2.6. *(*[9, Corollary 2.7]*) A Banach algebra A is left (right) character amenable if and only if $\ker\varphi$ has a bounded left (right) approximate identity for every $\varphi \in \sigma(A) \cup \{0\}$.*

Theorem 2.7. *By assumptions of Lemma 2.2, If $IA = I^2$, then IA has a bounded left approximate identity if and only if A is left φ-amenable modulo I.*

Proof. Let $\psi \in \sigma(\frac{A}{IA})$, then $ker\psi = IA$. Using 2.6, IA has a bounded left approximate identity if and only if $\frac{A}{IA}$ is left ψ-amenable, if and only if A is left φ-amenable modulo I (by Theorems 2.3, 2.4 and 2.5). □

3. APPROXIMATE CHARACTER AMENABILITY MODULO AN IDEAL

Let A be a Banach algebra and X be a Banach $A-$bimodule. A derivation $D: A \to X$ is called approximately inner if there exists a net $(\phi_\alpha)_\alpha \subseteq X$ such that $D(a) = \lim_\alpha (a.\phi_\alpha - \phi_\alpha.a)$ for all $a \in A$ .i.e. $D = \lim_\alpha ad_{\phi_\alpha}$.

Definition 3.1. Let I be a closed ideal of Banach algebra A and $\varphi \in (\sigma(A) \cup \{0\}) \cap I^\perp$. A is left approximately φ-amenable modulo I if for all $X \in \mathcal{M}_\varphi^A$ for which $I.X = 0$, every continuous derivation $D: A \to X^*$ is approximately inner on the set theoretic $A \backslash I$. Moreover, A is left approximately character amenable modulo I if it is left approximately $\varphi-$amenable modulo I for every $\varphi \in (\sigma(A) \cup \{0\}) \cap I^\perp$. Right approximate character amenability modulo an ideal is defined analogously by considering ${}_\varphi\mathcal{M}^A$. We call A is approximately character amenable modulo I if it is both left and right approximately character amenable modulo I.

Theorem 3.2. *Let A be a Banach algebra, I be a closed ideal of A and $\varphi \in (\sigma(A) \cup \{0\}) \cap I^\perp$. Then (i) implies (ii) and if $I^2 = I$, (ii) implies (i).*
(i) A is approximately $\varphi-$amenable modulo I,
(ii) there exist $m \in (\frac{A}{I})^{**}$ and a net $(m_\alpha) \subseteq (\frac{A}{I})^{**}$ such that $m(\varphi) = 1$ and

$$m(f.a) = \lim_\alpha m_\alpha(f.a) = \lim_\alpha \varphi(a) m_\alpha(f), \ (a \in A \backslash I, \ f \in (\frac{A}{I})^*).$$

Proof. $(i) \to (ii)$ Consider $(\frac{A}{I})^*$ as a Banach $A-$bimodule where module actions are defined as

$$\langle \bar{b}, a.f \rangle = \varphi(a) \langle \bar{b}, f \rangle, \ \langle \bar{b}, f.a \rangle = \langle \overline{ab}, f \rangle, \ (a \in A, \bar{b} \in \frac{A}{I}, f \in (\frac{A}{I})^*).$$

Thus $(\frac{A}{I})^* \in {}_\varphi\mathcal{M}^A$. Also, Consider $(\frac{A}{I})^{**}$ as a Banach $A-$module where module actions are defined as

$$\langle f, \Psi.a \rangle = \varphi(a) \langle f, \Psi \rangle, \ \langle f, a.\Psi \rangle = \langle f.a, \Psi \rangle, \ (a \in A, f \in (\frac{A}{I})^*, \Psi \in (\frac{A}{I})^{**}).$$

As $\varphi \in (\frac{A}{I})^* \simeq I^\perp$ and φ is multiplicative on A, $a.\varphi = \varphi.a = \varphi(a)\varphi$. So $\mathbb{C}\varphi$ is a closed $A-$submodule of $(\frac{A}{I})^* \simeq I^\perp$. Consider the canonical map $i : (\frac{A}{I})^* \to (\frac{A}{I})^* \backslash \mathbb{C}\varphi$ and let $D: A \to (\frac{A}{I})^{**}$ be a derivation. As $(\frac{A}{I})^{**} \in \mathcal{M}_\varphi^A$ and $I.(\frac{A}{I})^{**} = 0$, approximate $\varphi-$amenability modulo I of A implies that there exists a net $(\tilde{u}_\alpha)$ in $(\frac{A}{I})^{**}$ such that $D(a) = \lim_\alpha a.\tilde{u}_\alpha - \tilde{u}_\alpha.a$, $(a \in A \backslash I)$. We have;

$$\langle \varphi, D(a) \rangle = \lim_\alpha \langle \varphi, a.\tilde{u}_\alpha \rangle - \langle \varphi, \tilde{u}_\alpha.a \rangle = \lim_\alpha \langle \varphi.a, \tilde{u}_\alpha \rangle - \langle \varphi.a, \tilde{u}_\alpha \rangle = 0,$$

so $D(a) \in i^*((\frac{A}{I})^* \backslash \mathbb{C}\varphi)^*$. As i^* is monomorphism, there exists $d(a) \in ((\frac{A}{I})^* \backslash \mathbb{C}\varphi)^*$ such that $i^*(d(a)) = D(a)$. Therefore $d: A \hookrightarrow (\frac{A}{I}) \to ((\frac{A}{I})^* \backslash \mathbb{C}\varphi)^*$ is a derivation. It is clear that $((\frac{A}{I})^* \backslash \mathbb{C}\varphi)^* \in \mathcal{M}_\varphi^A$ and $I.((\frac{A}{I})^* \backslash \mathbb{C}\varphi)^* = 0$, so there exists a net $(\tilde{\zeta}_\beta) \in ((\frac{A}{I})^* \backslash \mathbb{C}\varphi)^*$ such that $d(a) = \lim_\beta a.\tilde{\zeta}_\beta - \tilde{\zeta}_\beta.a$, $(a \in A \backslash I)$. Thus for each $a \in A \backslash I$,

$$\begin{aligned} \lim_\beta a.(i^*\tilde{\zeta}_\beta) - (i^*\tilde{\zeta}_\beta).a &= \lim_\beta i^*(a.\tilde{\zeta}_\beta - \tilde{\zeta}_\beta.a) \\ &= i^*(d(a)) = D(a) \\ &= \lim_\alpha (a.\tilde{u}_\alpha - \tilde{u}_\alpha.a). \end{aligned}$$

Put $m_{\alpha,\beta} = \tilde{u}_\alpha - i^*\tilde{\zeta}_\beta \in (\frac{A}{I})^{**}$. We have $m_{\alpha,\beta}(\varphi) = 1$ and $a.m_{\alpha,\beta} = m_{\alpha,\beta}.a$, so $m_{\alpha,\beta}(f.a) = m_{\alpha,\beta}(a.f) = \varphi(a) m_{\alpha,\beta}(f)$, $(a \in A \backslash I, f \in (\frac{A}{I})^*)$. Set $L = I \times \prod_{\alpha \in I} J$ where I, J are the index set of nets $(\tilde{u}_\alpha), (\tilde{\zeta}_\beta)$ respectively, with the product ordering. Using an iterated limit construction [8], let $m_k = m_{\alpha,f(\alpha)}$, (for each $k = (\alpha, f) \in K$). By Setting $\lim_k m_k = m$, the proof is complete.

$(ii) \to (i)$ Let $X \in \mathcal{M}_{\varphi}^{A}$ with $I.X = 0$ and $d : A \to X^*$ be a bounded derivation. We can consider X as a Banach $(\frac{A}{I})-$module where the module actions are defined by $\bar{a}.x = a.x$ and $x.\bar{a} = \psi(\bar{a}).x$ where $\psi(\bar{a}) = \psi(a + I) = \varphi(a)$ is the induction character of A on $\frac{A}{I}$ as in Lemma 2.2 ($\bar{a} \in \frac{A}{I}, x \in X$). It is easy to see that the defined module actions of X are well defined. Set $D : \frac{A}{I} \to X^*$ by $D(a+I) = d(a)$. Now if $a, b \in A$ and $a - b \in I = I^2$, then $a - b = i.j$, (for some $i, j \in I$). As $I.X = X.I = 0$, $d(a-b) = d(i.j) = 0$. This imply that D is a well defined derivation. Suppose that (m_α) and m are as assumptions (ii). Set $D_1 = D^* \mid_X : X \to (\frac{A}{I})^*$ and $\bar{u}_\alpha = (D_1)^*(m_\alpha) \in X^*$ such that $\lim_\alpha \bar{u}_\alpha = \lim_\alpha (D_1)^*(m_\alpha) = (D_1)^*(m)$. We have;

$$\langle \bar{b}, D_1(\bar{a}.x)\rangle = \langle \bar{a}.x, D(\bar{b})\rangle = \psi(\bar{a})\langle x, D(\bar{b})\rangle = \psi(\bar{a})\langle \bar{b}, D_1(x)\rangle,\ (\bar{a}, \bar{b} \in \frac{A}{I}, x \in X),$$

so $D(\bar{a}.x) = \psi(\bar{a})D_1(x)$. Thus;

$$\langle x, \bar{u}_\alpha.\bar{a}\rangle = \langle \bar{a}.x, \bar{u}_\alpha\rangle = \langle D_1(\bar{a}.x), m_\alpha\rangle = \psi(\bar{a})\langle D_1(x), m_\alpha\rangle = \psi(\bar{a})\langle x, \bar{u}_\alpha\rangle.$$

As D is a derivation,

$$\begin{aligned}\langle \bar{b}, D_1(\bar{a}.x)\rangle &= \langle x.\bar{a}, D(\bar{b})\rangle = \langle x, \bar{a}.D(\bar{b})\rangle \\ &= \langle \bar{a}\bar{b}, D_1(x)\rangle - \langle \bar{b}.x, D(\bar{a})\rangle,\ (\bar{a}, \bar{b} \in \tfrac{A}{I}, x \in X) \\ &= \langle \bar{b}, D_1(x).\bar{a}\rangle - \psi(\bar{b})\langle x, D(\bar{a})\rangle.\end{aligned}$$

Thus $D_1(x.\bar{a}) = D_1(x).\bar{a} - \langle x, D(\bar{a})\rangle\varphi$, ($\bar{a} \in \frac{A}{I}, x \in X$). Therefore

$$\begin{aligned}\langle D_1(x.\bar{a}), m\rangle &= \langle x.\bar{a}, (D_1)^*(m)\rangle = \lim_\alpha \langle x, \bar{a}.u_\alpha\rangle \\ &= \lim_\alpha \langle x.\bar{a}, u_\alpha\rangle = \lim_\alpha \langle D_1(x.\bar{a}), m_\alpha\rangle \\ &= \lim_\alpha \langle D_1(x).\bar{a}, m_\alpha\rangle - \lim_\alpha \langle x, D(\bar{a})\rangle\langle \varphi, m_\alpha\rangle \\ &= \lim_\alpha \psi(\bar{a})\langle x, \bar{u}_\alpha\rangle - \langle x, D(\bar{a})\rangle.\end{aligned}$$

Thus

$$\begin{aligned}d(a) = D(\bar{a}) &= \lim_\alpha \psi(\bar{a})\bar{u}_\alpha - \bar{a}.\bar{u}_\alpha \\ &= \lim_\alpha \varphi(a)\bar{u}_\alpha - \bar{a}.\bar{u}_\alpha \\ &= \lim_\alpha a.(-\bar{u}_\alpha) - (-\bar{u}_\alpha).a.\end{aligned}$$

□

Theorem 3.3. *Let A be a Banach algebra and I be a closed ideal of A. If $\frac{A}{I^2}$ be left approximately $\psi-$amenable, then A is left approximately φ-amenable modulo I.*

Proof. Suppose that $\varphi \in (\sigma(A) \cup \{0\}) \cap I^{\perp}$, $X \in \mathcal{M}_{\varphi}^{A}$, such that $I.X = 0$ and $D : A \to X^*$ be a bounded derivation. Using assumptions, it is not far too see that the module actions $(a+I^2).x = a.x$ and $x.(a+I^2) = \psi(a+I^2).x = \varphi(a).x$, made X as a Banach $\frac{A}{I^2}$-bimodule. Thus $X \in \mathcal{M}_{\psi}^{\frac{A}{I^2}}$. Set $\tilde{D} : \frac{A}{I^2} \to X^*$ by $\tilde{D}(a+I^2) = D(a)$. Since $X^*.I = I.X^* = 0$, $\tilde{D}$ is a well-defined bounded derivation. Left approximately $\psi-$amenability of $\frac{A}{I^2}$ implies that there exist a net $\eta_\alpha \in X^*$ such that $\tilde{D}(a + I^2) = \lim_\alpha ad_{\eta_\alpha}(a + I^2)$ for all $a \in A\backslash I$ .Hence $D(a) = \tilde{D}(a + I^2) = \lim_\alpha ad_{\eta_\alpha}(a + I^2) = \lim_\alpha ad_{\eta_\alpha}(a)$. Thus A is left approximately $\varphi-$amenable modulo I. □

Theorem 3.4. *Let I be a closed ideal of Banach algebra A and $I^2 = I$. Then A is approximately character amenable modulo I if and only if $\frac{A}{I}$ is approximately character amenable.*

Proof. We show all assertion for left approximate character amenability. Similar statement holds for right approximate character amenability. Suppose that $\psi \in \sigma(\frac{A}{I})$, $X \in \mathcal{M}_{\psi}^{\frac{A}{I}}$ and $D : \frac{A}{I} \to X^*$ is a bounded derivation. Then X can be made in to a Banach $A-$bimodule by defined actions $x.a = \varphi(a).x$ and $a.x = \bar{a}.x(a \in A, x \in X)$. Thus $X \in \mathcal{M}_{\varphi}^{A}$ and $I.X = 0$. Now $D \circ \pi : A \to X^*$ is a bounded derivation on A. Left approximate φ-amenability modulo I of A implies that there exist a net $(\phi_\alpha)_\alpha \subseteq X^*$ such that $D \circ \pi(a) = \lim_\alpha ad_{\phi_\alpha}(a)$ for all $a \in A \backslash I$. Hence $D(\bar{a}) = D \circ \pi(a) = \lim_\alpha ad_{\phi_\alpha}(a) = \lim_\alpha ad_{\phi_\alpha}(\bar{a})$. The converse proved in Theorem 3.3. □

It is shown that character amenability of Banach algebras implies existence of bounded approximate identity for them [14, Theorem 2]. The same result is given for approximate character amenability of Banach algebras; for a Banach algebra A and $\varphi \in \sigma(A)$, if A is approximately φ-amenable, then A have right and left approximate identity [6, proposition 2.6]. In the following Theorem we show that approximate $\varphi-$amenability modulo I of A implies existence of approximate identity for $\frac{A}{I^2}$.

Theorem 3.5. *Let A be a Banach algebra and I be a closed ideal of A such that $I^2 = I$. If A is a left approximately φ-amenable modulo I, then $\frac{A}{I}$ has left approximate identity..*

Proof. Using Theorem 3.4, A is approximately φ-amenable modulo I if and only if $\frac{A}{I}$ is approximately ψ-amenable. Since $\frac{A}{I}$ is left approximately ψ-amenable, $\frac{A}{I}$ has left approximate identity. □

Theorem 3.6. *Let A be a Banach algebra, I be a closed ideal of A which $I^2 = I$, $\varphi \in (\sigma(A) \cup \{0\}) \cap I^\perp$ and let $\tilde{\varphi}$ be the extension of φ to A^{**}. Then A is approximately $\varphi-$amenable modulo I if and only if A^{**} is approximately $\tilde{\varphi}-$amenable modulo I^{**}.*

Proof. Using Theorem 3.2, there exist $m \in (\frac{A}{I})^{**}$ and $(m_\alpha) \subseteq (\frac{A}{I})^{**}$ such that $m(\varphi) = 1$ and $m(f.a) = \lim_\alpha m_\alpha(f.a) = \lim_\alpha \varphi(a) m_\alpha(f)$, $(a \in A \backslash I,\ f \in (\frac{A}{I})^*)$. Let $\hat{m}$ and $(\hat{m_\alpha})$ be the Gelfand transform of m and (m_α) respectively. Suppose that $\bar{a}^{**} \in (\frac{A}{I})^{**}$ and $\bar{u} \in (\frac{A}{I})^{***}$, then there exist nets $(\bar{a}_\gamma) \subseteq (\frac{A}{I})$ and $(\bar{u}_\beta) \subseteq (\frac{A}{I})^*$ such that $\bar{a}_\gamma \xrightarrow{w^*} \bar{a}^{**}$, $\bar{u_\beta} \xrightarrow{w^*} \bar{u}$. Thus

$$\begin{aligned}
\langle \bar{u}.\bar{a}^{**}, \hat{m} \rangle = \langle m, \bar{u}.\bar{a}^{**} \rangle \quad &= \lim_\beta \langle m, \bar{u}_\beta \bar{a}^{**} \rangle \\
&= \lim_\beta \lim_\gamma \langle m, \bar{u}_\beta \bar{a_\gamma}^{**} \rangle \\
&= \lim_\beta \lim_\gamma \lim_\alpha \langle \varphi(\bar{a}_\gamma) \langle \bar{u}_\beta, m_\alpha \rangle \\
&= \lim_\beta \lim_\alpha \tilde{\varphi}(\bar{a}^{**}) \langle \bar{u}_\beta, m_\alpha \rangle \\
&= \lim_\alpha \tilde{\varphi}(\bar{a}^{**}) \hat{m}_\alpha(\bar{u})
\end{aligned}$$

Conversely, let A^{**} be approximately $\tilde{\varphi}-$amenable modulo I, then there exists $\tilde{M} \subseteq (\frac{A}{I})^{****}$ and $(\tilde{M_\alpha}) \subseteq (\frac{A}{I})^{****}$ such that $\tilde{M}(f.a) = \lim_\alpha \tilde{\varphi}(\bar{a}) \tilde{M}_\alpha(f)$. Now with restriction of $\tilde{M}$ and $\tilde{M}_\alpha$ to $(\frac{A}{I})^{**}$, we conclude that $\frac{A}{I}$ is approximately amenable. Since $I^2 = I$, A is approximately amenable modulo I. □

Theorem 3.7. *Let A and B be Banach algebras, I and J be closed ideals of A and B and $\theta : A \to B$ is a continuous homomorphism with dense rang such that $\theta(I) \subset J$. If A is left approximately character amenable modulo I, then B is left approximately character amenable modulo J.*

Proof. Let $\varphi \in (\sigma(B) \cup \{0\}) \cup J^{\perp}$ and $X \in \mathcal{M}_{\varphi}^{B}$. If $\varphi \neq 0$, then $\varphi \circ \theta \neq 0$ and $\varphi \circ \theta \in (\sigma(A) \cup \{0\}) \cup I^{\perp}$. Then X can be identified as a Banach $A-$bimodule where the module actions are defined as $x.a = x.\theta(a) = \varphi(\theta(a))x$ and $a.x = \theta(a).x$, for all $(a \in A, x \in X)$. Let $D : B \to X^*$ be a bounded derivation. It is routine to check that $D \circ \theta : A \to X^*$ is a bounded derivation. Since A is left approximately character amenable modulo I, there exist a net $(u_\alpha)_\alpha \in X^*$ such that for all $a \in A \setminus I$,

$$D \circ \theta(a) = \lim_{\alpha} ad_{u_\alpha}(a) = \lim_{\alpha}(a.u_\alpha - u_\alpha.a),$$

since $\overline{\theta(A)} = B$, for every $b \in B \setminus J$, there exists a sequence $(a_n)_n \subset A \setminus I$ such that $\theta(a_n) \to b$. Using the continuity of module actions, we have

$$\begin{aligned} D(b) &= D(\lim_n \theta(a_n)) = \lim_n D \circ \theta(a_n) \\ &= \lim_n \lim_\alpha ad_{f_\alpha}(a_n) \\ &= \lim_{n,\alpha}(a_n.f_\alpha - f_\alpha.a_n) \\ &= \lim_{n,\alpha}(\theta(a_n).f_\alpha - f_\alpha.\theta(a_n)) \\ &= \lim_\alpha(\lim_n \theta(a_n).f_\alpha - f_\alpha. \lim_n \theta(a_n)) \\ &= \lim_\alpha(b.f_\alpha - f_\alpha.b) \\ &= \lim_\alpha ad_{f_\alpha}(b). \end{aligned}$$

□

4. Approximate character amenability of semigroup algebras

In the following, using the obtained results in the previous sections, we study approximately character amenability of semigroup algebras. Let us recall some basic preliminaries of semigroup theory. For more details we refer the reader to [2, 5]. For semigroup S, we denote the set of idempotents of S by $E(S)$. A congruence ρ on semigroup S is called a group congruence if S/ρ is group. We denote the least group congruence on S by σ. We recall that the semigroup algebra $l^1(S) = \{f | f : S \to \mathbb{C}, \sum_{s\in S} |f(s)| \leq \infty\}$ is a Banach algebra under $\|f\|_1 = \sum_{s\in S} |f(s)|$. Each member of $l^1(S)$ has a unique presentation as $f = \sum_{s\in S} c_s\delta_s$, where δ_s is the point mass at s. Suppose that $\pi : S \to S/\rho$, is the quotient map then one can extend π to an algebra epimorphism $\widehat{\pi} : l^1(S) \to l^1(S/\rho)$, whose $ker(\widehat{\pi}) = I_\rho$ is an ideal in $l^1(S)$ generated by the set $\{\delta_s - \delta_t : s, t \in S \text{ with}\, s\rho t\}$. Also, $l^1(\frac{S}{\rho}) \simeq \frac{l^1(S)}{I_\rho}$ and if S is an E-inversive semigroup with commuting idempotents or S is an eventually inverse semigroup and σ is the least group congruence on S, then $I_\sigma = I_\sigma^2$, [1, Lemma 2]. The kernel of congruence ρ on semigroup S is the set $\{a \in S : a\rho \in E(S/\rho)\} = \{a \in S : (a, a^2) \in \rho\}$[3]. It is shown that for semigroup S, if $Ker\rho$ is central then S is amenable if and only if S/ρ is amenable [1, Theorem 2].

It is shown that for a locally compact group G, approximate character amenability of $L^1(G)$ as a Banach algebra is equivalent to amenability of group G [13, Theorem 3.5]. Up to now a little works has been done on the character amenability and approximate character amenability for semigroup algebras as in the other notions of amenability. To see the presented results of character amenability of semigroup algebras we refer to [12, 13]. In the following we investigate to approximate character amenability of the semigroup algebra $l^1(S)$ in relation to the semigroup S.

Theorem 4.1. *Let S be a semigroup, ρ be a group congruence on S such that $Ker(\rho)$ is central and I_ρ has an approximate identity. Then S is amenable if and only $l^1(S)$ is approximately character amenable modulo I_ρ.*

Proof. The semigroup S is amenable if and only if S/ρ is amenable [1, Theorem2]. Since S/ρ is a group, S/ρ is amenable if and only if $l^1(S/\rho)$ is approximately character amenable [13, Theorem 3.5]. As I_ρ has an approximate identity, $I^2 = I$. Using Theorem 3.5, $l^1(S/\rho)$ is approximately character amenable if and only if $l^1(S)$ is character amenable modulo I_ρ. □

A semigroup S is called *E-inversive* if for all $x \in S$, there exists $y \in S$ such that $xy \in E(S)$ and S is called an *E-semigroup* if $E(S)$ forms a sub-semigroup of S. If S is an E-inversive E-semigroup with commutative idempotents then the relation $\sigma = \{(a,b) \in S \times S \,|\, ea = fb \; for\, some \; e, f \in E_S\}$ is the least group congruence on S [3].

Using the similar argument of Theorem 4.1, we have the following result.

Theorem 4.2. *Let S be an E-inversive E-semigroup with commuting idempotents. Then S is amenable if and only if $l^1(S)$ is approximately character amenable modulo I_σ.*

Example 4.3. (i) Consider the bicyclic semigroup $S = \{p^m q^n : m, n \geq 0\}$ generated by $p\,, q$ and define $x\sigma y\,(x, y \in S)$ if and only if $ex = ey$ for some $e \in E(S)$. Then σ is the least group congruence on S and $S/\sigma = \mathbb{Z}$ [2]. As S is an amenable inverse semigroup [1], $l^1(S)$ is approximately character amenable modulo I_σ.

(ii) Let $S = (\mathbb{N}, +)$. It is shown that $\rho_n = \{(k,l) \in \mathbb{N}\times\mathbb{N} : n \mid (k-l)\}\,(n > 0)$ is a group congruence on S [3]. As S is a commutative semigroup, S is amenable. Since S/ρ_n is amenable group. Using [13, Theorem 3.5], $l^1(S/\rho_n)$ is (approximately) character amenable. We clime that $l^1(S)$ is not approximate character amenable modulo I_σ. Suppose by contradiction that $l^1(S)$ is approximately character amenable modulo I_{ρ_n}. Suppose that $\varphi \in \sigma(l^1(S)) \cup \{0\}) \cap I^\perp$ and put $X = \frac{l^1(S)}{I_{\rho_n}} \simeq l^1(S/\rho_n)$. It is easy to check X becomes a Banach $l^1(S)-$bimodule by the following module actions:

$$\delta_x * \delta_{\bar{y}} = \delta_{\overline{x+y}},\, \delta_{\bar{y}} * \delta_x = \varphi(\delta_x)\delta_{\bar{y}}, (\delta_x \in l^1(S), \delta_{\bar{y}} \in l^1(S/\rho_n))$$

Thus $X \in \mathcal{M}_\varphi^{l^1(S)}$ and $I_{\rho_n}.X = 0$. It is routine to check $X^* \in {}_\varphi\mathcal{M}^{l^1(S)}$ and $X^*.I_{\rho_n} = 0$. Let $\psi \in \sigma(\frac{l^1(S)}{I_{\rho_n}})$ be the induction character of φ. Then $\varphi(\delta_x) = \psi(\delta_x + I_{\rho_n}) = \psi(\delta_{\bar{x}})$. It is not hard to see that $X \in \mathcal{M}_\psi^{\frac{l^1(S)}{I_{\rho_n}}}$ where right module action is $\delta_{\bar{x}}.\delta_{\bar{y}} = \psi(\bar{y})\delta_{\bar{x}}$ and left module action is it's multiplication. Let $D : l^1(S) \to X^*$ be a nonzero bounded derivation. Then there exists a net $\bar{u}_\alpha \in X^*$ such that $D(\delta_k) = \lim_\alpha(\delta_k.\bar{u}_\alpha - \bar{u}_\alpha.\delta_k) = \lim_\alpha(\varphi(\delta_k)\bar{u}_\alpha - \bar{u}_\alpha.\delta_k)$. Now for each α, we have

$$\begin{aligned}
\langle \delta_{\bar{z}}, D(\delta_k)\rangle = \;& \lim_\alpha \langle \delta_{\bar{z}}, \varphi(\delta_k)\bar{u}_\alpha - \bar{u}_\alpha.\delta_k\rangle \\
&= \lim_\alpha \varphi(\delta_k)\langle \delta_{\bar{z}}, \bar{u}_\alpha\rangle - \langle \delta_{\bar{z}}, \bar{u}_\alpha.\delta_k\rangle \\
&= \lim_\alpha \varphi(\delta_k)\langle \delta_{\bar{z}}, \bar{u}_\alpha\rangle - \langle \delta_{\overline{k+z}}, \bar{u}_\alpha\rangle \\
&= \lim_\alpha \varphi(\delta_k)\langle \delta_{\bar{z}}, \bar{u}_\alpha\rangle - \langle \delta_{\overline{z+k}}, \bar{u}_\alpha\rangle \\
&= \lim_\alpha \varphi(\delta_k)\langle \delta_{\bar{z}}, \bar{u}_\alpha\rangle - \langle \delta_{\bar{z}}, \delta_{\bar{k}}.\bar{u}_\alpha\rangle \\
&= \lim_\alpha \varphi(\delta_k)\langle \delta_{\bar{z}}, \bar{u}_\alpha\rangle - \psi(\delta_{\bar{k}})\langle \delta_{\bar{z}}, \bar{u}_\alpha\rangle \\
&= \lim_\alpha \varphi(\delta_k)\langle \delta_{\bar{z}}, \bar{u}_\alpha\rangle - \varphi(\delta_k)\langle \delta_{\bar{z}}, \bar{u}_\alpha\rangle = 0
\end{aligned}$$

Therefore D vanishes on X, which is a contradiction. We note that the congruence $\rho_n\,(n\in\mathbb{N})$ is not the least group congruence and $I_{\rho_n}^2 \neq I_{\rho_n}$.

By an eventually regular semigroup S, we mean a semigroup S which every element of S has some power that is regular and $E(S)$ is a commutative and idempotent semigroup. It is shown the relation $\sigma=\{(s,t): es=et,\ for\,some\ e\in E(S)\}$ is the least group congruence on eventually semigroup S [4]. Using an argument similar to that Theorem 4.1, we have the following result;

Corollary 4.4. *Let S be an eventually inverse semigroup. Then S is amenable if and only if $l^1(S)$ is character amenable modulo I_σ.*

Using Theorem 2.5 and the similar discussion of Theorem 4.1, we have the following Corollary.

Corollary 4.5. *If S is either*
(i) E-inverse E-semigroup with commutative idempotents, or
(ii) eventually inverse semigroup with commutative idempotents,
then S is amenable if and only if $l^1(S)$ is character amenable modulo I_σ.

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References

[1] M. Amini, H. Rahimi, Amenability of semigroups and their algebras modulo a group congruence, Acta Mathematica Hungarica, Vol 144, Issue 2, 407-415, (2014).
[2] A. H. Clifford and J. B. Preston, The Algebraic Theory of Semigroups I, American Mathematical Society Surveys **7**, American Mathematical Society, Providence (1961).
[3] R. S. Gigon, Congruences and group congruences on a semigroup, Semigroup Forum, 86, 431450, (2013).
[4] G. M. S. Gomes, Group congruences on eventually inverse semigroups, Portugaliae Mathematica, 49, 417428, (1992).
[5] J. M. Howie, Fundamentals of Semigroup Theory, Clarendon Press, Oxford (1995).
[6] A. Jabbari, Approximate Character amenability of Banach algebras,Stud. Univ.Babes-Bolyai Math.57, No.3,409-420, (2012).
[7] B. E. Johnson, Cohomology in Banach Algebras, Mem. Amer. Math. Soc., Vol. 127, (1972).
[8] J. L. Kelley, General topology, American Book, Van Nostrand, Reinhold, (1969). Anal. Appl., 344, 942-955, (2008).
[9] E. Kaniuth, A. T. Lau and J. Pym, On character amenability of Banach algebras, J. Math. Anal. Appl., 344, 942955, (2008).
[10] E. Kaniuth, A. T. Lau and J. Pym, On $\varphi-$amenability of Banach algebras, Math. Proc. Cambridge Philos. Soc., 144, 8596, (2008).
[11] Z. Hu, M. Sangani Monfared and T. Traynor, On character amenable Banach algebras,Studia Mathematica, 193, (2009).
[12] O. T. Mewomo and S.M. Maepa, On character amenability of semigroup algebras. Quaest. Math. 39, 3, 307318, (2015).
[13] O. T. Mewomo, S. M. Meapa and M. O. Uwala. On character and approximate character amenability of various Segal algebras, Acta Math. Academiae Paedagogicae Nyregyhaziensis 31, 273283, (2015).
[14] M. S. Monfared, Character amenability of Banach algebras, Math. Proc. Cambridge Philos. Soc.,144, 697-706, (2008).

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