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NUMERICAL SOLUTION OF 2D VOLTERRA–FREDHOLM INTEGRAL EQUATIONS VIA MODIFICATION OF HAT FUNCTIONS

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Abstract. Two–dimensional modification of hat functions (2D–MHFs) are presented as a set of basis functions for expanding 2D functions. Their properties are determined and an operational matrix for integration attended. By this concept, the primary equations will be changed in to the associated systems of algebraic equations. Also, an error analysis is provided under several mild conditions. Finally, examples are presented to show the pertinent features of the method and the results are examine.

1. Introduction

Integral equations are used as mathematical models for many different physical conditions, and they also happen as reformulations of other mathematical problems. The numerical methods for solution of Volterra–Fredholm integro–differential equations have been investigated in many studies [1]. Nonlinear integral equations have been studied in connection with many diverse topics such as vehicular traffic, biology, the theory of optimal control, economics, etc [2, 3].
We consider the Two–Dimensional Volterra–Fredholm integral equation (2D–VF–IE)

$$\phi(x_1,y_1) = g(x_1,y_1) + \lambda_1 \int_0^{y_1}\int_0^{x_1} k_1(x_1,y_1,x_2,y_2)u_1(\phi(x_2,y_2))\mathrm{d}x_2\mathrm{d}y_2 \tag{1.1}$$

$$+\lambda_2 \int_0^1\int_0^1 k_2(x_1,y_1,x_2,y_2)u_2(\phi(x_2,y_2))\mathrm{d}x_2\mathrm{d}y_2; \qquad (x_1,y_1)\in\Omega=[0,1]\times[0,1],$$

where λ_1 and λ_2 are arbitrary integers, $g(x_1,y_1), u_1(\phi(x_2,y_2)), u_2(\phi(x_2,y_2)) \in C^3(\Omega)$ and $k_1(x_1,y_1,x_2,y_2), k_2(x_1,y_1,x_2,y_2) \in C^3(\Omega\times\Omega)$ are known functions and $\phi(x_1,y_1)\in C^3(\Omega)$ is an unknown function. Classical theorems on the existence and uniqueness of the solution of 2D nonlinear integral equations can be found in Abdou et al. [4, 5]. Although several numerical methods for approximating the solutions of one-dimensional integral equations were presented [6, 7], for 2D ones

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only a few have been discussed in the literature [8, 9]. Recently, Mirzaee and E. Hadadiyan have been used modified hat functions in [10] to solve Eq. (1.1) by transforming it to a linear system which can solved by using numerical methods.

For solving 2D integral equations, a new set of 2D hat Functions are introduced. These functions are extensions of One–Dimensional Modification of hat Functions (1D–MHFs), introduced by Atkinson [7], to 2D ones. Furthermore, the main properties of 2D–MHFs, expansions of the functions in two and four variables with respect to 2D–MHFs, and the operational integration matrix are presented.
In our present method Eq. (1.1) is reduced to a linear or nonlinear system of algebraic equations by developing the unknown function with respect to 2D–MHFs with unknown coefficients. The properties of 2D–MHFs lead to this system being set up in very fast and simple manner. Finally, the proposed method is tested with the aid of some numerical examples.

2. A review of 1D modification of hat functions

In an $(m+1)$–set of 1D–MHFs over interval $[0, 1]$, the hat are defined as follows, [10].

For i odd,

$$h_i(\tau) = \begin{cases} \frac{-1}{h^2}(\tau-(i-1)h)(\tau-(i+1)h), & (i-1)h \leqslant \tau \leqslant (i+1)h, \\ \\ 0, & otherwise. \end{cases}$$

For i even,

$$h_i(\tau) = \begin{cases} \frac{1}{2h^2}(\tau-(i-1)h)(\tau-(i-2)h), & (i-2)h \leqslant \tau \leqslant ih, \\ \\ \frac{1}{2h^2}(\tau-(i+1)h)(\tau-(i+2)h), & ih \leqslant \tau \leqslant (i+2)h, \\ \\ 0, & otherwise, \end{cases}$$

where $m \geq 2$ is an even integer and $h = \dfrac{1}{m}$. The functions $h_0(\tau)$ and $h_m(\tau)$ are appropriate modifications of this last case. According to definition of MHFs, we have

$$h_i(jh) = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

$$h_i(\tau)h_j(\tau) = \begin{cases} 0, & \text{i is even and } |i-j| \geq 3, \\ 0, & \text{i is odd and } |i-j| \geq 2, \end{cases}$$

and

$$\sum_{i=0}^{m} h_i(\tau) = 1.$$

Let us write the 1D-MHFs vector $\mathbf{H}(x)$ as follows:

$$\mathbf{H}(\tau) = [h_0(\tau), h_1(\tau), ..., h_m(\tau)]^T; \qquad \tau \in [0, 1].$$

Simple calculation show that:

$$\int_0^\tau \mathbf{H}(\sigma)\mathrm{d}\sigma \simeq \Psi_1 \mathbf{H}(\tau), \tag{2.1}$$

where Ψ_1 is the $(m+1)\times(m+1)$ matrix as follows
and

$$\int_0^1 \mathbf{H}(\tau)\mathbf{H}^T(\tau)\mathrm{d}\tau = \Psi_2, \tag{2.2}$$

where Ψ_1 and Ψ_2 are the $(m+1)\times(m+1)$ matrices which have been defined in [10].

3. 2D–MHFs and their properties

Definitions. An $(m+1)^2$–set of 2D–MHFs consists of $(m+1)^2$ functions which are defined over district D as follows:

$$h_{i,j}(x_1,y_1) = h_i(x_1)h_j(y_1); \qquad i,j = 0,1,...,m,$$

where $h_i(x_1)$ and $h_j(y_1)$ are 1D-MHFs. According to definition of 2D–MHFs, we have:

$$h_{i,j}(sh,th) = \begin{cases} 1, & i = s \textit{ and } j = t, \\ 0, & \textit{otherwise}, \end{cases} \tag{3.1}$$

and

$$\sum_{i=0}^{m}\sum_{j=0}^{m} h_{i,j}(x_1,y_1) = 1.$$

Also, we can define $\mathbf{H}(x_1,y_1)$ as the 2D–MHFs vector forms as follows:

$$\mathbf{H}(x_1,y_1) = [h_{0,0}(x_1,y_1), h_{0,1}(x_1,y_1), ..., h_{0,m}(x_1,y_1), h_{1,0}(x_1,y_1), ..., h_{m,m}(x_1,y_1)]^T; \tag{3.2}$$

$$(x_1,y_1) \in \Omega.$$

It is clear that

$$\mathbf{H}(x_1,y_1) = \mathbf{H}(x_1) \otimes \mathbf{H}(y_1), \tag{3.3}$$

where $\otimes$ is Tensor product [17]. An arbitrary function $\phi(x_1,y_1)$ on Ω can be developed by the 2D–MHFs like in [7] as

$$\phi(x_1,y_1) \simeq \Phi^T\mathbf{H}(x_1,y_1) = \mathbf{H}^T(x_1,y_1)\Phi,$$

where

$$\Phi = [\phi_{0,0}, \phi_{0,1}, ..., \phi_{0,m}, ..., \phi_{m,m}]^T,$$

and

$$\phi_{i,j} = \phi(ih, jh); \qquad i,j = 0,1,...,m.$$

Similarly an arbitrary function of four variables, $k(x_1,y_1,x_2,y_2)$, on area $\Omega\times\Omega$ may be approximated with respect to 2D–MHFs such as:

$$k(x_1,y_1,x_2,y_2) \simeq \mathbf{H}^T(x_1,y_1)\mathbf{K}\mathbf{H}(x_2,y_2),$$

where $\mathbf{H}(x_1,y_1)$ and $\mathbf{H}(x_2,y_2)$ are 2D–MHFs vector of dimension $(m+1)^2$, and $\mathbf{K}$ is the $(m+1)^2\times(m+1)^2$ 2D–MHFs coefficients matrix.
From Eqs. (2.1) and (3.3), we have:

$$\int_0^{y_1}\int_0^{x_1} \mathbf{H}(x_2,y_2)\mathrm{d}x_2\mathrm{d}y_2 = (\int_0^{x_1} \mathbf{H}(x_2)\mathbf{H}^T(x_2)\mathrm{d}x_2) \otimes (\int_0^{y_1} \mathbf{H}(y_2)\mathbf{H}^T(y_2)\mathrm{d}y_2) \tag{3.4}$$

$$\simeq (\Psi_1\mathbf{H}(x_1)) \otimes (\Psi_1\mathbf{H}(y_1)) \simeq (\Psi_1 \otimes \Psi_1)(\mathbf{H}(x_1)\otimes\mathbf{H}(y_1)) = \mathbf{P}_1\mathbf{H}(x_1,y_1),$$

where Ψ_1 is defined in Eq. (2.1) and $\mathbf{P}_1 = \Psi_1 \otimes \Psi_1$.
From Eqs. (2.2) and (3.3), we have:

$$\int_0^1 \int_0^1 \mathbf{H}(x_2, y_2)\mathbf{H}^T(x_2, y_2)\mathrm{d}x_2\mathrm{d}y_2 = (\int_0^1 \mathbf{H}(x_2)\mathbf{H}^T(x_2)\mathrm{d}x_2) \otimes (\int_0^1 \mathbf{H}(y_2)\mathbf{H}^T(y_2)\mathrm{d}y_2) = \Psi_2 \otimes \Psi_2 = \mathbf{P}_2, \quad (3.5)$$

where Ψ_2 is defined in Eq. (2.2).
According to (3.1) and expansion of $\mathbf{H}(x_1, y_1)\mathbf{H}^T(x_1, y_1)$ by 2D–MHFs, we obtain

$$\mathbf{H}(x_1, y_1)\mathbf{H}^T(x_1, y_1) \simeq diag(\mathbf{H}^T(x_1, y_1)),$$

so we have

$$\mathbf{H}(x_1, y_1)\mathbf{H}^T(x_1, y_1)\Phi \simeq \tilde{\Phi}\mathbf{H}(x_1, y_1), \quad (3.6)$$

where Φ be an $(m+1)^2$–vector and $\tilde{\Phi} = diag(\Phi^T)$ and

$$\mathbf{H}^T(x_1, y_1)\mathbf{M}\mathbf{H}(x_1, y_1) \simeq \mathbf{H}^T(x_1, y_1)\hat{\mathbf{M}}, \quad (3.7)$$

where $\mathbf{M}$ is an $(m+1)^2 \times (m+1)^2$ matrix and $\hat{\mathbf{M}}$ is an $(m+1)^2$–vector with elements equal to the diagonal entries of matrix.

4. Description of the method

In this section, we present an useful method for solving a 2D nonlinear Volterra–Fredholm integral equation by using 2D–MHFs. we transform the model (1.1) to nonlinear systems of matrix equations which can be without difficulty solved. Let

$$v_i(x_2, y_2) = u_i(\phi(x_2, y_2)); \qquad i = 1, 2. \quad (4.1)$$

So we have

$$\phi(x_1, y_1) = g(x_1, y_1) + \lambda_1 \int_0^{y_1} \int_0^{x_1} k_1(x_1, y_1, x_2, y_2)v_1(x_2, y_2)\mathrm{d}x_2\mathrm{d}y_2 + \lambda_2 \int_0^1 \int_0^1 k_2(x_1, y_1, x_2, y_2)v_2(x_2, y_2)\mathrm{d}x_2\mathrm{d}y_2. \quad (4.2)$$

By combination of (4.1) and (4.2), we have

$$v_n(x_1, y_1) = u_n[g(x_1, y_1) + \lambda_1 \int_0^{y_1} \int_0^{x_1} k_1(x_1, y_1, x_2, y_2)v_1(x_2, y_2)\mathrm{d}x_2\mathrm{d}y_2 \quad (4.3)$$
$$+\lambda_2 \int_0^1 \int_0^1 k_2(x_1, y_1, x_2, y_2)v_2(x_2, y_2)\mathrm{d}x_2\mathrm{d}y_2]; \qquad n = 1, 2,$$

Now, by approximating the functions in (4.3) by 2D–MHFs, we have

$$v_i(x_1, y_1) \simeq \mathbf{V}_i^T\mathbf{H}(x_1, y_1) = \mathbf{H}^T(x_1, y_1)\mathbf{V}_i; \qquad i = 1, 2, \quad (4.4)$$

and

$$k_i(x_1, y_1, x_2, y_2) \simeq \mathbf{H}^T(x_1, y_1)\mathbf{K}_i\mathbf{H}(x_2, y_2); \qquad i = 1, 2, \quad (4.5)$$

where $\mathbf{H}(x_1, y_1)$ is defined in Eq. (3.2) and the vectors $\mathbf{V}_i$ and matrix $\mathbf{K}_i$ are 2D–MHFs coefficients of $v_i(x_1, y_1)$ and $k_i(x_1, y_1, x_2, y_2)$, respectively. Substituting (4.4) and (4.5) in (4.3) yields

$$\mathbf{H}^T(x_1,y_1)\mathbf{V}_n \simeq u_n[g(x_1,y_1) + \lambda_1\mathbf{H}^T(x_1,y_1)\mathbf{K}_1\int_0^{y_1}\int_0^{x_1}\mathbf{H}(x_2,y_2)\mathbf{H}^T(x_2,y_2)\mathbf{V}_1\mathrm{d}x_2\mathrm{d}y_2 + \lambda_2\mathbf{H}^T(x_1,y_1)\mathbf{K}_2\int_0^{1}\int_0^{1}\mathbf{H}(x_2,y_2)\mathbf{H}^T(x_2,y_2)\mathrm{d}x_2\mathrm{d}y_2\mathbf{V}_2]; \qquad n=1,2.$$

Using (3.5) and (3.6), yields

$$\mathbf{H}^T(x_1,y_1)\mathbf{V}_n \simeq u_n[g(x_1,y_1) + \lambda_1\mathbf{H}^T(x_1,y_1)\mathbf{K}_1\tilde{\mathbf{V}}_1\int_0^{y_1}\int_0^{x_1}\mathbf{H}(x_2,y_2)\mathrm{d}x_2\mathrm{d}y_2 + \lambda_2\mathbf{H}^T(x_1,y_1)\mathbf{K}_2\mathbf{P}_2\mathbf{V}_2]; \qquad n=1,2.$$

From (3.4), we have

$$\mathbf{H}^T(x_1,y_1)\mathbf{V}_n \simeq u_n[g(x_1,y_1) + \lambda_1\mathbf{H}^T(x_1,y_1)\mathbf{K}_1\tilde{\mathbf{V}}_1\mathbf{P}_1\mathbf{H}(x_1,y_1) + \lambda_2\mathbf{H}^T(x_1,y_1)\mathbf{K}_2\mathbf{P}_2\mathbf{V}_2]; \qquad n=1,2.$$

According to (3.7), we can rewrite above equations as follows

$$\mathbf{H}^T(x_1,y_1)\mathbf{V}_n \simeq u_n[g(x_1,y_1) + \lambda_1\mathbf{H}^T(x_1,y_1)\hat{\mathbf{M}} + \lambda_2\mathbf{H}^T(x_1,y_1)\mathbf{K}_2\mathbf{P}_2\mathbf{V}_2]; \qquad n=1,2,$$

where $\mathbf{M} = \mathbf{K}_1\tilde{\mathbf{V}}_1\mathbf{P}_1$. Now, using Newton–Cotes nodes as

$$x_{1,j} = y_{1,j} = \frac{2j+1}{2(m+1)}; \qquad j = 0, 1, ..., m-1,$$

and supplanting $\simeq$ by $=$, we have for $n = 1, 2$

$$\mathbf{H}^T(x_{1,i},y_{1,j})\mathbf{V}_n = u_n[g(x_{1,i},y_{1,j}) + \lambda_1\mathbf{H}^T(x_{1,i},y_{1,j})\hat{\mathbf{M}} + \lambda_2\mathbf{H}^T(x_{1,i},y_{1,j})\mathbf{K}_2\mathbf{P}_2\mathbf{V}_2].$$

We have a system of $2(m+1)^2$ nonlinear equations and $2(m+1)^2$ unknowns. After solving the above nonlinear system, we can find $\mathbf{V}_1$ and $\mathbf{V}_2$ and then

$$\phi(x_1,y_1) \simeq g(x_1,y_1) + \lambda_1\mathbf{H}^T(x_1,y_1)\hat{\mathbf{M}} + \lambda_2\mathbf{H}^T(x_1,y_1)\mathbf{K}_2\mathbf{P}_2\mathbf{V}_2.$$

Remark. Assume that we use 2D–MHFs method for solving

$$\phi(x_1,y_1) = g(x_1,y_1) + \lambda_1\int_0^{y_1}\int_0^{x_1} k_1(x_1,y_1,x_2,y_2)\phi(x_2,y_2)\mathrm{d}x_2\mathrm{d}y_2 + \lambda_2\int_0^{1}\int_0^{1} k_2(x_1,y_1,x_2,y_2)\phi(x_2,y_2)\mathrm{d}x_2\mathrm{d}y_2. \tag{4.6}$$

We approximate functions $\phi(x_1, y_1), g(x_1, y_1), k(x_1, y_1, x_2, y_2)$ with respect to 2D–MHFs as:

$$\phi(x_1,y_1) \simeq \Phi^T\mathbf{H}(x_1,y_1) = \mathbf{H}^T(x_1,y_1)\Phi,$$
$$g(x_1,y_1) \simeq \mathbf{G}^T\mathbf{H}(x_1,y_1) = \mathbf{H}^T(x_1,y_1)\mathbf{G},$$
$$k(x_1,y_1,x_2,y_2) \simeq \mathbf{H}^T(x_1,y_1)\mathbf{K}\mathbf{H}(x_2,y_2),$$

these equations transform to the following linear system

$$\phi(x_1, y_1) \simeq \mathbf{G} + \lambda_1 \hat{\mathbf{M}} + \lambda_2 \mathbf{K}_2 \mathbf{P}_2 \Phi.$$

5. Convergence analysis

In this sections, we show that the 2D–MHFs method in the section 4 is convergent and its order of convergence is $\mathbf{O}(h^3)$. Let χ_m be the set of all continuous function that are quadratic polynomials regard to x_1 and y_1 when regulated to each of the subintervals $[x_{1,i-2}, x_{1,i}] \times [y_{1,j-2}, y_{1,j}]$ for $i, j = 2, 3, ..., m$. The dimension of χ_m is $d_m = (m+1)^2$, based on each element of χ_m being absolutely determined by its values at the $(m+1)^2$ nodes $\{(x_{1,i}, y_{1,j}) | i, j = 0, 1, ..., m\}$.

Theorem 5.1. *Suppose $x_j = y_j = jh, j = 0, 1, ..., m, \phi(x_1, y_1) \in C^3(\Omega)$ and $\phi_m(x_1, y_1)$ be the 2D–MHFs expansion of $\phi(x_1, y_1)$ that defined as*

$$\phi_m(x_1, y_1) = \sum_{i=0}^{m} \sum_{j=0}^{m} \phi_{i,j} h_{i,j}(x_1, y_1).$$

Also, assume that $e_m(x_1, y_1) = |\phi(x_1, y_1) - \phi_m(x_1, y_1)|$ where $(x_1, y_1) \in \Omega$. Then

$$\|e_m(x_1, y_1)\|_\infty = \boldsymbol{O}(h^3) \ .$$

Proof. for the interpolation error on $\omega = [x_{1,i-2}, x_{1,i}] \times [y_{1,j-2}, y_{1,j}]; i, j = 2, 3, ..., m$, we have [11],

$$e_m(x_1, y_1) = \frac{1}{6} \frac{\partial^3 \phi(\xi_{1,x}, y)}{\partial x_1^3} \prod_{k=i-2}^{i} (x_1 - x_{1,k}) + \frac{1}{6} \frac{\partial^3 \phi(x_1, \zeta_{1,y})}{\partial y_1^3} \prod_{l=j-2}^{j} (y_1 - y_{1,l}) \tag{5.1}$$

$$- \frac{1}{36} \frac{\partial^6 \phi(\xi_{1,x}, \zeta_{1,y})}{\partial x_1^3 \partial y_1^3} \prod_{k=i-2}^{i} (x_1 - x_{1,k}) \prod_{l=j-2}^{j} (y_1 - y_{1,l}),$$

where $\xi_{1,x} \in [x_{1,i-2}, x_{1,i}]$ and $\zeta_{1,y} \in [y_{1,j-2}, y_{1,j}]$. Therefore it is not hard to check that

$$\|e_m(x_1, y_1)\|_\infty \leq \frac{h^3}{9\sqrt{3}} \left\| \frac{\partial^3 \phi(x_1, y_1)}{\partial x_1^3} \right\|_\infty + \frac{h^3}{9\sqrt{3}} \left\| \frac{\partial^3 \phi(x_1, y_1)}{\partial y_1^3} \right\|_\infty + \frac{h^6}{243} \left\| \frac{\partial^6 \phi(x_1, y_1)}{\partial x_1^3 \partial y_1^3} \right\|_\infty, \tag{5.2}$$

consequently, $\|e_m(x_1, y_1)\|_\infty = \mathbf{O}(h^3)$. □

6. Numerical examples

In this section, seven examples are provided to examine the accuracy and effectiveness of the proposed method and all of them are performed on a computer using programs written in MATHEMATICA. In this regard, we describe in tables, the absolute error function:

$$e(x_1, y_1) = |\phi(x_1, y_1) - \phi_m(x_1, y_1)| = \mathbf{O}(h^3),$$

at any considered points.

Nodes (x_1, y_1)	$e(x_1, y_1)$ $m = 4$	$m = 8$
$(0.0, 0.0)$	$0.0000e-0$	$0.0000e-0$
$(0.1, 0.1)$	$8.8472e-8$	$6.2382e-6$
$(0.2, 0.2)$	$1.9753e-4$	$4.9945e-5$
$(0.3, 0.3)$	$8.9379e-4$	$7.5135e-5$
$(0.4, 0.4)$	$1.5951e-3$	$1.2453e-4$
$(0.5, 0.5)$	$1.3181e-5$	$2.1665e-7$
$(0.6, 0.6)$	$2.3691e-3$	$1.1319e-4$
$(0.7, 0.7)$	$6.3358e-4$	$5.2516e-4$
$(0.8, 0.8)$	$4.1595e-3$	$5.9645e-4$
$(0.9, 0.9)$	$7.5833e-3$	$5.1962e-4$
$(1.0, 1.0)$	$8.4382e-4$	$4.8459e-5$

TABLE 1. Numerical results of Example 6.1 with 2D–MHFs.

Nodes	Present method		Method of [15]	
$(x_1, y_1) = (2^{-\lambda}, 2^{-\lambda})$	$m = 4$	$m = 8$	$m = 4$	$m = 8$
$\lambda = 1$	$7.7e-6$	$2.2e-7$	$2.4e-1$	$1.2e-1$
$\lambda = 2$	$4.5e-7$	$2.8e-8$	$2.5e-1$	$1.2e-1$
$\lambda = 3$	$2.3e-6$	$1.5e-5$	$7.0e-5$	$1.2e-1$
$\lambda = 4$	$3.0e-6$	$4.8e-7$	$1.2e-1$	$2.2e-6$
$\lambda = 5$	$1.2e-6$	$1.0e-7$	$1.9e-1$	$6.2e-2$
$\lambda = 6$	$3.6e-7$	$2.5e-8$	$2.2e-1$	$9.4e-2$

TABLE 2. Comparison of the errors $e(x_1, y_1)$ of Example 6.1

Example 6.1. [15] *As the first example, consider the following equation,*

$$\phi(x_1, y_1) = g(x_1, y_1) + \int_0^{y_1}\int_0^{x_1}(x_1 + y_1 - x_2 - y_2)\phi^2(x_2, y_2)\mathrm{d}x_2\mathrm{d}y_2; \qquad (x_1, y_1) \in \Omega, \quad (6.1)$$

where $g(x_1, y_1) = x_1 + y_1 - \frac{1}{12}x_1y_1(x_1^3 + 4x_1^2y_1 + 4x_1y_1^2 + y_1^3)$, *with the exact solution* $\phi(x_1, y_1) = x_1 + y_1$.

Table 1 illustrates the absolute error function for this example. Moreover, we compare the absolute error function computed for the chosen grid points by the present method and rationalized Haar functions [15] in Table 2. This fact is obvious from Table 2 that the results attended by the present method is superior than that attended in [15].

Nodes (x_1, y_1)	$e(x_1, y_1)$ $m=4$	$m=6$
$(0.0, 0.0)$	$3.0530e-3$	$8.2969e-4$
$(0.1, 0.1)$	$3.1600e-3$	$8.5159e-4$
$(0.2, 0.2)$	$3.5068e-3$	$8.1982e-4$
$(0.3, 0.3)$	$2.7251e-3$	$6.1412e-4$
$(0.4, 0.4)$	$3.9148e-5$	$2.8998e-3$
$(0.5, 0.5)$	$1.9116e-3$	$1.0222e-3$
$(0.6, 0.6)$	$3.3582e-2$	$5.1522e-3$
$(0.7, 0.7)$	$2.1199e-2$	$1.1896e-2$
$(0.8, 0.8)$	$3.8519e-2$	$8.7792e-3$
$(0.9, 0.9)$	$8.0612e-2$	$2.8361e-2$
$(1.0, 1.0)$	$8.4613e-2$	$2.4083e-2$

TABLE 3. Numerical results of Example 6.2 with 2D–MHFs.

Nodes (x_1, y_1)	Present method $m=4$	Method of [12] $m_1=m_2=4$
$(0.0, 0.0)$	$3.0e-3$	$2.2e-4$
$(0.1, 0.1)$	$3.2e-3$	$7.6e-3$
$(0.2, 0.2)$	$3.5e-3$	$5.2e-3$
$(0.3, 0.3)$	$2.7e-3$	$6.4e-3$
$(0.4, 0.4)$	$3.9e-5$	$7.3e-3$
$(0.5, 0.5)$	$1.9e-3$	$7.2e-3$
$(0.6, 0.6)$	$3.4e-2$	$8.4e-3$
$(0.7, 0.7)$	$2.1e-2$	$3.1e-2$
$(0.8, 0.8)$	$3.9e-2$	$5.1e-2$
$(0.9, 0.9)$	$8.1e-2$	$6.4e-2$

TABLE 4. Comparison of the errors $e(x_1, y_1)$ of Example 6.2

Example 6.2. [12, 13] *regard as the following 2D–VF–IE:*

$$\phi(x_1,y_1) = g(x_1,y_1) - \int_0^{y_1}\int_0^{x_1}(x_1+y_1+x_2+y_2)\phi^2(x_2,y_2)\mathrm{d}x_2\mathrm{d}y_2 \qquad (6.2)$$
$$-\int_0^1\int_0^1(x_1y_1+x_2y_2^2)\phi(x_2,y_2)\mathrm{d}x_2\mathrm{d}y_2; \qquad (x_1,y_1)\in\Omega,$$

where $g(x_1,y_1) = \frac{1}{4} + \frac{17}{6}x_1y_1 + x_1^2 + \frac{7}{9}x_1^3y_1^4 + \frac{29}{18}x_1^4y_1^3 + \frac{6}{5}x_1^5y_1^2 + \frac{11}{30}x_1^6y_1$, *with the exact solution* $\phi(x_1,y_1) = x_1^2 + 2x_1y_1$.

Table 3 illustrate the absolute error function for this example. Moreover, we compare the absolute error function computed for the chosen grid points by the present method and triangular functions method [12] in Table 4.

Nodes (x_1, y_1)	$e(x_1, y_1)$ $m=4$	$m=8$
$(0.0, 0.0)$	$0.0000e-0$	$0.0000e-0$
$(0.1, 0.1)$	$1.0565e-4$	$1.9142e-5$
$(0.2, 0.2)$	$2.3768e-4$	$5.7281e-5$
$(0.3, 0.3)$	$4.5092e-4$	$4.7659e-5$
$(0.4, 0.4)$	$6.3371e-4$	$7.9754e-5$
$(0.5, 0.5)$	$5.6335e-4$	$8.4978e-5$
$(0.6, 0.6)$	$5.6306e-4$	$9.0114e-5$
$(0.7, 0.7)$	$6.5708e-4$	$1.1702e-4$
$(0.8, 0.8)$	$7.9303e-4$	$1.0485e-4$
$(0.9, 0.9)$	$8.8986e-4$	$1.2526e-4$
$(1.0, 1.0)$	$8.4502e-4$	$1.2747e-4$

TABLE 5. Numerical results of Example 6.3 with 2D–MHFs.

Nodes	Present method		Method of [15]	
$(x_1, y_1) = (2^{-\lambda}, 2^{-\lambda})$	$m=4$	$m=8$	$m=4$	$m=8$
$\lambda = 1$	$5.6e-4$	$8.5e-5$	$5.1e-2$	$2.8e-2$
$\lambda = 2$	$3.4e-4$	$5.1e-5$	$1.1e-1$	$6.5e-2$
$\lambda = 3$	$1.3e-4$	$2.8e-5$	$1.3e-2$	$1.1e-1$
$\lambda = 4$	$7.3e-5$	$9.9e-6$	$1.4e-1$	$4.9e-3$
$\lambda = 5$	$4.3e-5$	$5.4e-6$	$2.3e-1$	$9.1e-2$
$\lambda = 6$	$2.4e-5$	$3.2e-6$	$2.9e-1$	$1.4e-1$

TABLE 6. Comparison of the errors $e(x_1, y_1)$ of Example 6.3

Example 6.3. [14] *regard as the following 2D nonlinear Fredholm integral equation:*

$$\phi(x_1, y_1) = g(x_1, y_1) + \int_0^1 \int_0^1 \frac{x_1}{1+y_1}(1 + x_2 + y_2)\phi^2(x_2, y_2)\mathrm{d}x_2\mathrm{d}y_2, \tag{6.3}$$

where $g(x_1, y_1) = \dfrac{1}{(1 + x_1 + y_1)^2} - \dfrac{x_1}{6(1 + y_1)}$, *with the exact solution* $\phi(x_1, y_1) = \dfrac{1}{(1 + x_1 + y_1)^2}$.

Table 5 illustrates the absolute error function for this example. Moreover, we compare the absolute error function computed for the chosen grid points by the present method and rationalized Haar functions [15] in Table 6. This fact is obvious from Table 6 that the results attended by the present method is superior than that attended in [15].

Nodes	$e(x_1, y_1)$	
(x_1, y_1)	$m = 4$	$m = 8$
$(0.0, 0.0)$	$0.0000e-0$	$0.0000e-0$
$(0.1, 0.1)$	$3.3540e-6$	$6.2945e-8$
$(0.2, 0.2)$	$2.8929e-6$	$1.3033e-6$
$(0.3, 0.3)$	$1.6829e-5$	$3.7432e-6$
$(0.4, 0.4)$	$7.8801e-5$	$4.3715e-6$
$(0.5, 0.5)$	$7.6579e-6$	$1.1269e-6$
$(0.6, 0.6)$	$3.4009e-4$	$2.2234e-5$
$(0.7, 0.7)$	$4.2208e-4$	$5.9167e-5$
$(0.8, 0.8)$	$2.4244e-4$	$1.5786e-4$
$(0.9, 0.9)$	$1.3552e-3$	$3.5586e-6$
$(1.0, 1.0)$	$1.3347e-5$	$3.0415e-4$

TABLE 7. Numerical results of Example 6.4 with 2D–MHFs.

Nodes	Present method		Method of [15]	
$(x_1, y_1) = (2^{-\lambda}, 2^{-\lambda})$	$m = 4$	$m = 8$	$m = 4$	$m = 8$
$\lambda = 1$	$7.7e-6$	$1.1e-6$	$1.2e-1$	$6.0e-2$
$\lambda = 2$	$4.5e-7$	$1.2e-7$	$7.5e-2$	$3.4e-2$
$\lambda = 3$	$2.3e-6$	$2.1e-9$	$3.8e-5$	$1.9e-2$
$\lambda = 4$	$3.0e-6$	$2.0e-8$	$1.2e-2$	$2.5e-6$
$\lambda = 5$	$1.2e-6$	$4.0e-8$	$1.4e-2$	$2.9e-3$
$\lambda = 6$	$3.6e-7$	$1.7e-8$	$1.5e-2$	$3.6e-3$

TABLE 8. Comparison of the errors $e(x_1, y_1)$ of Example 6.4

Example 6.4. [15] *regard as the following 2D nonlinear Volterra integral equation:*

$$\phi(x_1, y_1) = g(x_1, y_1) + \int_0^{y_1} \int_0^{x_1} (x_1.y_1^2 + cos(y_2))\phi^2(x_2, y_2)\mathrm{d}x_2\mathrm{d}y_2; \qquad (x_1, y_1) \in \Omega, \quad (6.4)$$

where $g(x_1, y_1) = \frac{x_1^6}{20}(sin(2y_1) - 2y_1) + \frac{x_1}{9} sin(y_1)(9 - x_1^2.sin^2(y_1))$*, with the exact solution* $\phi(x_1, y_1) = x_1.sin(y_1)$*.*

Table 7 illustrates the absolute error function for this example. Moreover, we compare the absolute error function computed for the chosen grid points by the present method and rationalized Haar functions [15] in Table 8. This fact is obvious from Table 8 that the results attended by the present method is superior than that attended in [15].

Example 6.5. [16, 17] *regard as the following 2D linear Fredholm integral equation:*

$$\phi(x_1, y_1) = g(x_1, y_1) + \int_0^1 \int_0^1 (x_2.sin(y_2) + 1)\phi(x_2, y_2)\mathrm{d}x_2\mathrm{d}y_2, \qquad (6.5)$$

Nodes	$e(x_1, y_1)$	
(x_1, y_1)	$m = 8$	$m = 16$
$(0.0, 0.0)$	$9.5019e-6$	$5.9127e-7$
$(0.1, 0.1)$	$1.0242e-5$	$4.7909e-7$
$(0.2, 0.2)$	$5.9235e-6$	$2.9528e-7$
$(0.3, 0.3)$	$2.2569e-5$	$1.3047e-6$
$(0.4, 0.4)$	$2.0717e-7$	$3.1855e-6$
$(0.5, 0.5)$	$9.5019e-6$	$5.9127e-7$
$(0.6, 0.6)$	$3.1235e-5$	$4.4814e-6$
$(0.7, 0.7)$	$4.2969e-5$	$2.8917e-6$
$(0.8, 0.8)$	$8.4974e-5$	$5.1145e-6$
$(0.9, 0.9)$	$3.3863e-5$	$1.1846e-5$
$(1.0, 1.0)$	$9.5019e-6$	$5.9127e-7$

TABLE 9. Numerical results of Example 6.5 with 2D–MHFs.

Nodes	Present method		Method of [18]	
$(x_1, y_1) = (2^{-\lambda}, 2^{-\lambda})$	$m = 8$	$m = 16$	$m = 8$	$m = 16$
$\lambda = 1$	$9.5e-6$	$5.9e-7$	$2.3e-02$	$1.1e-02$
$\lambda = 2$	$9.5e-6$	$5.9e-7$	$3.2e-02$	$1.6e-02$
$\lambda = 3$	$9.5e-6$	$5.9e-7$	$1.6e-02$	$8.0e-03$
$\lambda = 4$	$1.0e-5$	$5.9e-7$	$9.5e-03$	$4.1e-02$
$\lambda = 5$	$9.8e-6$	$6.2e-7$	$2.3e-03$	$2.1e-04$
$\lambda = 6$	$9.6e-6$	$6.0e-7$	$3.2e-02$	$1.4e-02$

TABLE 10. Comparison of the errors $e(x_1, y_1)$ of Example 6.5

where $g(x_1, y_1) = x_1.cos(y_1) - \dfrac{sin(1)}{6}(sin(1)+3)$, *with the exact solution* $\phi(x_1, y_1) = x_1.cos(y_1)$.

Table 9 illustrates the absolute error function for this example. Moreover, we compare the absolute error function computed for the chosen grid points by the present method and Haar wavelets method [18] in Table 10. This fact is obvious from Table 10 that the results attended by the present method is superior than that attended in [18].

Example 6.6. *regard as the following 2D nonlinear Fredholm integral equation:*

$$\phi(x_1, y_1) = g(x_1, y_1) - \int_0^1 \int_0^1 e^{3|x_1 - y_2|} e^{-\phi^2(x_2, y_2)} \mathrm{d}x_2 \mathrm{d}y_2, \tag{6.6}$$

where $g(x_1, y_1) = 2e^{3x_1} + e^{-3x_1}$, *where the exact solution is unknown.*

Since the exact solutions of some problems are unknown, the maximum absolute error for these example is calculated using the following double mesh convention

$$\widetilde{E}_m = \sup_{x \in \Omega} R_m(x), \tag{6.7}$$

where $R_m(x) = |\phi_{2m}(x_1, y_1) - \phi_m(x_1, y_1)|$.

Example 6.7. *regard as the following 2D nonlinear Volterra integral equation:*

$$\phi(x_1, y_1) = g(x_1, y_1) - \int_0^{y_1} \int_0^{x_1} \sinh(3(x_1 - y_2)) \frac{\sin(\phi(x_2, y_2))}{\phi(x_2, y_2)} \mathrm{d}x_2 \mathrm{d}y_2, \qquad (6.8)$$

where $g(x_1, y_1) = \cosh(3x_1) + 1$*, where the exact solution is unknown.*

Table 11 illustrates the maximum absolute error function for examples 6.6 and 6.7. The obtained results show the efficiency and accuracy of this method, when we do not know the exact solution.

	$m = 8$	$m = 16$	$m = 32$
Example 6.6	$1.7e-5$	$1.3e-6$	$2.6e-7$
Example 6.7	$5.3e-3$	$5.2e-4$	$1.2e-4$

TABLE 11. Comparison of the errors $\widetilde{E}_m$

7. Discussion and Conclusions

In this paper, we presented a numerical method for solving 2D linear and nonlinear Volterra–Fredholm integral equations of the second kind based on 2D–MHFs. This method transforms the linear or nonlinear Volterra–Fredholm integral equations in to a linear or nonlinear system of algebraic equations. in addition, it is determined that 2D–MHFs method is convergent and the order of convergence of this method is $\mathbf{O}(h^3)$. The accuracy and convergence of the method are presented through seven examples. The numerical results present that the accuracy of the solutions attended is superior. This method can be without difficulty extended and applied to a system of linear or nonlinear 2D integral equations, but some changes are required.

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