Journal of Mathematical Analysis
ISSN: 2217-3412, URL: www.ilirias.com/jma
Volume 10 Issue 5 (2019), Pages 14-33.

SOME FIXED POINT RESULTS IN EXTENDED PARAMETRIC B-METRIC SPACES WITH APPLICATION TO INTEGRAL EQUATIONS

V. PARVANEH*, N. HUSSAIN, M. A. KUTBI AND M. KHORSHIDI

Abstract. In this paper, we introduce the concept of extended parametric b-metric space and investigate the existence of fixed points for some contractive type mappings in such spaces. Consequently, we derive some new fixed point results in triangular ordered extended fuzzy b-metric spaces. Moreover, some examples and an application are provided here to illustrate the applicability of the obtained results.

1. Introduction and preliminaries

The notion of an extended b-metric space was studied by Parvaneh and Ghoncheh in [12].

Definition 1.1. [12] *Let X be a (nonempty) set. A function $d : X \times X \to R^+$ is an extended b-metric (p-metric, for short) if there exists a strictly increasing continuous function $\Omega : [0, \infty) \to [0, \infty)$ with $\Omega^{-1}(t) \leq t \leq \Omega(t)$ such that for all $x, y, z \in X$, the following conditions hold:*

(p$_1$) $\widetilde{d}(x, y) = 0$ *iff* $x = y$,
(p$_2$) $\widetilde{d}(x, y) = \widetilde{d}(y, x)$,
(p$_3$) $\widetilde{d}(x, z) \leq \Omega(\widetilde{d}(x, y) + \widetilde{d}(y, z))$.

In this case, the pair (X, d) is called an extended b-metric space, or, briefly, a p-metric space.

It should be noted that the class of p-metric spaces is considerably larger than the class of b-metric spaces, since a b-metric is a p-metric when $\Omega(t) = st$ for fixed $s \geq 1$, while a metric is a p-metric when $\Omega(t) = t$. Fore more details on b-metric spaces and fixed point results in these spaces we refer the reader to [8], [13] [15] and [16].

More generally, several examples of p-metrics can be constructed using the following easy proposition.

Proposition 1.2. *Let (X, d) be a b-metric space with coefficient $s \geq 1$ and let $\rho(x, y) = \xi(d(x, y))$ where $\xi : [0, \infty) \to [0, \infty)$ is a strictly increasing continuous*

2000 *Mathematics Subject Classification.* Primary 47H10; Secondary 54H25.
Key words and phrases. metric space, b-metric space, parametric b-metric space, fixed point.
©2019 Ilirias Research Institute, Prishtinë, Kosovë.
Submitted Maj 3, 2019. Published Jul 8,2019.
Communicated by guest Editor Jamshaid Jasim.

function with $t \leq \xi(t)$ for $t \geq 0$ and $\xi(0) = 0$. Then ρ is a p-metric with $\Omega(t) = \xi(st)$.

Hussain et al. [6] defined and studied the concept of parametric metric space as follows.

Definition 1.3. *Let X be a nonempty set and $\mathcal{P} : X \times X \times (0, \infty) \to [0, \infty)$ be a function. We say that $\mathcal{P}$ is a parametric metric on X if,*

(i): $\mathcal{P}(x, y, t) = 0$ *for all $t > 0$ if and only if $x = y$;*
(ii): $\mathcal{P}(x, y, t) = \mathcal{P}(y, x, t)$ *for all* $t > 0$;
(iii): $\mathcal{P}(x, y, t) \leq \mathcal{P}(x, z, t) + \mathcal{P}(z, y, t)$ *for all $x, y, z \in X$ and all* $t > 0$.

and we say that the pair $(X, \mathcal{P})$ is a parametric metric space.

In this paper, we introduce a new type of generalized metric space, which we call it extended parametric b-metric space, as a generalization of parametric metric and parametric b-metric spaces ([5]). Then, we prove some fixed point theorems for contractive mappings in extended parametric b-metric spaces. As application, we derive some new fixed point results in triangular extended fuzzy b-metric spaces. We illustrate these results by appropriate examples.

Definition 1.4. *Let X be a non-empty set, $\Omega : [0, \infty) \to [0, \infty)$ be a strictly increasing continuous function with $\Omega^{-1}(t) \leq t \leq \Omega(t)$ and let $\widetilde{\mathcal{P}} : X^2 \times (0, \infty) \to [0, \infty)$ be a map satisfying the following conditions:*

($\widetilde{\mathcal{P}}_b1$) $\widetilde{\mathcal{P}}(x, y, t) = 0$ *if and only if* $x = y$,
($\widetilde{\mathcal{P}}_b2$) $\widetilde{\mathcal{P}}(x, y, t) = \widetilde{\mathcal{P}}(y, x, t)$,
($\widetilde{\mathcal{P}}_b3$) $\widetilde{\mathcal{P}}(x, z, t) \leq \Omega[\widetilde{\mathcal{P}}(x, y, t) + \widetilde{\mathcal{P}}(y, z, t)]$.

Then $\widetilde{\mathcal{P}}$ is called an extended parametric b-metric on X and $(X, \widetilde{\mathcal{P}})$ is called an EPbMS with control function Ω.

Obviously, for $\Omega(x) = sx$, the extended parametric b-metric reduces to parametric b-metric.

Definition 1.5. *Let $\{x_n\}$ be a sequence in an EPbMS $(X, \widetilde{\mathcal{P}})$.*

1. $\{x_n\}$ is said to be convergent to $x \in X$, written as $\lim_{n \to infty} x_n = x$, if $\lim_{n \to \infty} \widetilde{\mathcal{P}}(x_n, x, t) = 0$, for all $t > 0$.

2. $\{x_n\}$ is said to be a Cauchy sequence in X if $\lim_{n \to \infty} \widetilde{\mathcal{P}}(x_n, x_m, t) = 0$, for all $t > 0$.

3. $(X, \widetilde{\mathcal{P}})$ is said to be complete if every Cauchy sequence in it is a convergent sequence.

The following are some easy examples of EPbMSs.

Example 1.6. *Let $X = [0, +\infty)$ and $\widetilde{\mathcal{P}}(x, y, t) = e^{t(x-y)^p} - 1$ $(p \geq 1)$. Then $\widetilde{\mathcal{P}}$ is an extended parametric b-metric with $\Omega(t) = e^{2^{p-1}t} - 1$.*

More generally, several examples of EPb metrics can be constructed using the following easy proposition.

Proposition 1.7. *Let $(X, \mathcal{P})$ be a parametric b-metric space with coefficient $s \geq 1$ and let $\widetilde{\mathcal{P}}(x, y, t) = \xi(\mathcal{P}(x, y, t))$ where $\xi : [0, \infty) \to [0, \infty)$ is a strictly increasing* *continuous function with $t \leq \xi(t)$ for $t \geq 0$ and $\xi(0) = 0$. Then $\widetilde{\mathcal{P}}$ is an extended parametric b-metric with $\Omega(t) = \xi(st)$.*

Taking various functions ξ in the previous proposition, we can obtain a lot of examples of extended parametric b-metrics. We state just a few of them which we will use later in the text.

Example 1.8. (1) *If $\xi(t) = e^t - 1$, we get $\widetilde{\mathcal{P}}(x, y, t) = e^{\mathcal{P}(x,y,t)} - 1$ and $\Omega(t) = e^{st} - 1$. Note that $\Omega^{-1}(u) = \frac{1}{s}\ln(1 + u)$.*

(2) *If $\xi(t) = te^t$, then $\widetilde{\mathcal{P}}(x, y, t) = \mathcal{P}(x, y, t)e^{\mathcal{P}(x,y,t)}$ and $\Omega(t) = ste^{st}$. Note that in this case $\Omega^{-1}(u) = \frac{1}{s}W(u)$, for $u \geq 0$, where W is the Lambert W-function (see, e.g., [2]).*

(3) *If $\xi(t) = t + \ln(1 + t)$, then $\widetilde{\mathcal{P}}(x, y, t) = \mathcal{P}(x, y, t) + \ln(1 + \mathcal{P}(x, y, t))$ and $\Omega(t) = st+\ln(1+st)$. Here, again W-function is used to express the inverse: $\Omega^{-1}(u) = \frac{1}{s}(W(e^{u+1}) - 1)$ for $u \geq 0$.*

In general, an extended parametric b-metric function for nontrivial function Ω is not jointly continuous in all its variables. So, we need the following simple lemma about the convergent sequences in the proof of our main result.

Lemma 1.9. *Let $(X, \widetilde{\mathcal{P}})$ be an EPbMS with a strictly increasing continuous function $\Omega : [0, \infty) \to [0, \infty)$, and suppose that $\{x_n\}$ and $\{y_n\}$ convergent to x, y, respectively. Then, we have*

$$(\Omega^2)^{-1}(\widetilde{\mathcal{P}}(x, y, t)) \leq \liminf_{n\longrightarrow\infty} \widetilde{\mathcal{P}}(x_n, y_n, t) \leq \limsup_{n\longrightarrow\infty,t} \widetilde{\mathcal{P}}(x_n, y_n, t) \leq \Omega^2(\widetilde{\mathcal{P}}(x, y, t)).$$

In particular, if $x = y$, then, $\lim_{n\longrightarrow\infty} \widetilde{\mathcal{P}}(x_n, y_n, t) = 0$. Moreover, for each $z \in X$ we have

$$\Omega^{-1}(\widetilde{\mathcal{P}}(x, z, t)) \leq \liminf_{n\longrightarrow\infty} \widetilde{\mathcal{P}}(x_n, z, t) \leq \limsup_{n\longrightarrow\infty} \widetilde{\mathcal{P}}(x_n, z, t) \leq \Omega(\widetilde{\mathcal{P}}(x, z, t)).$$

2. Main results

2.1. **Results under Geraghty-type conditions.** In 1973, Geraghty [4] proved a fixed point result, generalizing Banach contraction principle. Fixed point results of this kind in b-metric spaces were obtained by Đukić et al. in [3].

Following [3], for a function Ω, let $\mathcal{F}_\Omega$ denote the class of all functions $\beta : [0, \infty) \to [0, \Omega^{-1}(1))$ satisfying the following condition:

$$\beta(t_n) \to \Omega^{-1}(1) \text{ as } n \to \infty \text{ implies } \; t_n \to 0 \text{ as } n \to \infty.$$

Theorem 2.1. *Let $(X, \preceq, \widetilde{\mathcal{P}})$ be a complete ordered EPbMS. Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that*

$$\Omega^2[\widetilde{\mathcal{P}}(fx, fy, t)] \leq \beta(\widetilde{\mathcal{P}}(x, y, t))M(x, y, t) \tag{2.1}$$

for all $t > 0$ and for all comparable elements $x, y \in X$, where

$$M(x, y, t) = \max\left\{\widetilde{\mathcal{P}}(x, y, t), \frac{\widetilde{\mathcal{P}}(x, fx, t)\widetilde{\mathcal{P}}(y, fy, t)}{[1 + \widetilde{\mathcal{P}}(x, y, t)][1 + \widetilde{\mathcal{P}}(fx, fy, t)]}\right\}.$$

If f is continuous, then f has a fixed point.

Proof. Starting with the given x_0, put $x_n = f^n x_0$. Since $x_0 \preceq fx_0$ and f is an increasing function we obtain by induction that

$$x_0 \preceq fx_0 \preceq f^2x_0 \preceq \cdots \preceq f^nx_0 \preceq f^{n+1}x_0 \preceq \cdots .$$

Step I: We will show that $\lim_{n\to\infty} \widetilde{\mathcal{P}}(x_n, x_{n+1}, t) = 0$. Since $x_n \preceq x_{n+1}$ for each $n \in \mathbb{N}$, then by (2.1) we have

$$\begin{aligned}\Omega^2[\widetilde{\mathcal{P}}(x_n, x_{n+1}, t)] &= \Omega^2[\widetilde{\mathcal{P}}(fx_{n-1}, fx_n, t)] \leq \beta(\widetilde{\mathcal{P}}(x_{n-1}, x_n, t))M(x_{n-1}, x_n, t)\\ &< \Omega^{-1}(1)\widetilde{\mathcal{P}}(x_{n-1}, x_n, t) \leq \widetilde{\mathcal{P}}(x_{n-1}, x_n, t), \end{aligned}\tag{2.2}$$

because

$$\begin{aligned}&M(x_{n-1}, x_n, t)\\ &= \max\left\{\widetilde{\mathcal{P}}(x_{n-1}, x_n, t), \frac{\widetilde{\mathcal{P}}(x_{n-1}, fx_{n-1}, t)\widetilde{\mathcal{P}}(x_n, fx_n, t)}{[1+\widetilde{\mathcal{P}}(x_{n-1}, x_n, t)][1+\widetilde{\mathcal{P}}(fx_{n-1}, fx_n, t)]}\right\}\\ &= \max\left\{\widetilde{\mathcal{P}}(x_{n-1}, x_n, t), \frac{\widetilde{\mathcal{P}}(x_{n-1}, x_n, t)\widetilde{\mathcal{P}}(x_n, x_{n+1}, t)}{[1+\widetilde{\mathcal{P}}(x_{n-1}, x_n, t)][1+\widetilde{\mathcal{P}}(x_n, x_{n+1}, t)]}\right\}\\ &\leq \max\left\{\widetilde{\mathcal{P}}(x_{n-1}, x_n, t), \frac{\widetilde{\mathcal{P}}(x_n, x_{n+1}, t)}{[1+\widetilde{\mathcal{P}}(x_n, x_{n+1}, t)]}\right\}.\end{aligned}$$

If $\max\{\widetilde{\mathcal{P}}(x_{n-1}, x_n, t), \frac{\widetilde{\mathcal{P}}(x_n,x_{n+1},t)}{[1+\widetilde{\mathcal{P}}(x_n,x_{n+1},t)]}\} = \frac{\widetilde{\mathcal{P}}(x_n,x_{n+1},t)}{[1+\widetilde{\mathcal{P}}(x_n,x_{n+1},t)]}$, then from (2.2) we have,

$$\begin{aligned}\widetilde{\mathcal{P}}(x_n, x_{n+1}, t) \quad &\leq \beta(\widetilde{\mathcal{P}}(x_{n-1}, x_n, t))\tfrac{\widetilde{\mathcal{P}}(x_n,x_{n+1},t)}{[1+\widetilde{\mathcal{P}}(x_n,x_{n+1},t)]}\\ &< \Omega^{-1}(1)\tfrac{\widetilde{\mathcal{P}}(x_n,x_{n+1},t)}{[1+\widetilde{\mathcal{P}}(x_n,x_{n+1},t)]}\\ &\leq \tfrac{\widetilde{\mathcal{P}}(x_n,x_{n+1},t)}{[1+\widetilde{\mathcal{P}}(x_n,x_{n+1},t)]},\end{aligned}\tag{2.3}$$

which is a contradiction.

Hence, $\max\{\widetilde{\mathcal{P}}(x_{n-1}, x_n, t), \frac{\widetilde{\mathcal{P}}(x_n,x_{n+1},t)}{[1+\widetilde{\mathcal{P}}(x_n,x_{n+1},t)]}\} = \widetilde{\mathcal{P}}(x_{n-1}, x_n, t)$, so from (2.3),

$$\widetilde{\mathcal{P}}(x_n, x_{n+1}, t) \leq \beta(\widetilde{\mathcal{P}}(x_{n-1}, x_n, t))\widetilde{\mathcal{P}}(x_{n-1}, x_n, t) \leq \widetilde{\mathcal{P}}(x_{n-1}, x_n, t). \tag{2.4}$$

Therefore, the sequence $\{\widetilde{\mathcal{P}}(x_n, x_{n+1}, t)\}$ is decreasing, so there exists $r \geq 0$ such that $\lim_{n\to\infty} \widetilde{\mathcal{P}}(x_n, x_{n+1}, t) = r$. Suppose that $r > 0$. Now, letting $n \to \infty$, from (2.4) we have

$$\Omega^{-1}(1)r < r \leq \lim_{n\to\infty} \beta(\widetilde{\mathcal{P}}(x_{n-1}, x_n, t))r \leq r.$$

So, we have $\lim_{n\to\infty} \beta(\widetilde{\mathcal{P}}(x_{n-1}, x_n, t)) \geq \Omega^{-1}(1)$ and since $\beta \in \mathcal{F}_\Omega$ we deduce that $\lim_{n\to\infty} \widetilde{\mathcal{P}}(x_{n-1}, x_n, t) = 0$ which is a contradiction. Hence, $r = 0$, that is,

$$\lim_{n\to\infty} \widetilde{\mathcal{P}}(x_n, x_{n+1}, t) = 0. \tag{2.5}$$

Step II: Now, we prove that the sequence $\{x_n\}$ is a Cauchy sequence. Using the triangle inequality and by (2.1) we have

$$\begin{aligned}\widetilde{\mathcal{P}}(x_n, x_m, t) &\leq \Omega[\widetilde{\mathcal{P}}(x_n, x_{n+1}, t) + \Omega[\widetilde{\mathcal{P}}(x_{n+1}, x_{m+1}, t) + \widetilde{\mathcal{P}}(x_{m+1}, x_m, t)]]\\ &\leq \Omega[\widetilde{\mathcal{P}}(x_n, x_{n+1}, t) + \Omega[\widetilde{\mathcal{P}}(x_m, x_{m+1}, t) + \Omega^{-2}[\beta(\widetilde{\mathcal{P}}(x_n, x_m, t))M(x_n, x_m, t)]]].\end{aligned}$$

Letting $m,n\to\infty$ in the above inequality and applying (2.5) we have

$$\lim_{m,n\to\infty}\widetilde{\mathcal{P}}(x_n,x_m,t)\le\lim_{m,n\to\infty}\beta(\widetilde{\mathcal{P}}(x_n,x_m,t))\lim_{m,n\to\infty}M(x_n,x_m,t). \tag{2.6}$$

Here,

$$\begin{aligned}\widetilde{\mathcal{P}}(x_n,x_m,t)&\le M(x_n,x_m,t)\\&=\max\left\{\widetilde{\mathcal{P}}(x_n,x_m,t),\frac{\widetilde{\mathcal{P}}(x_n,fx_n,t)\widetilde{\mathcal{P}}(x_m,fx_m,t)}{[1+\widetilde{\mathcal{P}}(x_n,x_m,t)][1+\widetilde{\mathcal{P}}(fx_n,fx_m,t)]}\right\}\\&=\max\left\{\widetilde{\mathcal{P}}(x_n,x_m,t),\frac{\widetilde{\mathcal{P}}(x_n,x_{n+1},t)\widetilde{\mathcal{P}}(x_m,x_{m+1},t)}{[1+\widetilde{\mathcal{P}}(x_n,x_m,t)][1+\widetilde{\mathcal{P}}(x_{n+1},x_{m+1},t)]}\right\}.\end{aligned}$$

Letting $m,n\to\infty$ in the above inequality we get

$$\lim_{m,n\to\infty}M(x_n,x_m,t)=\lim_{m,n\to\infty}\widetilde{\mathcal{P}}(x_n,x_m,t). \tag{2.7}$$

From (2.6) and (2.7), we obtain

$$\lim_{m,n\to\infty}\widetilde{\mathcal{P}}(x_n,x_m,t)\le\lim_{m,n\to\infty}\beta(\widetilde{\mathcal{P}}(x_n,x_m,t))\lim_{m,n\to\infty}\widetilde{\mathcal{P}}(x_n,x_m,t). \tag{2.8}$$

Now we claim that, $\lim_{m,n\to\infty}\widetilde{\mathcal{P}}(x_n,x_m,t)=0$. On the contrary, if $\lim\limits_{m,n\to\infty}\widetilde{\mathcal{P}}(x_n,x_m,t)\neq 0$, then we get

$$\frac{\lim_{m,n\to\infty}\widetilde{\mathcal{P}}(x_n,x_m,t)}{\lim_{m,n\to\infty}\widetilde{\mathcal{P}}(x_n,x_m,t)}\le\lim_{m,n\to\infty}\beta(\widetilde{\mathcal{P}}(x_n,x_m,t)).$$

Since $\beta\in\mathcal{F}_\Omega$ we deduce that

$$\lim_{m,n\to\infty}\widetilde{\mathcal{P}}(x_n,x_m,t)=0. \tag{2.9}$$

which is a contradiction. Consequently, $\{x_n\}$ is a Cauchy sequence in X. Since $(X,\widetilde{\mathcal{P}})$ is complete, the sequence $\{x_n\}$ converges to some $z\in X$, that is, $\lim\limits_{n\to\infty}\widetilde{\mathcal{P}}(x_n,z,t)=0$.

Step III: Now, we show that z is a fixed point of f.

Using the triangle inequality, we get

$$\widetilde{\mathcal{P}}(fz,z,t)\le\Omega[\widetilde{\mathcal{P}}(ft,fx_n,t)+\widetilde{\mathcal{P}}(fx_n,z,t)].$$

Letting $n\to\infty$ and using the continuity of f, we have $fz=z$. Thus, z is a fixed point of f. $\square$

Example 2.2. *Let $X=[0,3]$ be endowed with the extended parametric b-metric*

$$\widetilde{\mathcal{P}}(x,y,t)=\begin{cases}e^{t(x+y)^2}-1, & \text{if } x\neq y,\\ 0 & \text{if } x=y,\end{cases}$$

for all $x,y\in X$ and all $t>0$. So, $\Omega(t)=e^{2t}-1$. Define $T:X\to X$ by

$$Tx=\begin{cases}\frac{1}{8\sqrt{2}}x^{10}, & \text{if } x\in[0,1),\\ \frac{1}{8\sqrt{2}}x, & \text{if } x\in[1,3].\end{cases}$$

Also, define, $\beta : [0,\infty) \to [0,\Omega^{-1}(1)) = [0, \frac{\ln(1+1)}{2} = 0.34657359028)$ by $\beta(t) = \frac{1}{4}$. Clearly, $(X, \widetilde{\mathcal{P}}, e^{2t} - 1)$ is a complete EPbMS, T is a continuous mapping and $\beta \in \mathcal{F}_{e^{2t}-1}$.

- Let $x, y \in [0,1)$ with $x \le y$, then,

$$\begin{aligned}
\Omega^2[\widetilde{\mathcal{P}}(fx,fy,t)] &= e^{2e^{2\widetilde{\mathcal{P}}(Tx,Ty,t)}-1} - 1 = e^{2e^{2t(\frac{1}{8\sqrt{2}}x^{10}+\frac{1}{8\sqrt{2}}y^{10})^2}-1} - 1 \\
&= e^{2e^{\frac{1}{64}t(x^{10}+y^{10})^2}-1} - 1 \le e^{2e^{\frac{1}{64}t(x+y)^2}-1} - 1 \\
&\le e^{\frac{1}{32}e^{t(x+y)^2}-1} - 1 \le e^{\frac{1}{32}[\widetilde{\mathcal{P}}(x,y,t)]} - 1 \\
&\le \tfrac{1}{32}[e^{\widetilde{\mathcal{P}}(x,y,t)} - 1] \le \tfrac{1}{4}[\widetilde{\mathcal{P}}(x,y,t)] \\
&= \beta(\widetilde{\mathcal{P}}(x,y,t))\widetilde{\mathcal{P}}(x,y,t) \\
&\le \beta(\widetilde{\mathcal{P}}(x,y,t))M(x,y,t).
\end{aligned}$$

In other possible cases the above result can also be obtained similarly. Therefore,

$$\Omega^2[\widetilde{\mathcal{P}}(fx,fy,t)] \le \beta(\widetilde{\mathcal{P}}(x,y,t))M(x,y,t)$$

for all $x, y \in X$ with $x \le y$ and all $t > 0$. Hence, all conditions of Theorem 2.1 holds and T has a unique fixed point.

Example 2.3. *Let $X = [0,5]$ be equipped with the $\widetilde{\mathcal{P}}$-metric*

$$\widetilde{\mathcal{P}}(x,y,t) = \sinh(t\,|x-y|)$$

for all $x, y \in X$, where $\Omega(x) = \sinh x$ and hence, $\Omega^{-1}(x) = \sinh^{-1}(x)$.

Define a relation $\preceq$ on X by $x \preceq y$ iff $y \le x$, the function $f : [0,5] \to [0,2]$ by

$$fx = \sqrt{2 + \frac{x}{4}}$$

and the function β given by $\beta(t) = \frac{1}{2} < 0.88137358702 = \Omega^{-1}(1)$.
For all comparable elements $x, y \in X$, we have,

$$\begin{aligned}
\Omega^2(\widetilde{\mathcal{P}}(fx,fy,t)) &= \sinh(\sinh(\sinh(t|\sqrt{2+\frac{x}{4}} - \sqrt{2+\frac{y}{4}}|))) \\
&\le \sinh(\sinh(\sinh(t|\frac{x}{4} - \frac{y}{4}|))) \\
&\le \sinh(\sinh(\frac{\widetilde{\mathcal{P}}(x,y,t)}{4})) \\
&\le \frac{\widetilde{\mathcal{P}}(x,y,t)}{2} = \beta(\widetilde{\mathcal{P}}(x,y,t))\widetilde{\mathcal{P}}(x,y,t) \le \beta(M(x,y,t))M(x,y,t),
\end{aligned}$$

So, from Theorem 2.1 f has a fixed point.

Note that the continuity of f in Theorem 2.1 is not necessary and can be dropped.

Theorem 2.4. *Under the hypotheses of Theorem 2.1, without the continuity assumption on f, assume that whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to u$, one has $x_n \preceq u$ for all $n \in \mathbb{N}$. Then f has a fixed point.*

Proof. Repeating the proof of Theorem 2.1, we construct an increasing sequence $\{x_n\}$ in X such that $x_n \to z \in X$. Using the assumption on X we have $x_n \preceq z$. Now, we show that $z = fz$. By (2.1) and Lemma 1.9,

$$\begin{aligned}\Omega^3\left[\Omega^{-1}\widetilde{\mathcal{P}}(z, fz, t)\right] &\leq \Omega^2[\limsup_{n\to\infty}\widetilde{\mathcal{P}}(x_{n+1}, fz, t)]\\ &\leq \limsup_{n\to\infty}\beta(\widetilde{\mathcal{P}}(x_n, z, t))\limsup_{n\to\infty}M(x_n, z, t),\end{aligned}$$

where,

$$\begin{aligned}&\lim_{n\to\infty} M(x_n, z, t)\\ &= \lim_n \max\left\{\widetilde{\mathcal{P}}(x_n, z, t), \frac{\widetilde{\mathcal{P}}(x_n, fx_n, t)\widetilde{\mathcal{P}}(z, fz, t)}{[1+\widetilde{\mathcal{P}}(x_n, z, t)][1+\widetilde{\mathcal{P}}(fx_n, fz, t)]}\right\} = 0.\end{aligned}$$

Therefore, we deduce that $\widetilde{\mathcal{P}}(z, fz, t) \leq 0$. As t is arbitrary, hence, we have $z = fz$. □

If in the above theorems we take $\beta(t) = r$, where $0 \leq r < \Omega^{-1}(1)$, then we have the following corollary.

Corollary 2.5. *Let $(X, \preceq, \widetilde{\mathcal{P}})$ be an ordered complete EPbMS. Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that for some r, with $0 \leq r < \Omega^{-1}(1)$,*

$$\widetilde{\mathcal{P}}(fx, fy, t) \leq \Omega^{-2}[rM(x, y, t)]$$

holds for each $t > 0$ and all comparable elements $x, y \in X$, where

$$M(x, y, t) = \max\left\{\widetilde{\mathcal{P}}(x, y, t), \frac{\widetilde{\mathcal{P}}(x, fx, t)\widetilde{\mathcal{P}}(y, fy, t)}{[1+\widetilde{\mathcal{P}}(x, y, t)][1+\widetilde{\mathcal{P}}(fx, fy, t)]}\right\}.$$

If f is continuous, or, for any nondecreasing sequence $\{x_n\}$ in X such that $x_n \to u \in X$ one has $x_n \preceq u$ for all $n \in \mathbb{N}$, then f has a fixed point.

Corollary 2.6. *Let $(X, \preceq, \widetilde{\mathcal{P}})$ be an ordered complete EPbMS. Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that*

$$\widetilde{\mathcal{P}}(fx, fy, t) \leq \alpha\widetilde{\mathcal{P}}(x, y, t) + \beta\frac{\widetilde{\mathcal{P}}(x, fx, t)\widetilde{\mathcal{P}}(y, fy, t)}{[1+\widetilde{\mathcal{P}}(x, y, t)][1+\widetilde{\mathcal{P}}(fx, fy, t)]}$$

for each $t > 0$ and all comparable elements $x, y \in X$, where $\alpha, \beta \geq 0$ and $\alpha + \beta \leq \Omega^{-1}(1)$.

If f is continuous, or, for any nondecreasing sequence $\{x_n\}$ in X such that $x_n \to u \in X$ one has $x_n \preceq u$ for all $n \in \mathbb{N}$, then f has a fixed point.

2.2. **Results using comparison functions.** Let Ψ denote the family of all nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\lim_n \psi^n(t) = 0$ for all $t > 0$, where ψ^n denotes the n-th iterate of ψ. It is easy to show that, for each $\psi \in \Psi$, the following is satisfied:

(a) $\psi(t) < t$ for all $t > 0$;
(b) $\psi(0) = 0$.

Theorem 2.7. *Let $(X, \preceq, \widetilde{\mathcal{P}})$ be an ordered complete EPbMS. Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that*

$$\Omega[\widetilde{\mathcal{P}}(fx, fy, t)] \leq \psi(N(x, y, t)) \tag{2.10}$$

where

$$N(x,y,t) = \max\left\{\widetilde{\mathcal{P}}(x,y,t), \frac{[\widetilde{\mathcal{P}}(x,fx,t)\mathcal{P}(x,fy,t) + \widetilde{\mathcal{P}}(y,fy,t)\widetilde{\mathcal{P}}(y,fx,t)]^2}{1+\Omega[\widetilde{\mathcal{P}}(x,fx,t)+\widetilde{\mathcal{P}}(y,fy,t)][1+\widetilde{\mathcal{P}}(x,fy,t)+\widetilde{\mathcal{P}}(y,fx,t)]}\right\},$$

for some $\psi \in \Psi$ and for all comparable elements $x, y \in X$ and all $t > 0$. If f is continuous, then f has a fixed point.

Proof. Since $x_0 \preceq fx_0$ and f is an increasing function, we obtain by induction that

$$x_0 \preceq fx_0 \preceq f^2x_0 \preceq \cdots \preceq f^nx_0 \preceq f^{n+1}x_0 \preceq \cdots .$$

Putting $x_n = f^nx_0$, we have

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots .$$

We assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

Step I. We will prove that $\lim_{n\to\infty} \widetilde{\mathcal{P}}(x_n, x_{n+1}, t) = 0$. Using condition (2.10), we obtain

$$\widetilde{\mathcal{P}}(x_n, x_{n+1}, t) \leq \Omega[\widetilde{\mathcal{P}}(x_n, x_{n+1}, t)] = \Omega[\widetilde{\mathcal{P}}(fx_{n-1}, fx_n, t)] \leq \psi(N(x_{n-1}, x_n, t)).$$

Here,

$$\begin{aligned}
&N(x_{n-1}, x_n, t)\\
&= \max\{\widetilde{\mathcal{P}}(x_{n-1}, x_n, t),\\
&\frac{[\widetilde{\mathcal{P}}(x_{n-1}, fx_{n-1}, t)\widetilde{\mathcal{P}}(x_{n-1}, fx_n, t) + \widetilde{\mathcal{P}}(x_n, fx_n, t)\widetilde{\mathcal{P}}(x_n, fx_{n-1}, t)]^2}{1+\Omega[\widetilde{\mathcal{P}}(x_{n-1}, fx_{n-1}, t)+\widetilde{\mathcal{P}}(x_n, fx_n, t)][1+\widetilde{\mathcal{P}}(x_{n-1}, fx_n, t)+\widetilde{\mathcal{P}}(x_n, fx_{n-1}, t)]}\}\\
&= \max\{\widetilde{\mathcal{P}}(x_{n-1}, x_n, t),\\
&\frac{[\widetilde{\mathcal{P}}(x_{n-1}, x_n, t)\widetilde{\mathcal{P}}(x_{n-1}, x_{n+1}, t) + \widetilde{\mathcal{P}}(x_n, x_{n+1}, t)\widetilde{\mathcal{P}}(x_n, x_n, t)]^2}{1+\Omega[\widetilde{\mathcal{P}}(x_{n-1}, x_n, t)+\widetilde{\mathcal{P}}(x_n, x_{n+1}, t)][1+\widetilde{\mathcal{P}}(x_{n-1}, x_{n+1}, t)+\widetilde{\mathcal{P}}(x_n, x_n, t)]}\}\\
&= \widetilde{\mathcal{P}}(x_{n-1}, x_n, t).
\end{aligned}$$

Hence,

$$\widetilde{\mathcal{P}}(x_n, x_{n+1}, t) \leq \Omega[\widetilde{\mathcal{P}}(x_n, x_{n+1}, t)] \leq \psi(\widetilde{\mathcal{P}}(x_{n-1}, x_n, t)).$$

By induction, we get that

$$\widetilde{\mathcal{P}}(x_n, x_{n+1}, t) \leq \psi(\widetilde{\mathcal{P}}(x_{n-1}, x_n, t)) \leq \psi^2(\widetilde{\mathcal{P}}(x_{n-2}, x_{n-1}, t)) \leq \cdots \leq \psi^n(\widetilde{\mathcal{P}}(x_0, x_1, t)).$$

As $\psi \in \Psi$, we conclude that

$$\lim_{n\to\infty} \widetilde{\mathcal{P}}(x_n, x_{n+1}, t) = 0. \tag{2.11}$$

Step II. We will prove that $\{x_n\}$ is a Cauchy sequence. Suppose the contrary. Then there exist $t_0 > 0$ and $\varepsilon > 0$ for them we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i \text{ and } \widetilde{\mathcal{P}}(x_{m_i}, x_{n_i}, t_0) \geq \varepsilon. \tag{2.12}$$

This means that

$$\widetilde{\mathcal{P}}(x_{m_i}, x_{n_i-1}, t_0) < \varepsilon. \tag{2.13}$$

From (2.12) and using the triangle inequality, we get

$$\varepsilon \leq \widetilde{\mathcal{P}}(x_{m_i}, x_{n_i}, t_0) \leq \Omega[\widetilde{\mathcal{P}}(x_{m_i}, x_{m_i+1}, t_0) + \widetilde{\mathcal{P}}(x_{m_i+1}, x_{n_i}, t_0)].$$

Taking the upper limit as $i \to \infty$, we get

$$\Omega^{-1}[\varepsilon] \leq \limsup_{i\to\infty} \widetilde{\mathcal{P}}(x_{m_i+1}, x_{n_i}, t_0). \tag{2.14}$$

From the definition of $M(x, y, t_0)$ we have

$$\begin{aligned}
&M(x_{m_i}, x_{n_i-1}, t_0)\\
&= \max\{\widetilde{\mathcal{P}}(x_{m_i}, x_{n_i-1}, t_0),\\
&\frac{[\widetilde{\mathcal{P}}(x_{m_i}, fx_{m_i}, t_0)\widetilde{\mathcal{P}}(x_{m_i}, fx_{n_i-1}, t_0) + \widetilde{\mathcal{P}}(x_{n_i-1}, fx_{n_i-1}, t_0)\widetilde{\mathcal{P}}(x_{n_i-1}, fx_{m_i}, t_0)]^2}{1 + \Omega[\widetilde{\mathcal{P}}(x_{m_i}, fx_{m_i}, t_0) + \widetilde{\mathcal{P}}(x_{n_i-1}, fx_{n_i-1}, t_0)][1 + \widetilde{\mathcal{P}}(x_{m_i}, fx_{n_i-1}, t_0) + \widetilde{\mathcal{P}}(x_{n_i-1}, fx_{m_i}, t_0)]}\}\\
&= \max\{\widetilde{\mathcal{P}}(x_{m_i}, x_{n_i-1}, t_0),\\
&\frac{[\widetilde{\mathcal{P}}(x_{m_i}, x_{m_i+1}, t_0)\widetilde{\mathcal{P}}(x_{m_i}, x_{n_i}, t_0) + \widetilde{\mathcal{P}}(x_{n_i-1}, x_{n_i}, t_0)\widetilde{\mathcal{P}}(x_{n_i-1}, x_{m_i+1}, t_0)]^2}{1 + \Omega[\widetilde{\mathcal{P}}(x_{m_i}, x_{m_i+1}, t_0) + \widetilde{\mathcal{P}}(x_{n_i-1}, x_{n_i}, t_0)][1 + \widetilde{\mathcal{P}}(x_{m_i}, x_{n_i}, t_0) + \widetilde{\mathcal{P}}(x_{n_i-1}, x_{m_i+1}, t_0)]}\}
\end{aligned}$$

and if $i \to \infty$, by (2.11) and (2.13) we have

$$\limsup_{i\to\infty} M(x_{m_i}, x_{n_i-1}, t_0) \leq \varepsilon.$$

Now, from (2.10) we have

$$\Omega[\widetilde{\mathcal{P}}(x_{m_i+1}, x_{n_i}, t_0)] = \Omega[\widetilde{\mathcal{P}}(fx_{m_i}, fx_{n_i-1}, t_0)] \leq \psi(M(x_{m_i}, x_{n_i-1}, t_0)).$$

Again, if $i \to \infty$ by (2.14) we obtain

$$\varepsilon = \Omega[\Omega^{-1}[\varepsilon]] \leq \Omega[\limsup_{i\to\infty} \widetilde{\mathcal{P}}(x_{m_i+1}, x_{n_i}, a)] \leq \psi(\varepsilon) < \varepsilon,$$

which is a contradiction. Consequently, $\{x_n\}$ is a Cauchy sequence in X. Therefore, the sequence $\{x_n\}$ converges to some $z \in X$, that is, $\lim_n \widetilde{\mathcal{P}}(x_n, z, t) = 0$ for all $t > 0$.

Step III. Now we show that z is a fixed point of f.

Using the triangle inequality, we get

$$\widetilde{\mathcal{P}}(z, fz, t) \leq \Omega[\widetilde{\mathcal{P}}(z, fx_n, t) + \widetilde{\mathcal{P}}(fx_n, fz, t)].$$

Letting $n \to \infty$ and using the continuity of f, we get

$$\widetilde{\mathcal{P}}(z, fz, t) \leq 0.$$

Hence, we have $fz = z$. Thus, z is a fixed point of f. □

Theorem 2.8. *Under the hypotheses of Theorem 2.7, without the continuity assumption on f, assume that whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to u \in X$, one has $x_n \preceq u$ for all $n \in \mathbb{N}$. Then f has a fixed point.*

Proof. Following the proof of Theorem 2.7, we construct an increasing sequence $\{x_n\}$ in X such that $x_n \to z \in X$. Using the given assumption on X we have $x_n \preceq z$. Now, we show that $z = fz$. By (3.2) we have

$$\Omega[\widetilde{\mathcal{P}}(fz, x_n, t)] = \Omega[\widetilde{\mathcal{P}}(fz, fx_{n-1}, t)] \leq \psi(M(z, x_{n-1}, t)), \tag{2.15}$$

where

$$\begin{aligned}
&M(z,x_{n-1},t)\\
&=\max\{\widetilde{\mathcal{P}}(x_{n-1},z,t),\\
&\frac{[\widetilde{\mathcal{P}}(x_{n-1},fx_{n-1},t)\widetilde{\mathcal{P}}(x_{n-1},fz,t)+\widetilde{\mathcal{P}}(z,fz,t)\widetilde{\mathcal{P}}(z,fx_{n-1},t)]^2}{1+\Omega[\widetilde{\mathcal{P}}(x_{n-1},fx_{n-1},t)+\widetilde{\mathcal{P}}(z,fz,t)][1+\widetilde{\mathcal{P}}(x_{n-1},fz,t)+\widetilde{\mathcal{P}}(z,fx_{n-1},t)]}\}\\
&=\max\{\widetilde{\mathcal{P}}(x_{n-1},z,t),\\
&\frac{[\widetilde{\mathcal{P}}(x_{n-1},x_n,t)\widetilde{\mathcal{P}}(x_{n-1},fz,t)+\widetilde{\mathcal{P}}(z,fz,t)\widetilde{\mathcal{P}}(z,x_n,t)]^2}{1+\Omega[\widetilde{\mathcal{P}}(x_{n-1},x_n,t)+\widetilde{\mathcal{P}}(z,fz,t)][1+\widetilde{\mathcal{P}}(x_{n-1},fz,t)+\widetilde{\mathcal{P}}(z,x_n,t)]}\}.
\end{aligned}$$

Letting $n\to\infty$ in the above relation, we get

$$\limsup_{n\to\infty} M(z,x_{n-1},t)=0. \tag{2.16}$$

Again, taking the upper limit as $n\to\infty$ in (2.15) and using Lemma 1.9 and (2.16) we get

$$\begin{aligned}
\Omega^2[\Omega^{-1}[\widetilde{\mathcal{P}}(z,fz,t)]] &\le \Omega[\limsup_{n\to\infty}\widetilde{\mathcal{P}}(x_n,fz,t)]\\
&\le \limsup_{n\to\infty}\psi(M(z,x_{n-1},t))=0.
\end{aligned}$$

So, we get $\widetilde{\mathcal{P}}(z,fz,t)=0$, i.e., $fz=z$. □

Corollary 2.9. *Let $(X,\preceq,\widetilde{\mathcal{P}})$ be an ordered complete EPbMS. Let $f:X\to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0\in X$ with $x_0\preceq fx_0$. Suppose that*

$$\Omega[\widetilde{\mathcal{P}}(fx,fy,t)]\le rN(x,y,t)$$

where $0\le r<1$ and

$$N(x,y,t)=\max\left\{\widetilde{\mathcal{P}}(x,y,t),\frac{[\widetilde{\mathcal{P}}(x,fx,t)\mathcal{P}(x,fy,t)+\widetilde{\mathcal{P}}(y,fy,t)\widetilde{\mathcal{P}}(y,fx,t)]^2}{1+\Omega[\widetilde{\mathcal{P}}(x,fx,t)+\widetilde{\mathcal{P}}(y,fy,t)][1+\widetilde{\mathcal{P}}(x,fy,t)+\widetilde{\mathcal{P}}(y,fx,t)]}\right\},$$

for all comparable elements $x,y\in X$ and all $t>0$. If f is continuous, or, whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n\to u\in X$, one has $x_n\preceq u$ for all $n\in\mathbb{N}$, then f has a fixed point.

Example 2.10. *Let $X=[0,\infty]$ be equipped with the*

$$\widetilde{\mathcal{P}}(x,y,t)=t\,|x-y|+\ln(t\,|x-y|)$$

for all $x,y,t\in X$, where $\Omega(x)=x+\ln x$.

Define a relation $\preceq$ on X by $x\preceq y$ iff $y\le x$, the function $f:X\to X$ by

$$fx=\ln(\frac{x}{5}+2)$$

and the function ψ *given by* $\psi(t) = \ln(1+t)$.
For all comparable elements $x, y \in X$*, by mean value theorem, we have,*

$$\begin{aligned}
\Omega[\widetilde{\mathcal{P}}(fx, fy, t)] &= \widetilde{\mathcal{P}}(fx, fy, t) + \ln[1 + \widetilde{\mathcal{P}}(fx, fy, t)] \\
&= t\left|\ln 2 + \frac{x}{5} - \ln 2 + \frac{y}{5}\right| \\
&+ \ln[1 + t\left|\ln 2 + \frac{x}{5} - \ln 2 + \frac{y}{5}\right|] \\
&\leq t\left|\frac{x}{5} - \frac{y}{5}\right| \\
&+ \ln[1 + t\left|\frac{x}{5} - \frac{y}{5}\right|] \\
&\leq \frac{1}{5}\widetilde{\mathcal{P}}(x, y, t) + \ln[1 + \frac{1}{5}\widetilde{\mathcal{P}}(x, y, t)] \\
&\leq \ln[1 + \widetilde{\mathcal{P}}(x, y, t)] = \psi(\widetilde{\mathcal{P}}(x, y, t)) \leq \psi(M(x, y, t)).
\end{aligned}$$

So, from Theorem 2.14 f *has a fixed point.*

2.3. **Results for generalized weakly contractive mappings.** In this section, we define the notion of generalized $(\psi, \varphi)_{\Omega,t}$-contractive mapping and prove our new results.

Definition 2.11. [9] *A function* $\varphi : [0, +\infty) \to [0, +\infty)$ *is called an altering distance function, if the following properties hold:*

(1) φ *is continuous and non-decreasing.*
(2) $\varphi(t) = 0$ *if and only if* $t = 0$.

Let $(X, \widetilde{\mathcal{P}})$ be an EPbMS and let $f : X \to X$ be a mapping. For all $x, y \in X$ and for all $t > 0$, let

$$M_t(x, y) = \max\left\{\widetilde{\mathcal{P}}(x, y, t), \widetilde{\mathcal{P}}(x, fx, t), \widetilde{\mathcal{P}}(y, fy, t), \frac{1}{2}\Omega^{-1}[\widetilde{\mathcal{P}}(x, fy, t) + \widetilde{\mathcal{P}}(y, fx, t)]\right\}.$$

Definition 2.12. *Let* $(X, \widetilde{\mathcal{P}})$ *be an EPbMS. We say that a mapping* $f : X \to X$ *is an ordered generalized* $(\psi, \varphi)_{\Omega,t}$*-contractive mapping if there exist two altering distance functions* ψ *and* φ *such that*

$$\psi(\Omega[\widetilde{\mathcal{P}}(fx, fy, t)]) \leq \psi(M_t(x, y)) - \varphi(M_t(x, y)) \tag{2.17}$$

for all comparable elements $x, y \in X$ *and for all* $t > 0$.

Recall that a real function Ω is called super-additive if

$$\Omega(s + t) \geq \Omega(s) + \Omega(t)$$

for all $t, s \in D(\Omega)$. If Ω is a super-additive function, and if $0 \in D(\Omega)$, then $\Omega(0) \leq 0$. Indeed, super-additivity of Ω yields that $\Omega(s) \leq \Omega(s + t) - \Omega(t)$ for all $s, t \in D(\Omega)$. Taking $s = 0$ one has $\Omega(0) \leq \Omega(0 + t) - \Omega(t) = 0$. Also, it is easy to see that $2\Omega(t) \leq \Omega(2t)$ for each $t \in D(\Omega)$.

Now, let us prove another result.

Theorem 2.13. *Let* $(X, \preceq, \widetilde{\mathcal{P}})$ *be an ordered complete EPbMS. Let* $f : X \to X$ *be an ordered generalized* $(\psi, \varphi)_{\Omega,t}$*-contractive and continuous non-decreasing mapping with respect to* $\preceq$ *where* Ω *is super-additive. If there exists* $x_0 \in X$ *such that* $x_0 \preceq fx_0$*, then* f *has a fixed point.*

Proof. Define a sequence $\{x_n\}$ in X such that $x_{n+1} = fx_n$, for all $n \geq 0$. Since $x_0 \preceq fx_0 = x_1$ and f is non-decreasing, we have $x_1 = fx_0 \preceq x_2 = fx_1$, and by induction

$$x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots .$$

We assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. By (2.17), we have

$$\begin{aligned}\psi(\widetilde{\mathcal{P}}(x_n, x_{n+1}, t)) &\leq \psi(\Omega[\widetilde{\mathcal{P}}(x_n, x_{n+1}, t)]) \\ &= \psi(\Omega[\widetilde{\mathcal{P}}(fx_{n-1}, fx_n, t)]) \\ &\leq \psi(M_t(x_{n-1}, x_n)) - \varphi(M_t(x_{n-1}, x_n)), \end{aligned} \tag{2.18}$$

where

$$\begin{aligned}&M_t(x_{n-1}, x_n) \\ &= \max\left\{\widetilde{\mathcal{P}}(x_{n-1}, x_n, t), \widetilde{\mathcal{P}}(x_{n-1}, fx_{n-1}, t), \widetilde{\mathcal{P}}(x_n, fx_n, t), \frac{1}{2}\Omega^{-1}[\widetilde{\mathcal{P}}(x_{n-1}, fx_n, t) + \widetilde{\mathcal{P}}(x_n, fx_{n-1}, t)]\right\} \\ &= \max\left\{\widetilde{\mathcal{P}}(x_{n-1}, x_n, t), \widetilde{\mathcal{P}}(x_n, x_{n+1}, t), \frac{1}{2}\Omega^{-1}[\widetilde{\mathcal{P}}(x_{n-1}, x_{n+1}, t)]\right\} \\ &\leq \max\left\{\widetilde{\mathcal{P}}(x_{n-1}, x_n, t), \widetilde{\mathcal{P}}(x_n, x_{n+1}, t), \frac{\widetilde{\mathcal{P}}(x_{n-1}, x_n, t) + \widetilde{\mathcal{P}}(x_n, x_{n+1}, t)}{2}\right\} \end{aligned} \tag{2.19}$$

From (2.18) and (2.19) and the properties of ψ and φ, we get

$$\begin{aligned}&\psi(\widetilde{\mathcal{P}}(x_n, x_{n+1}, t)) \\ &\leq \psi\left(\max\left\{\widetilde{\mathcal{P}}(x_{n-1}, x_n, t), \widetilde{\mathcal{P}}(x_n, x_{n+1}, t)\right\}\right) \\ &\quad - \varphi\left(\max\left\{\widetilde{\mathcal{P}}(x_{n-1}, x_n, t), \widetilde{\mathcal{P}}(x_n, x_{n+1}, t), \frac{1}{2}\Omega^{-1}[\widetilde{\mathcal{P}}(x_{n-1}, x_{n+1}, t)]\right\}\right). \end{aligned} \tag{2.20}$$

If

$$\max\left\{\widetilde{\mathcal{P}}(x_{n-1}, x_n, t), \widetilde{\mathcal{P}}(x_n, x_{n+1}, t)\right\} = \widetilde{\mathcal{P}}(x_n, x_{n+1}, t),$$

then by (2.20) we have

$$\begin{aligned}&\psi(\widetilde{\mathcal{P}}(x_n, x_{n+1}, t)) \\ &\leq \psi(\widetilde{\mathcal{P}}(x_n, x_{n+1}, t)) - \varphi\left(\max\left\{\widetilde{\mathcal{P}}(x_{n-1}, x_n, t), \widetilde{\mathcal{P}}(x_n, x_{n+1}, t), \frac{1}{2}\Omega^{-1}[\widetilde{\mathcal{P}}(x_{n-1}, x_{n+1}, t)]\right\}\right),\end{aligned}$$

which gives that $x_n = x_{n+1}$, a contradiction.

Thus, $\{\widetilde{\mathcal{P}}(x_n, x_{n+1}, t) : n \in \mathbb{N} \cup \{0\}\}$ is a non-increasing sequence of positive numbers. Hence, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} \widetilde{\mathcal{P}}(x_n, x_{n+1}, t) = r.$$

Letting $n \to \infty$ in (2.20), we get

$$\psi(r) \leq \psi(r) - \varphi\left(\max\left\{r, r, \lim_n \frac{1}{2}\Omega^{-1}[\widetilde{\mathcal{P}}(x_{n-1}, x_{n+1}, t)]\right\}\right) \leq \psi(r).$$

Therefore,

$$\varphi\left(\max\left\{r, r, \lim_{n\to\infty} \Omega^{-1}[\widetilde{\mathcal{P}}(x_{n-1}, x_{n+1}, t)]\right\}\right) = 0,$$

and hence,

$$\lim_{n\to\infty} \widetilde{\mathcal{P}}(x_n, x_{n+1}, t) = 0, \tag{2.21}$$

for each $t > 0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence in X.

Suppose the contrary, that is, $\{x_n\}$ is not a Cauchy sequence. Then there exist $t_0 > 0$ and $\varepsilon > 0$ for them we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i, \text{and } \widetilde{\mathcal{P}}(x_{m_i}, x_{n_i}, t_0) \geq \varepsilon. \tag{2.22}$$

This means that

$$\widetilde{\mathcal{P}}(x_{m_i}, x_{n_i-1}, t_0) < \varepsilon. \tag{2.23}$$

The following relations are easy to obtain:

$$\Omega^{-1}[\varepsilon] \leq \limsup_{i\to\infty} \widetilde{\mathcal{P}}(x_{m_i+1}, x_{n_i}, t_0). \tag{2.24}$$

$$\limsup_{i10\to\infty} \widetilde{\mathcal{P}}(x_{m_i+1}, x_{n_i-1}, t_0) \leq \Omega[\varepsilon]. \tag{2.25}$$

$$\limsup_{i\to\infty} \widetilde{\mathcal{P}}(x_{m_i}, x_{n_i}, t_0) \leq \Omega[\varepsilon]. \tag{2.26}$$

From (2.17), we have

$$\begin{aligned} \psi(\Omega[\widetilde{\mathcal{P}}(x_{m_i+1}, x_{n_i}, t_0)]) &= \psi(\Omega[\widetilde{\mathcal{P}}(fx_{m_i}, fx_{n_i-1}, t_0)]) \\ &\leq \psi(M_{t_0}(x_{m_i}, x_{n_i-1})) - \varphi(M_{t_0}(x_{m_i}, x_{n_i-1})), \end{aligned} \tag{2.27}$$

where

$$\begin{aligned} &M_{t_0}(x_{m_i}, x_{n_i-1}) \\ &= \max\Big\{\widetilde{\mathcal{P}}(x_{m_i}, x_{n_i-1}, t_0), \widetilde{\mathcal{P}}(x_{m_i}, fx_{m_i}, t_0), \widetilde{\mathcal{P}}(x_{n_i-1}, fx_{n_i-1}, t_0), \\ &\frac{1}{2}\Omega^{-1}[\widetilde{\mathcal{P}}(x_{m_i}, fx_{n_i-1}, t_0) + \widetilde{\mathcal{P}}(fx_{m_i}, x_{n_i-1}, t_0)]\Big\} \\ &= \max\Big\{\widetilde{\mathcal{P}}(x_{m_i}, x_{n_i-1}, t_0), \widetilde{\mathcal{P}}(x_{m_i}, x_{m_i+1}, t_0), \widetilde{\mathcal{P}}(x_{n_i-1}, x_{n_i}, t_0), \\ &\frac{1}{2}\Omega^{-1}[\widetilde{\mathcal{P}}(x_{m_i}, x_{n_i}, t_0) + \widetilde{\mathcal{P}}(x_{m_i+1}, x_{n_i-1}, t_0)]\Big\}. \end{aligned} \tag{2.28}$$

Taking the upper limit as $i \to \infty$ in (2.28) and using the above mentioned limits, we get

$$\limsup_{i\to\infty} M_{t_0}(x_{m_i-1}, x_{n_i-1}) \leq \varepsilon. \tag{2.29}$$

Now, taking the upper limit as $i \to \infty$ in (2.27) and using (2.22) and (2.29) we have

$$\begin{aligned} \psi\Big(\varepsilon\Big) &\leq \psi(\Omega[\limsup_{i\to\infty} \widetilde{\mathcal{P}}(x_{m_i+1}, x_{n_i}, t_0)]) \\ &\leq \psi(\limsup_{i\to\infty} M_t(x_{m_i}, x_{n_i-1}, t_0)) - \liminf_{i\to\infty} \varphi(M_t(x_{m_i}, x_{n_i-1})) \\ &\leq \psi(\varepsilon) - \varphi(\liminf_{i\to\infty} M_t(x_{m_i}, x_{n_i-1})), \end{aligned}$$

which further implies that

$$\varphi(\liminf_{i\to\infty} M_t(x_{m_i}, x_{n_i-1})) = 0,$$

so $\liminf_{i\to\infty} M_t(x_{m_i}, x_{n_i-1}) = 0$, a contradiction to (2.22). Thus, $\{x_{n+1} = fx_n\}$ is a Cauchy sequence in X.

As X is complete, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$, that is,

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} fx_n = u.$$

Now, suppose that f is continuous. Using the triangular inequality, we get

$$\widetilde{\mathcal{P}}(u, fu, t) \leq \Omega[\widetilde{\mathcal{P}}(u, fx_n, t) + \widetilde{\mathcal{P}}(fx_n, fu, t)].$$

Letting $n \to \infty$, we get

$$\widetilde{\mathcal{P}}(u, fu, t) \leq \Omega[\lim_{n\to\infty} \widetilde{\mathcal{P}}(u, fx_n, t) + \lim_{n\to\infty} \widetilde{\mathcal{P}}(fx_n, fu, t)].$$

So, we have $fu = u$. Thus, u is a fixed point of f. □

Note that the continuity of f in Theorem 2.13 is not necessary and can be dropped.

Theorem 2.14. *Under the hypotheses of Theorem 2.13, without the continuity assumption on f, assume that whenever $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x \in X$, one has $x_n \preceq x$, for all $n \in \mathbb{N}$. Then f has a fixed point in X.*

Proof. Following similar arguments to those given in the proof of Theorem 2.13, we construct an increasing sequence $\{x_n\}$ in X such that $x_n \to u$, for some $u \in X$. Using the assumption on X, we have that $x_n \preceq u$, for all $n \in \mathbb{N}$. By (2.17), we have

$$\psi(\Omega[\widetilde{\mathcal{P}}(x_{n+1}, fu, t)]) \leq \psi(M_t(x_n, u)) - \varphi(M_t(x_n, u)), \tag{2.30}$$

where

$$\begin{aligned} M_t(x_n, u) &= \max\left\{\widetilde{\mathcal{P}}(x_n, u, t), \widetilde{\mathcal{P}}(x_n, fx_n, t), \widetilde{\mathcal{P}}(u, fu, t), \frac{1}{2}\Omega^{-1}[\widetilde{\mathcal{P}}(x_n, fu, t) + \widetilde{\mathcal{P}}(fx_n, u, t)]\right\} \\ &= \max\left\{\widetilde{\mathcal{P}}(x_n, u, t), \widetilde{\mathcal{P}}(x_n, x_{n+1}, t), \widetilde{\mathcal{P}}(u, fu, t), \frac{1}{2}\Omega^{-1}[\widetilde{\mathcal{P}}(x_n, fu, t) + \widetilde{\mathcal{P}}(x_{n+1}, u, t)]\right\}. \end{aligned} \tag{2.31}$$

Letting $n \to \infty$ in (2.31) and using Lemma 1.9, we get

$$\frac{\Omega^{-2}[\widetilde{\mathcal{P}}(u, fu, t)]}{2} \leq \liminf_{n\to\infty} M_t(x_n, u) \leq \limsup_{n\to\infty} M_t(x_n, u) \leq \widetilde{\mathcal{P}}(u, fu, t). \tag{2.32}$$

Again, taking the upper limit as $i \to \infty$ in (2.30) and using Lemma 1.9 and (2.32) we get

$$\begin{aligned} \psi(\widetilde{\mathcal{P}}(u, fu, t) &\leq \psi(\Omega[\limsup_{n\to\infty} \widetilde{\mathcal{P}}(x_{n+1}, fu, t)]) \\ &\leq \psi(\limsup_{n\to\infty} M_t(x_n, u)) - \liminf_{n\to\infty} \varphi(M_t(x_n, u)) \\ &\leq \psi(\widetilde{\mathcal{P}}(u, fu, t)) - \varphi(\liminf_{n\to\infty} M_t(x_n, u)). \end{aligned}$$

Therefore, $\varphi(\liminf_{n\to\infty} M_t(x_n, u)) \leq 0$, equivalently, $\liminf_{n\to\infty} M_t(x_n, u) = 0$. Thus, from (2.32) we get $u = fu$ and hence u is a fixed point of f. □

Example 2.15. *Let* $X = [0, \frac{7}{10}]$ *and* $\widetilde{\mathcal{P}}$ *on* X *be given by*

$$\widetilde{\mathcal{P}}(x, y, t) = \sinh[t\,|x - y|],$$

for all $x, y \in X$. *We define an ordering "* $\preceq$*"on* X *as follows:*

$$x \preceq y \Longleftrightarrow y \leq x, \quad \forall x, y \in X.$$

Define self-map f *on* X *by*

$$fx = \sinh^{-1} \tfrac{x}{15}.$$

Furthermore,

$$fX = [0, 0.65266656608236] \subseteq X,$$

.

Define $\psi, \varphi : [0, \infty) \to [1, \infty)$ *as* $\psi(t) = \frac{2\cosh t}{1+\cosh t} - 1$ *and* $\varphi(t) = [\frac{2\cosh t}{1+\cosh t}]^2 - 1$. *Using the mean value theorem for the function* sinh *we have,*

$$\begin{aligned}
\psi\Big(\Omega\Big(\widetilde{\mathcal{P}}(fx, fyt)\Big)\Big) &= \frac{2\cosh\Big(\Omega\Big(\sinh\Big(t|fx - fy|\Big)\Big)\Big)}{1 + \cosh\Big(\Omega\Big(t|fx - fy|\Big)\Big)} - 1 \\
&\leq \frac{2\cosh(\Omega(\sinh(t|\sinh^{-1}\frac{x}{15} - \sinh^{-1}\dfrac{y}{15}|)))}{1 + \cosh(\Omega(\sinh(t|\sinh^{-1}\frac{x}{15} - \sinh^{-1}\dfrac{y}{15}|)))} - 1 \\
&\leq \frac{2\cosh\Big(\Omega\Big(\Big(\frac{1}{15}t|x - y|\Big)\Big)\Big)}{1 + \cosh\Big(\Omega\Big(\Big(\frac{1}{15}t|x - y|\Big)\Big)\Big)} - 1 \\
&\leq \frac{2\cosh\Big(\Big(\sinh\Big(\frac{(t|x-y|)}{15}\Big)\Big)\Big)}{1 + \cosh\Big(\Big(\sinh\Big(\frac{(t|x-y|)}{15}\Big)\Big)\Big)} - 1 \\
&= \frac{2\cosh\Big(\Big(\frac{1}{15}\sinh\Big((t|x - y|)\Big)\Big)\Big)}{1 + \cosh\Big(\Big(\frac{1}{15}\sinh\Big((t|x - y|)\Big)\Big)\Big)} - 1 \\
&\leq [\frac{2\cosh[\sinh(t|x - y|)]}{1 + 2\cosh[\sinh(t|x - y|)]}]^4 - 1 \\
&\leq \frac{2\cosh[\widetilde{\mathcal{P}}(x, y, t)]}{1 + \cosh[\widetilde{\mathcal{P}}(x, y, t)]} - 1 - [\frac{2\cosh[\sinh(t|x - y|)]}{1 + 2\cosh[\sinh(t|x - y|)]}]^2 - 1 \\
&= \psi(M_t(x, y)) - \varphi(M_t(x, y)).
\end{aligned}$$

Thus, (2.1) is satisfied for all $x, y \in X$. *Therefore, all the conditions of Theorem 2.13 are satisfied. Moreover,* 0 *is a fixed point of* f. □

3. Fuzzy *b*-metric Spaces

In [5], Hussain et al. presented the relationship between parametric *b*-metrics and fuzzy *b*-metrics and deduce certain fixed point results in triangular partially ordered fuzzy *b*-metric space.

Definition 3.1. *(Schweizer and Sklar [14]) A binary operation* $\star : [0,1] \times [0,1] \to [0,1]$ *is called a continuous t-norm if it satisfies the following assertions:*

(T1) $\star$ *is commutative and associative;*
(T2) $\star$ *is continuous;*
(T3) $a \star 1 = a$ *for all* $a \in [0,1]$;
(T4) $a \star b \leq c \star d$ *when* $a \leq c$ *and* $b \leq d$, *with* $a, b, c, d \in [0,1]$.

Definition 3.2. *A 3-tuple* $(X, M, *)$ *is said to be a fuzzy metric space if* X *is an arbitrary set,* $*$ *is a continuous t-norm and* M *is a fuzzy set on* $X^2 \times (0,\infty)$ *satisfying the following conditions, for all* $x, y, z \in X$ *and for all* $t, s > 0$,

(i) $M(x, y, t) > 0$;
(ii) $M(x, y, t) = 1$ *for all* $t > 0$ *if and only if* $x = y$;
(iii) $M(x, y, t) = M(y, x, t)$;
(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(v) $M(x, y, .) : (0,\infty) \to [0,1]$ *is continuous;*

The function $M(x, y, t)$ *denotes the degree of nearness between* x *and* y *with respect to* t.

Definition 3.3. [5] *A fuzzy b-metric space is an ordered triple* $(X, B, \star)$ *such that* X *is a nonempty set,* $\star$ *is a continuous t-norm and* B *is a fuzzy set on* $X \times X \times (0, +\infty)$ *satisfying the following conditions, for all* $x, y, z \in X$ *and for all* $t, s > 0$:

(F1) $B(x, y, t) > 0$;
(F2) $B(x, y, t) = 1$ *if and only if* $x = y$;
(F3) $B(x, y, t) = B(y, x, t)$;
(F4) $B(x, y, t) \star B(y, z, s) \leq B(x, z, b(t + s))$ *where* $b \geq 1$;
(F5) $B(x, y, \cdot) : (0, +\infty) \to (0, 1]$ *is left continuous.*

Definition 3.4. *An extended fuzzy b-metric space is an ordered quadruple* $(X, B, \star, \Omega)$ *such that* X *is a nonempty set,* $\star$ *is a continuous t-norm and* B *is a fuzzy set on* $X \times X \times (0, +\infty)$ *satisfying the following conditions, for all* $x, y, z \in X$ *and for all* $t, s > 0$:

(F1) $B(x, y, t) > 0$;
(F2) $B(x, y, t) = 1$ *if and only if* $x = y$;
(F3) $B(x, y, t) = B(y, x, t)$;
(F4) $B(x, y, t) \star B(y, z, s) \leq B(x, z, \Omega(t + s))$;
(F5) $B(x, y, \cdot) : (0, +\infty) \to (0, 1]$ *is left continuous.*

Definition 3.5. *Let* $(X, B, \star, \Omega)$ *be an extended fuzzy b-metric space. Then*

(i) *a sequence* $\{x_n\}$ *converges to* $x \in X$, *if and only if* $\lim_{n\to+\infty} B(x_n, x, t) = 1$ *for all* $t > 0$;
(ii) *a sequence* $\{x_n\}$ *in* X *is a Cauchy sequence if and only if for all* $\epsilon \in (0, 1)$ *and for all* $t > 0$, *there exists* n_0 *such that* $B(x_n, x_m, t) > 1 - \epsilon$ *for all* $m, n \geq n_0$;
(iii) *the extended fuzzy b-metric space is called complete if every Cauchy sequence converges to some* $x \in X$.

Definition 3.6. *Let* $(X, B, *, \Omega)$ *be an extended fuzzy b-metric space. The extended fuzzy b-metric* B *is called triangular whenever,*

$$\frac{1}{B(x,y,t)} - 1 \leq \Omega[\frac{1}{B(x,z,t)} - 1 + \frac{1}{B(z,y,t)} - 1)]$$

for all $x, y, z \in X$ and for all $t > 0$.

Motivated by Example 8 of [10] we present the following example.

Example 3.7. *Let (X, d, s) be a b-metric space. Define $B : X \times X \times (0, \infty) \to [0, \infty)$ by $B(x, y, t) = \frac{s\Omega^{-1}[t]}{s\Omega^{-1}[t] + \mathcal{P}(x,y)}$ where $\Omega : [0, \infty) \to [0, \infty)$ is a strictly increasing continuous function with $\Omega^{-1}(t) \leq t \leq \Omega(t)$. Also suppose that $a * b = \min\{a, b\}$. Then $(X, B, *, \Omega)$ is an extended fuzzy b-metric space.*

Remark. *Notice that $\widetilde{\mathcal{P}}(x, y, t) = \frac{1}{B(x,y,t)} - 1$ is an extended parametric b-metric whenever B is a triangular extended fuzzy b-metric.*

As an applications of Remark 3 and the results established in section 2, we can deduce the following results in ordered extended fuzzy b-metric spaces.

Theorem 3.8. *Let $(X, \preceq, B, *, \Omega)$ be an ordered triangular complete extended fuzzy b-metric space. Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that*

$$\Omega^2[\frac{1}{B(fx, fy, t))} - 1] \leq \beta(\frac{1}{B(x, y, t)} - 1)\mathcal{M}(x, y, t) \tag{3.1}$$

for all $t > 0$ and for all comparable elements $x, y \in X$, where

$$\mathcal{M}(x, y, t) = \max\left\{\frac{1}{B(x, y, t)} - 1, \frac{[\frac{1}{B(x,fx,t)} - 1][\frac{1}{B(y,fy,t)} - 1]}{\frac{1}{B(fx,fy,t)} \cdot \frac{1}{B(x,y,t)}}\right\}$$

If f is continuous, then f has a fixed point.

Theorem 3.9. *Under the hypotheses of Theorem 3.8, without the continuity assumption on f, assume that whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to u$, one has $x_n \preceq u$ for all $n \in \mathbb{N}$. Then f has a fixed point.*

Theorem 3.10. *Let $(X, \preceq, B, *, \Omega)$ be an ordered triangular complete extended fuzzy b-metric space (where Ω is super-additive). Let $f : X \to X$ be a continuous non-decreasing mapping with respect to $\preceq$. Also suppose that there exist two altering distance functions ψ and φ such that*

$$\psi(\Omega[\frac{1}{B(fx, fy, t))} - 1]) \leq \psi(\mathcal{M}_t(x, y)) - \varphi(\mathcal{M}_t(x, y))$$

for all comparable elements $x, y \in X$ where,

$$\mathcal{M}_t(x, y) = \max\left\{\frac{1}{B(x, y, t)} - 1, \frac{1}{B(x, fx, t)} - 1, \frac{1}{B(y, fy, t)} - 1, \frac{1}{2}\Omega^{-1}[\frac{1}{B(x, fy, t)} + \frac{1}{B(y, fx, t)} - 2]\right\}$$

If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Theorem 3.11. *Under the hypotheses of Theorem 3.10, without the continuity assumption on f, assume that whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to u \in X$, one has $x_n \preceq u$ for all $n \in \mathbb{N}$. Then f has a fixed point.*

Theorem 3.12. *Let $(X, \preceq, B, *, \Omega)$ be an ordered triangular complete extended fuzzy b-metric space. Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that*

$$\Omega[\frac{1}{B(fx, fy, t))} - 1] \leq \psi(\mathcal{N}(x, y, t)) \tag{3.2}$$

where

$$\mathcal{N}(x,y,t)$$
$$= \max\left\{\frac{1}{B(x,y,t))}-1, \frac{[[\frac{1}{B(x,fx,t)}-1][\frac{1}{B(x,fy,t)}-1]+[\frac{1}{B(y,fy,t)}-1][\frac{1}{B(y,fx,t)}-1]]^2}{[1+\Omega[\frac{1}{B(x,fx,t)}+\frac{1}{B(y,fy,t)}-2]][\frac{1}{B(x,fy,t)}+\frac{1}{B(y,fx,t)}-1]}\right\},$$

for some $\psi \in \Psi$ and for all comparable elements $x, y \in X$ and all $t > 0$. If f is continuous, then f has a fixed point.

4. Application to existence of solutions of integral equations

Let $X = C([0,T],\mathbb{R})$ be the set of real continuous functions defined on $[0,T]$ and $\widetilde{\mathcal{P}} : X \times X \times (0,\infty) \to [0,+\infty)$ be defined by $\widetilde{\mathcal{P}}(x,y,\alpha) = \xi[\sup_{t\in[0,T]} e^{-\alpha t}|x(t) - y(t)|^p]$ for all $x, y \in X$ and all $t > 0$. Then $(X, \widetilde{\mathcal{P}}, \xi(2^{p-1}t))$ is a complete extended parametric b-metric space. Let $\preceq$ be the partial order on X defined by $x \preceq y$ if and only if $x(t) \le y(t)$ for all $t \in [0,T]$. Then $(X, \widetilde{\mathcal{P}}, \preceq)$ is a complete partially ordered metric space. Consider the following integral equation

$$x(t) = p(t) + \int_0^T S(t,s)f(s,x(s))ds \tag{4.1}$$

where

(A) $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous,
(B) $p : [0,T] \to \mathbb{R}$ is continuous,
(C) $S : [0,T] \times [0,T] \to [0,+\infty)$ is continuous and

$$2^{p-1} \sup_{t\in[0,T]} (e^{-\alpha t}[\int_0^T [S(t,s)]^q ds]^{\frac{1}{q}}) \le 1,$$

(D) there exist $k \in [0,1)$ such that

$$0 \le \xi[[f(s,y(s)) - f(s,x(s))]]$$
$$\le k(e^{-\alpha s}\max\left\{|x(s)-y(s)|^p, |x(s)-Hx(s)|^p, |y(s)-Hy(s)|^p,\right.$$
$$\left. 2^{1-p}\xi^{-1}[\frac{|x(s)-Hy(s)|^p + |y(s)-Hx(s)|^p}{2}]\right\}$$

for all $x, y \in X$ with $x \preceq y$, $s \in [0,T]$ and $\alpha > 0$ where

$$Hx(t) = p(t) + \int_0^T S(t,s)f(s,x(s))ds, \quad t \in [0,T], \quad \text{for all} \quad x \in X.$$

(E) there exist $x_0 \in X$ such that

$$x_0(t) \le p(t) + \int_0^T S(t,s)f(s,x_0(s))ds.$$

We have the following result of existence of solutions for integral equations.

Theorem 4.1. *Under assumptions $(A)-(E)$, the integral equation (4.1) has a unique solution in $X = C([0,T],R)$.*

Proof. Let $H : X \to X$ be defined by

$$Hx(t) = p(t) + \int_0^T S(t,s)f(s,x(s))ds, \quad t \in [0,T], \quad \text{for all} \quad x \in X.$$

First, we will prove that H is a non-decreasing mapping with respect to $\preceq$. Let $x \preceq y$ then by (D) we have $0 \leq f(s,y(s)) - f(s,x(s))$ for all $s \in [0,T]$. On the other hand by definition of H we have

$$Hy - Hx = \int_0^T S(t,s)[f(s,y(s)) - f(s,x(s))]ds \geq 0 \quad \text{for all} \quad t \in [0,T].$$

Then $Hx \preceq Hy$, that is, H is a non-decreasing mapping with respect to $\preceq$. Now suppose that $x, y \in X$ with $x \preceq y$. Then by (C), (D) and the definition of H we get

$$\xi[2^{p-1}\widetilde{\mathcal{P}}(Hx,Hy,\alpha)] = \xi[2^{p-1} \sup_{t\in[0,T]} \xi[e^{-\alpha t}|Hx(t) - Hy(t)|^p]]$$

$$\leq \xi[2^{p-1} \sup_{t\in[0,T]} \xi[(e^{-\alpha t} \int_0^T S(t,s)|f(s,x(s)) - f(s,y(s))|ds)^p]]$$

$$\leq \xi[2^{p-1} \sup_{t\in[0,T]} \xi[e^{-p\alpha t}([\int_0^T [S(t,s)]^q ds]^{\frac{1}{q}} [\int_0^T |f(s,x(s)) - f(s,y(s))|^p ds]^{\frac{1}{p}})^p]]$$

$$\leq k(e^{-\alpha s} \max\Big\{|x(s) - y(s)|^p, |x(s) - Hx(s)|^p, |y(s) - Hy(s)|^p,$$

$$2^{1-p}\xi^{-1}[\frac{|x(s) - Hy(s)|^p + |y(s) - Hx(s)|^p}{2}]\Big\})$$

$$\leq k(e^{-\alpha s} \max\Big\{\sup_{s\in[0,T]} |x(s) - y(s)|^p, \sup_{s\in[0,T]} |x(s) - Hx(s)|^p, \sup_{s\in[0,T]} |y(s) - Hy(s)|^p,$$

$$2^{1-p}\xi^{-1}[\frac{\sup_{s\in[0,T]} |x(s) - Hy(s)|^p + \sup_{s\in[0,T]} |y(s) - Hx(s)|^p}{2}]\Big\})$$

$$\leq k[\max\Big\{\widetilde{\mathcal{P}}(x,y,\alpha), \widetilde{\mathcal{P}}(x,Hx,\alpha), \widetilde{\mathcal{P}}(y,Hy,\alpha), 2^{1-p}\xi^{-1}[\frac{\widetilde{\mathcal{P}}(x,Hy,\alpha) + \widetilde{\mathcal{P}}(y,Hx,\alpha)}{2}]\Big\}]$$

Now, by (E) there exists $x_0 \in X$ such that $x_0 \preceq Hx_0$. Then, the conditions of Theorem 2.13 are satisfied and hence the integral equation (4.1) has a unique solution in $X = C([0,T],\mathbb{R})$. □

Acknowledgments. This article was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The authors acknowledge with thanks DSR, KAU for financial support.

References

[1] I. Altun and D. Turkoglu, *Some fixed point theorems on fuzzy metric spaces with implicit relations*, Commun. Korean Math. Soc., **23(1)** (2008) 111–124.

[2] T. P. Dence, *A brief look into Lambert W function*, Applied Math., **4** (2013) 887–892.

[3] D. Đukić, Z. Kadelburg and S. Radenović, *Fixed points of Geraghty-type mappings in various generalized metric spaces*, Abstr. Appl. Anal., **2011**, Article ID 561245, 13 pages.

[4] M. Geraghty, *On contractive mappings*, Proc. Amer. Math. Soc. **40** (1973) 604–608.

[5] N Hussain, P Salimi and V Parvaneh, *Fixed point results for various contractions in parametric and fuzzy b-metric spaces*, J. Nonlinear Sci. Appl., **8(5)** (2015) 719–739.

[6] N. Hussain, S. Khaleghizadeh, P. Salimi and A. A. N. Abdou, *A new approach to fixed point results in triangular intuitionistic fuzzy metric spaces*, Abstr. Appl. Anal., 2014 (2014) 16 pages.
[7] N. Hussain and P. Salimi, *Implicit contractive mappings in Modular metric and Fuzzy Metric Spaces, The Scientific World Journal*, Volume 2014, Article ID 981578, 13 pp.
[8] N. Hussaina and M.H. Shah, *KKM mappings in cone b-metric spaces*, Computers and Mathematics with Applications, **62** (2011) 1677–1684.
[9] M. S. Khan, M. Swaleh and S. Sessa, *Fixed point theorems by altering distancces between the points*, Bull. Aust. Math. Soc., **30** (1984) 1–9.
[10] S. Nădăban, *Fuzzy b-Metric Spaces*, International Journal of Computers Communications and Control, **11(2)** (2016) 273–281.
[11] V. Parvaneh, A. Dinmohammadi and Z. Kadelburg, *Coincidence point results for weakly α-admissible pairs in extended b-metric spaces*, J. Math. Analysis, **8(6)** (2017) 74–89.
[12] V. Parvaneh and S.J. Hosseini Ghoincheh, *Fixed points of $(\psi, \varphi)_\Omega$-contractive mappings in ordered p-metric spaces*, Accepted in GADM.
[13] J.R. Roshan, N. Hussain, S. Sedghi and N. Shobkolaei, *Suzuki-type fixed point results in b-metric spaces*, Math. Sci., **9** (2015) 153–160.
[14] B. Schweizer and A. Sklar, *Statistical metric spaces*, Pacific Journal of Mathematics, **10(1)** (1960) 314–334.
[15] W. Shatanawi, *Fixed and Common Fixed Point for Mapping Satisfying Some Nonlinear Contraction in b-metric Spaces*, Journal of Mathematical Analysis, **7(4)** (2016) 1–12.
[16] W. Shatanawi, A. Pitea and R. Lazović, *Contraction conditions using comparison functions on b-metric spaces*, Fixed Point Theory and Applications, 2014, 2014:135 doi:10.1186/1687-1812-2014-135.

V. Parvaneh*
Department of Mathematics, Gilan-E-Gharb Branch, Islamic Azad University, Gilan-E-Gharb, Iran
E-mail address: zam.dalahoo@gmail.com

N. Hussain
Department of Mathematics, King Abdulaziz University P.O. Box 80203, Jeddah 21589, Saudi Arabia
E-mail address: nhusain@kau.edu.sa

M.A. Kutbi
Department of Mathematics, King Abdulaziz University P.O. Box 80203, Jeddah 21589, Saudi Arabia
E-mail address: mkutbi@kau.edu.sa

M. Khorshidi
Department of Mathematics, Firouzabad Institute of Higher Education, Firouzabad, Fars, Iran.
E-mail address: maryam_khorshidii@yahoo.com