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NEW FIXED POINT RESULTS IN MODULAR METRIC AND FUZZY METRIC SPACES

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Abstract. The notion of modular metric spaces being a natural generalization of classical modulars over linear spaces like Lebesgue, Orlicz, Musielak-Orlicz, Lorentz, Orlicz-Lorentz, and Calderon-Lozanovskii spaces was recently introduced. In this paper, in the setting of modular metric spaces, we introduce the class of (JS)-ω-contractions and establish certain fixed point results. As application of our results, we deduce some Suzuki type theorems in modular metric space. Moreover, we introduce some notions of continuity in the setting of fuzzy metric spaces and obtain some results of fixed point for self-mappings defined on a fuzzy metric space as consequence of those given for modular metric space. An example is furnished to demonstrate the validity of the obtained results.

1. Introduction and Preliminaries

Modular metric spaces were introduced in [4, 5]. The introduction of this new concept is justified by the physical interpretation of the modular. A metric on a set represents nonnegative finite distances between any two points of the set, whereas a modular on a set attributes a nonnegative (possibly, infinite valued) "field of (generalized) velocities": to each "time" $\lambda > 0$ (the absolute value of) an average velocity $\omega_\lambda(x, y)$ is associated in such a way that in order to cover the "distance" between points $x, y \in X$ it takes time λ to move from x to y with velocity $\omega_\lambda(x, y)$. But in this paper, we look at these spaces as the nonlinear version of the classical modular spaces introduced by Nakano [23] on vector spaces and modular function spaces introduced by Musielak [22] and Orlicz [24].

We remark that the usual approach in dealing with the Dirichlet energy problem [10, 17] is to convert the energy functional, naturally defined by a modular, to a convoluted and complicated problem which involves the Luxemburg norm.

Recently, there arise a strong interest to study the existence of fixed points and their applications in the setting of modular function spaces after the first paper [20] was published in 1990, see also [2, 11, 13, 14, 16, 18, 21].

In this paper, in the setting of modular metric spaces, we introduce the class of $(\alpha, \eta, \Theta)-\omega$-contraction and we establish certain fixed point results. As application

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of our results, we deduce some Suzuki type theorems. Moreover, we introduce some notions of continuity in the setting of fuzzy metric spaces and obtain some results of fixed point for self-mappings defined on a fuzzy metric spaces as consequence of those given for modular metric spaces. An example is furnished to demonstrate the validity of the obtained results.

Let X be a nonempty set and $\omega : (0, +\infty) \times X \times X \to [0, +\infty]$ be a function, for semplicity, we will write

$$\omega_\lambda(x, y) = \omega(\lambda, x, y),$$

for all $\lambda > 0$ and $x, y \in X$.

Definition 1.1. [4, 5] *A function $\omega : (0, +\infty) \times X \times X \to [0, +\infty]$ is called a modular metric on X if the following axioms hold:*

(i) *$x = y$ if and only if $\omega_\lambda(x, y) = 0$ for all $\lambda > 0$;*
(ii) *$\omega_\lambda(x, y) = \omega_\lambda(y, x)$ for all $\lambda > 0$ and $x, y \in X$;*
(iii) *$\omega_{\lambda+\mu}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$.*

If in the Definition 1.1, we use the condition

(i') $\omega_\lambda(x, x) = 0$ for all $\lambda > 0$ and $x \in X$;

instead of (i) then ω is said to be a pseudomodular metric on X. A modular metric ω on X is called regular if the following weaker version of (i) is satisfied

$$x = y \quad \text{if and only if} \quad \omega_\lambda(x, y) = 0 \quad \text{for some} \quad \lambda > 0.$$

Again, ω is called convex if for $\lambda, \mu > 0$ and $x, y, z \in X$ holds the inequality

$$\omega_{\lambda+\mu}(x, y) \leq \frac{\lambda}{\lambda + \mu}\omega_\lambda(x, z) + \frac{\mu}{\lambda + \mu}\omega_\mu(z, y).$$

Remark 1. *If ω is a pseudomodular metric on a set X, then the function $\lambda \to \omega_\lambda(x, y)$ is nonincreasing on $(0, +\infty)$ for all $x, y \in X$. Indeed, if $0 < \mu < \lambda$, then*

$$\omega_\lambda(x, y) \leq \omega_{\lambda-\mu}(x, x) + \omega_\mu(x, y) = \omega_\mu(x, y).$$

Definition 1.2. [4, 5] *Let ω be a pseudomodular on X and $x_0 \in X$ fixed. Consider the two sets*

$$X_\omega = X_\omega(x_0) = \{x \in X : \omega_\lambda(x, x_0) \to 0 \quad as \quad \lambda \to +\infty\}$$

and

$$X^*_\omega = X^*_\omega(x_0) = \{x \in X : \exists\lambda = \lambda(x) > 0 \quad such\ that \quad \omega_\lambda(x, x_0) < +\infty\}.$$

X_ω and X^*_ω are called modular spaces (around x_0).

It is clear that $X_\omega \subset X^*_\omega$ but this inclusion may be proper in general. Let ω be a modular on X, from [4, 5], we deduce that the modular space X_ω which can be equipped with a (nontrivial) metric, induced by ω and is defined by

$$d_\omega(x, y) = \inf\{\lambda > 0 : \omega_\lambda(x, y) \leq \lambda\} \quad \text{for all} \quad x, y \in X_\omega.$$

If ω is a convex modular on X, according to [4, 5] the two modular spaces coincide, that is, $X^*_\omega = X_\omega$, and this common set can be endowed with the metric d^*_ω defined by

$$d^*_\omega(x, y) = \inf\{\lambda > 0 : \omega_\lambda(x, y) \leq 1\} \quad \text{for all} \quad x, y \in X_\omega.$$

These distances will be called Luxemburg distances.

Definition 1.3. *Let X_ω be a modular metric space, M a subset of X_ω and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X_ω. Then*

(1) *$(x_n)_{n\in\mathbb{N}}$ is called ω-convergent to $x \in X_\omega$ if and only if $\omega_1(x_n, x) \to 0$, as $n \to +\infty$. x will be called the ω-limit of (x_n).*
(2) *$(x_n)_{n\in N}$ is called ω-Cauchy if $\omega_1(x_m, x_n) \to 0$, as $m, n \to +\infty$.*
(3) *M is called ω-closed if the ω-limit of a ω-convergent sequence of M always belong to M.*
(4) *M is called ω-complete if any ω-Cauchy sequence in M is ω-convergent to a point of M.*
(5) *M is called ω-bounded if we have $\delta_\omega(M) = \sup\{\omega_1(x, y); x, y \in M\} < +\infty$.*

Definition 1.4. [26] *Let T be a self-mapping on X and let $\alpha : X \times X \to [0, +\infty)$ be a function. We say that T is an α-admissible mapping if*

$$x, y \in X, \quad \alpha(x, y) \geq 1 \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq 1.$$

Definition 1.5. [25] *Let T be a self-mapping on X and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. We say that T is an α-admissible mapping with respect to η if*

$$x, y \in X, \quad \alpha(x, y) \geq \eta(x, y) \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq \eta(Tx, Ty).$$

Definition 1.6. [15] *Let (X, d) be a metric space. Let $\alpha, \eta : X \times X \to [0, +\infty)$ and $T : X \to X$ be functions. We say that T is an α-η-continuous mapping on (X, d) if, for given $x \in X$ and sequence $\{x_n\}$ with $x_n \to x$ as $n \to +\infty$*

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \quad \text{for all} \, n \in \mathbb{N} \Longrightarrow \; Tx_n \to Tx.$$

Definition 1.7. *Let X_ω be a modular metric space. Let $\alpha, \eta : X_\omega \times X_\omega \to [0, +\infty)$ be two functions and let $T : X_\omega \to X_\omega$ be a mapping. We say that T is an $\alpha\eta - \omega$-continuous mapping on X_ω, if, for given $x \in X_\omega$ and sequence $\{x_n\}$ with $\omega_1(x_n, x) \to 0$ as $n \to +\infty$*

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \quad \text{for all} \, n \in \mathbb{N} \Longrightarrow \; \omega_1(Tx_n, Tx) \to 0.$$

If $\eta(x, y) = 1$ for all $x, y \in X_\omega$, then T is called $\alpha - \omega$-continuous.

Example 1.8. Let $X = [0, +\infty)$ and $\omega_\lambda(x, y) = \frac{1}{\lambda}|x - y|$ be a modular metric on X_ω. Assume that $T : X \to X$ and $\alpha, \eta : X \times X \to [0, +\infty)$ are defined by

$$Tx = \begin{cases} x^5, & \text{if } x \in [0,1] \\ \\ 10, & \text{if } (1, +\infty) \end{cases}, \quad \alpha(x, y) = \begin{cases} x^2 + y^2 + 1, & \text{if } x, y \in [0, 1] \\ \\ 0, & \text{otherwise} \end{cases}$$

and $\eta(x, y) = x^2$. Then T is an $\alpha\eta - \omega$-continuous mapping.

2. Main Results

Jleli and Samet [9], defined the class, Δ_Θ of all functions $\Theta : (0, +\infty) \to (1, +\infty)$ satisfying the following conditions:

(Θ_1) Θ is increasing;
(Θ_2) for all sequence $\{\alpha_n\} \subseteq (0, +\infty)$, $\lim\limits_{n\to+\infty} \alpha_n = 0$ if and only if $\lim\limits_{n\to+\infty} \Theta(\alpha_n) = 1$;
(Θ_3) there exist $0 < r < 1$ and $\ell \in (0, +\infty]$ such that $\lim\limits_{t\to0^+} \frac{\Theta(t)-1}{t^r} = \ell$.

Definition 2.1. *Let X_ω be a complete modular metric space and T be a self-mapping on X_ω. Also suppose that $\alpha, \eta : X_\omega \times X_\omega \to [0, +\infty)$ are two functions. We say that T is an (JS)-ω-contraction if for all $x, y \in X_\omega$ with $\eta(x, Tx) \leq \alpha(x, y)$, we have*

$$\Theta\big(\omega_\lambda(Tx, Ty)\big) \leq \Big[\Theta\big(\omega_\lambda(x, y)\big)\Big]^k \tag{2.1}$$

for all $\lambda > 0$ such that $\omega_\lambda(Tx, Ty) > 0$, where $0 \leq k < 1$ and $\Theta \in \Delta_\Theta$.

Recall that a self-mapping T is said to have the property P if $Fix(T^n) = Fix(T)$ for every $n \in \mathbb{N}$. Now we state and prove our main result of this section.

Theorem 2.2. *Let X_ω be a complete modular metric space with ω regular and let $T : X_\omega \to X_\omega$ be a self-mapping. Assume that there exist two functions $\alpha, \eta : X_\omega \times X_\omega \to [0, +\infty)$ and a function $\Theta \in \Delta_\Theta$ such that the following assertions hold:*

- (i) *T is an α-admissible mapping with respect to η;*
- (ii) *T is an (JS)-ω-contraction;*
- (iii) *there exists $x_0 \in X_\omega$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;*
- (iv) *T is $\alpha\eta - \omega$-continuous mapping.*

Then T has a fixed point. Moreover, T has the property P if for all $n \in \mathbb{N}$ and for all $w \in Fix(T^n)$, we have $\alpha(T^j w, T^{j+1} w) \geq \eta(T^j w, T^{j+1} w)$ for each $j \in \{1, \dots, n-1\}$. Further, $Fix(T^n) = Fix(T) = \{x^*\}$ for all $n \in \mathbb{N}$, when $\alpha(x, y) \geq \eta(x, x)$ for all $x, y \in Fix(T)$.

Proof. Let $x_0 \in X_\omega$ be such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$. Let $\{x_n\}$ be a Picard sequence starting at x_0, that is, $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in \mathbb{N}$. Since T is an α-admissible mapping with respect to η, then $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0) = \eta(x_0, x_1)$. Continuing this process, we deduce

$$\eta(x_{n-1}, Tx_{n-1}) = \eta(x_{n-1}, x_n) \leq \alpha(x_{n-1}, x_n) \quad \text{for all} \quad n \in \mathbb{N}.$$

If there exists $n_0 \in \mathbb{N} \cup \{0\}$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of T and we have nothing to prove. Hence, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Now, ω regular implies $\omega_\lambda(Tx_{n-1}, Tx_n) > 0$ for all $n \in \mathbb{N}$ and $\lambda > 0$. Since T is an $(\alpha, \eta, \Theta) - \omega$-contraction, we derive

$$\Theta\big(\omega_\lambda(Tx_{n-1}, Tx_n)\big) \leq [\Theta\big(\omega_\lambda(x_{n-1}, x_n)\big)]^k$$

which implies that

$$\Theta\big(\omega_\lambda(x_n, x_{n+1})\big) \leq [\Theta\big(\omega_\lambda(x_{n-1}, x_n)\big)]^k. \tag{2.2}$$

Therefore,

$$\begin{aligned} 1 \quad &< \Theta\big(\omega_\lambda(x_n, x_{n+1})\big) \leq [\Theta\big(\omega_\lambda(x_{n-1}, x_n)\big)]^k \\ &\leq [\Theta\big(\omega_\lambda(x_{n-2}, x_{n-1})\big)]^{k^2} \leq \cdots \leq [\Theta(\omega_\lambda(x_0, x_1))]^{k^n}. \end{aligned} \tag{2.3}$$

Taking the limit as $n \to +\infty$ in (2.3), we get

$$\lim_{n \to +\infty} \Theta\big(\omega_\lambda(x_n, x_{n+1})\big) = 1 \text{ for all } \lambda > 0$$

and since $\Theta \in \Delta_\Theta$, we obtain

$$\lim_{n \to +\infty} \omega_\lambda(x_n, x_{n+1}) = 0 \text{ for all } \lambda > 0. \tag{2.4}$$

Thus there exist $0 < r < 1$ and $0 < \ell \leq +\infty$ such that

$$\lim_{n\to+\infty} \frac{\Theta\big(\omega_\lambda(x_n, x_{n+1})\big) - 1}{[\omega_\lambda(x_n, x_{n+1})]^r} = \ell. \tag{2.5}$$

Now, let $B^{-1} \in (0, \ell)$. From the definition of limit, there exists $n_\lambda \in \mathbb{N}$ such that

$$\frac{\Theta\big(\omega_\lambda(x_n, x_{n+1})\big) - 1}{[\omega_\lambda(x_n, x_{n+1})]^r} \geq B^{-1} \quad \text{for all} \quad n \geq n_\lambda$$

and so

$$n[\omega_\lambda(x_n, x_{n+1})]^r \leq nB[\Theta\big(\omega_\lambda(x_n, x_{n+1})\big) - 1] \quad \text{for all} \quad n \geq n_\lambda.$$

From (2.3), we deduce

$$n[\omega_\lambda(x_n, x_{n+1})]^r \leq nB[(\Theta(\omega_\lambda(x_0, x_1)))^{k^n} - 1] \quad \text{for all} \quad n \geq n_\lambda.$$

Taking the limit as $n \to +\infty$ in the above inequality, we have

$$\lim_{n\to+\infty} n[\omega_\lambda(x_n, x_{n+1})]^r = 0 \text{ for all } \lambda > 0. \tag{2.6}$$

From (2.6), it follows that for all $\lambda > 0$ there exists $N_\lambda \in \mathbb{N}$ such that

$$n[\omega_\lambda(x_n, x_{n+1})]^r \leq 1 \quad \text{for all} \quad n \geq N_\lambda.$$

Thus

$$\omega_\lambda(x_n, x_{n+1}) \leq \frac{1}{n^{1/r}} \quad \text{for all} \quad n \geq N_\lambda,\ \lambda > 0. \tag{2.7}$$

Now, for $\lambda = \frac{1}{m-n}$ and $m > n \geq N_\lambda$, by (2.7), we get

$$\omega_1(x_n, x_m) = \omega_{(m-n)\frac{1}{m-n}}(x_n, x_m) \leq \sum_{i=n}^{m-1} \omega_{\frac{1}{m-n}}(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} \frac{1}{i^{1/r}}.$$

Since $0 < r < 1$, then

$$\lim_{n\to+\infty} \sum_{i=n}^{\infty} \frac{1}{i^{1/r}} = 0$$

and hence $\omega_1(x_n, x_m) \to 0$ as $m, n \to +\infty$. Thus, we have proved that $\{x_n\}$ is a $\omega-$Cauchy sequence. The hypothesis of $\omega-$completeness of X_ω ensures that there exists $x^* \in X_\omega$ such that $\omega_1(x_n, x^*) \to 0$ as $n \to +\infty$. Now, since T is an $\alpha\eta - \omega$-continuous mapping and $\eta(x_{n-1}, x_n) \leq \alpha(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$, then $\omega_1(x_{n+1}, Tx^*) = \omega_1(Tx_n, Tx^*) \to 0$ as $n \to +\infty$. From

$$\omega_2(x^*, Tx^*) \leq \omega_1(x^*, x_{n+1}) + \omega_1(x_{n+1}, Tx^*),$$

taking limit as $n \to +\infty$, we get $\omega_2(x^*, Tx^*) = 0$ and hence $x^* = Tx^*$, because ω is regular. Thus, T has a fixed point and $Fix(T^n) = Fix(T)$ for $n = 1$. Let $n > 1$. Assume on the contrary that there is $w \in Fix(T^n) \setminus Fix(T)$. Then $\omega_\lambda(w, Tw) > 0$ for all $\lambda > 0$. Now, we have

$$\begin{aligned} 1 < \Theta(\omega_\lambda(w, Tw)) &= \Theta(\omega_\lambda(T(T^{n-1}w)), T(T^n w))) \\ &\leq [\Theta(\omega_\lambda(T^{n-1}w, T^n w))]^k \\ &\leq \cdots \\ &\leq [\Theta(\omega_\lambda(w, Tw))]^{k^n} \\ &< \Theta(\omega_\lambda(w, Tw)), \end{aligned}$$

which is a contradiction. Therefore, $Fix(T^n) = Fix(T)$ for all $n \in \mathbb{N}$. Let $x, y \in Fix(T)$ with $x \neq y$. Then

$$\Theta\big(\omega_\lambda(Tx, Ty)\big) \leq [\Theta\big(\omega_\lambda(x, y)\big)]^k$$

which is a contradiction. Hence, $x = y$. Therefore, $Fix(T^n) = Fix(T) = \{x^*\}$ for all $n \in \mathbb{N}$. □

Theorem 2.3. *Let X_ω be a complete modular metric space with ω regular and let $T : X_\omega \to X_\omega$ be a self-mapping. Assume that there exist two functions $\alpha, \eta : X_\omega \times X_\omega \to [0, +\infty)$ and a function $\Theta \in \Delta_\Theta$ such that the following assertions hold:*

(i) *Conditions (i)-(iii) of Theorem 2.2 hold*
(ii) *if $\{x_n\}$ is a sequence in X_ω such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ with $x_n \to x$ as $n \to +\infty$, then either*

$$\eta(Tx_n, T^2x_n) \leq \alpha(Tx_n, x) \text{ or } \eta(T^2x_n, T^3x_n) \leq \alpha(T^2x_n, x)$$

holds for all $n \in \mathbb{N}$.

Then T has a fixed point. Moreover, T has the property P if for all $n \in \mathbb{N}$ and for all $w \in Fix(T^n)$, we have $\alpha(T^jw, T^{j+1}w) \geq \eta(T^jw, T^{j+1}w)$ for each $j \in \{1, \ldots, n-1\}$. Further, $Fix(T^n) = Fix(T) = \{x^*\}$ for all $n \in \mathbb{N}$, when $\alpha(x, y) \geq \eta(x, x)$ for all $x, y \in Fix(T)$.

Proof. Let $x_0 \in X_\omega$ be such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$. As in the proof of Theorem 2.2, we deduce that a Picard sequence $\{x_n\}$ starting at x_0 is $\omega-$Cauchy and so converges to a point $x^* \in X_\omega$. Also we have

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \quad \text{for all} \quad n \in \mathbb{N} \cup \{0\}.$$

So, from (iv), either

$$\eta(Tx_n, T^2x_n) \leq \alpha(Tx_n, x^*) \quad \text{or} \quad \eta(T^2x_n, T^3x_n) \leq \alpha(T^2x_n, x^*)$$

holds for all $n \in \mathbb{N} \cup \{0\}$. This implies that

$$\eta(x_{n+1}, x_{n+2}) \leq \alpha(x_{n+1}, x^*) \quad \text{or} \quad \eta(x_{n+2}, x_{n+3}) \leq \alpha(x_{n+2}, x^*)$$

holds for all $n \in \mathbb{N} \cup \{0\}$. Equivalently, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\eta(x_{n_k}, Tx_{n_k}) = \eta(x_{n_k}, x_{n_k+1}) \leq \alpha(x_{n_k}, x^*).$$

It is not restrictive to assume $\omega_\lambda(Tx_{n_k}, Tx^*) > 0$ for all $k \in \mathbb{N}$ and so from (2.1), we deduce that

$$\Theta\big(\omega_\lambda(Tx_{n_k}, Tx^*)\big) \leq \big(\Theta\big(\omega_\lambda(x_{n_k}, x^*)\big)\big)^r, \tag{2.8}$$

which implies

$$\omega_\lambda(x_{n_k+1}, Tx^*) \leq \omega_\lambda(x_{n_k}, x^*) \quad \text{for all } \lambda > 0.$$

Then

$$\lim_{n\to+\infty} \omega_\lambda(x_{n_k+1}, Tx^*) = 0 \quad \text{for all } \lambda > 0$$

and hence

$$\omega_2(x^*, Tx^*) \leq \lim_{n\to+\infty} [\omega_1(x^*, x_{n_k+1}) + \omega_1(x_{n_k+1}, Tx^*)] = 0.$$

Thus, we get $x^* = Tx^*$, since ω is regular. The other statements follow as in the proof of Theorem 2.2. □

Example 2.4. Let $X=\mathbb{R}$ be endowed with the modular metric

$$\omega_\lambda(x,y)=\begin{cases}\frac{1}{\lambda}(|x|+|y|), & \text{if } x\neq y\\ 0 & \text{if } x=y\end{cases}$$

for all $x,y\in X$ and $\lambda>0$. Define $T:X\to X$, $\alpha,\eta:X\times X\to[0,+\infty)$ and $\Theta:(0,+\infty)\to(1,+\infty)$ by

$$Tx=\begin{cases}16x^2, & \text{if } x\in(-\infty,0)\\ \frac{1}{8}x^2, & \text{if } x\in[0,1]\\ \frac{1}{8}x, & \text{if } x\in(1,2)\\ \frac{1}{4} & \text{if } x\in[2,+\infty),\end{cases}$$

$$\alpha(x,y)=\begin{cases}\dfrac{1}{2}, & \text{if } x,y\in[0,1]\\ \dfrac{1}{9}, & \text{otherwise,}\end{cases}\qquad \eta(x,y)=\frac{1}{4},\quad \text{and}\quad \Theta(r)=e^{\sqrt{r}}.$$

Let $\alpha(x,y)\geq\eta(x,y)$, then $x,y\in[0,1]$. On the other hand, $Tw\in[0,1]$ for all $w\in[0,1]$. Thus $\alpha(Tx,Ty)\geq\eta(Tx,Ty)$, that is, T is an α-admissible mapping with respect to η. If $\{x_n\}$ be a sequence in X such that $\alpha(x_n,x_{n+1})\geq\eta(x_n,x_{n+1})$ with $x_n\to x$ as $n\to+\infty$, then $x,Tx_n,T^2x_n,T^3x_n\in[0,1]$ for all $n\in\mathbb{N}$. This ensures that

$$\eta(Tx_n,T^2x_n)\leq\alpha(Tx_n,x)\quad\text{and}\quad\eta(T^2x_n,T^3x_n)\leq\alpha(T^2x_n,x)$$

hold for all $n\in\mathbb{N}$. Clearly, $\alpha(0,T0)\geq\eta(0,T0)$. Let $\alpha(x,y)\geq\eta(x,Tx)$. So $x,y\in[0,1]$ and hence

$$\omega_\lambda(Tx,Ty)=\tfrac{1}{\lambda}(\tfrac{1}{8}x^2+\tfrac{1}{8}y^2)\leq\tfrac{1}{8}\omega_\lambda(x,y).$$

Therefore, if $\omega_\lambda(Tx,Ty)>0$, we get

$$\begin{aligned}\Theta(\omega_\lambda(Tx,Ty)) &= e^{\sqrt{\omega_\lambda(Tx,Ty)}}\\ &\leq e^{\sqrt{\frac{1}{8}\omega_\lambda(x,y)}}\\ &= \left[e^{\sqrt{\omega_\lambda(x,y)}}\right]^{\frac{1}{\sqrt{8}}}\\ &= \left[\Theta(\omega_\lambda(x,y))\right]^{\frac{1}{\sqrt{8}}}.\end{aligned}$$

Hence T is an $(\alpha,\eta,\Theta)-\omega$-contraction. Thus all conditions of Theorem 2.3 hold and T has a fixed point and $Fix(T^n)=Fix(T)=\{0\}$ for all $n\in\mathbb{N}$.

By taking $\alpha(x,y)=\eta(x,y)=1$ for all $x,y\in X_\omega$ in Theorem 2.3 we deduce that the property P holds without continuity of the mapping T.

Corollary 2.5. *Let X_ω be a complete modular metric space with ω regular and let $T:X_\omega\to X_\omega$ be a self-mapping. Assume that there exist a real number $r\in[0,1)$ and a function $\Theta\in\Delta_\Theta$ such that for all $x,y\in X_\omega$ and all $\lambda>0$ with $\omega_\lambda(Tx,Ty)>0$, we have*

$$\Theta\big(\omega_\lambda(Tx,Ty)\big)\leq\Big[\Theta\big(\omega_\lambda(x,y)\big)\Big]^r.$$

Then there exist $x^* \in X_\omega$ *such that* $Fix(T^n) = Fix(T) = \{x^*\}$ *for all* $n \in \mathbb{N}$.

Theorem 2.6. *Let* X_ω *be a complete modular metric space with* ω *regular and let* $T : X_\omega \to X_\omega$ *be a self-mapping. Assume that there exist a real number* $k \in [0, 1)$ *and a function* $\Theta \in \Delta_\Theta$ *such that*

$$\Theta\big(\omega_\lambda(Tx, T^2x)\big) \leq [\Theta\big(\omega_\lambda(x, Tx)\big)]^k \tag{2.9}$$

holds for all $\lambda > 0$ *and* $x \in X_\omega$ *with* $\omega_\lambda(Tx, T^2x) > 0$. *Assume also that there exists a function* $\alpha : X_\omega \times X_\omega \to [0, +\infty)$ *such that the following assertions hold:*

(i) T *is an* α*-admissible mapping;*
(ii) *there exists* $x_0 \in X_\omega$ *such that* $\alpha(x_0, Tx_0) \geq 1$;
(iii) T *is an* $\alpha - \omega$*-continuous mapping.*

Then T *has the property* P.

Proof. First we prove that T has a fixed point. We assume the contrary. Since ω is regular, we have $\omega_\lambda(x, Tx) > 0$ for all $\lambda > 0$ and $x \in X_\omega$. Let $x_0 \in X_\omega$ be such that $\alpha(x_0, Tx_0) \geq 1$ and $\{x_n\}$ a Picard sequence starting at x_0. Conditions (i) and (ii) ensure that

$$\alpha(x_{n-1}, x_n) \geq 1 \quad \text{for all} \quad n \in \mathbb{N}.$$

From (2.9) with $x = x_{n-1}$, we have

$$\Theta\big(\omega_\lambda(Tx_{n-1}, T^2x_{n-1})\big) \leq [\Theta\big(\omega_\lambda(x_{n-1}, Tx_{n-1})\big)]^k$$

which implies that

$$\Theta\big(\omega_\lambda(x_n, x_{n+1})\big) \leq [\Theta\big(\omega_\lambda(x_{n-1}, x_n)\big)]^k. \tag{2.10}$$

Therefore,

$$\begin{aligned} 1 \quad &< \Theta\big(\omega_\lambda(x_n, x_{n+1})\big) \leq [\Theta\big(\omega_\lambda(x_{n-1}, x_n)\big)]^k \\ &\leq [\Theta\big(\omega_\lambda(x_{n-2}, x_{n-1})\big)]^{k^2} \leq \cdots \leq [\Theta(\omega_\lambda(x_0, x_1))]^{k^n}. \end{aligned} \tag{2.11}$$

By taking the limit as $n \to +\infty$ in (2.11), we have $\lim_{n\to+\infty} \Theta\big(\omega_\lambda(x_n, x_{n+1})\big) = 1$ and since $\Theta \in \Delta_\Theta$ we obtain that

$$\lim_{n\to+\infty} \omega_\lambda(x_n, x_{n+1}) = 0. \tag{2.12}$$

Proceeding as in the proof of Theorem 2.2, we deduce that $\{x_n\}$ is a $\omega-$Cauchy sequence. The hypothesis of $\omega-$completeness of X_ω ensures that there exists $x^* \in X_\omega$ such that $\omega_1(x_n, x^*) \to 0$ as $n \to +\infty$. Now, conditions (i) and (ii) ensure that

$$\alpha(x_{n-1}, x_n) \geq 1 \quad \text{for all} \quad n \in \mathbb{N}.$$

Since T is $\alpha - \omega$-continuous, then $\omega_1(x_{n+1}, Tx^*) = \omega_1(Tx_n, Tx^*) \to 0$ as $n \to +\infty$. From

$$\omega_2(x^*, Tx^*) \leq \omega_1(x^*, x_{n+1}) + \omega_1(x_{n+1}, Tx^*),$$

taking limit as $n \to +\infty$, we get $\omega_2(x^*, Tx^*) = 0$ and hence $x^* = Tx^*$, because ω is regular. Thus, T has a fixed point. Finally, the proof that $Fix(T^n) = Fix(T)$ for all $n \in \mathbb{N}$ is as in Theorem 2.2. □

3. Suzuki type fixed point results

In this section, as applications of our results proved above, we deduce certain Suzuki type fixed point theorems.

Theorem 3.1. *Let X_ω be a complete modular metric space with ω regular and let $T : X_\omega \to X_\omega$ be a $\omega-$continuous self-mapping. Assume that there exist a real number $r \in [0,1)$ and a function $\Theta \in \Delta_\Theta$ such that for all $x, y \in X_\omega$ with $\omega_1(x,Tx) \leq \omega_1(x,y)$ and $\omega_\lambda(Tx,Ty) > 0$, we have*

$$\Theta\big(\omega_\lambda(Tx,Ty)\big) \leq \Big[\Theta\big(\omega_\lambda(x,y)\big)\Big]^r \tag{3.1}$$

for all $\lambda > 0$. Then there exists $x^* \in X_\omega$ such that $Fix(T^n) = Fix(T) = \{x^*\}$ for all $n \in \mathbb{N}$.

Proof. Define $\alpha, \eta : X_\omega \times X_\omega \to [0,+\infty)$ by

$$\alpha(x,y) = \eta(x,y) = \omega_1(x,y) \quad \text{for all} \quad x, y \in X_\omega.$$

Thus $\eta(x,y) \leq \alpha(x,y)$ for all $x, y \in X_\omega$. That is conditions (i) and (iii) of Theorem 2.2 hold true. Since T is ω-continuous, then T is $\alpha\eta - \omega$-continuous. Let $\eta(x,Tx) \leq \alpha(x,y)$ with $\omega_\lambda(Tx,Ty) > 0$, equivalently, let $\omega_1(x,Tx) \leq \omega_1(x,y)$ with $\omega_\lambda(Tx,Ty) > 0$. Then, from (3.1), we have

$$\Theta\big(\omega_\lambda(Tx,Ty)\big) \leq \Big[\Theta\big(\omega_\lambda(x,y)\big)\Big]^r, \tag{3.2}$$

that is, T is an $(\alpha, \eta, \Theta) - \omega$-contraction. Hence, all conditions of Theorem 2.2 hold and there exist $x^* \in X_\omega$ such that, $Fix(T^n) = Fix(T) = \{x^*\}$ for all $n \in \mathbb{N}$. □

Theorem 3.2. *Let X_ω be a complete modular metric space with ω regular and let $T : X_\omega \to X_\omega$ be a self-mapping. Assume that there exist a real number $r \in [0,1)$ and a function $\Theta \in \Delta_\Theta$ such that for all $x, y \in X_\omega$ with $\omega_\lambda(Tx,Ty) > 0$ and $\frac{1}{2}\omega_1(x,Tx) \leq \omega_{\frac{1}{2}}(x,y)$, we have*

$$\Theta\big(\omega_\lambda(Tx,Ty)\big) \leq \Big[\Theta\big(\omega_\lambda(x,y)\big)\Big]^r. \tag{3.3}$$

Then there exists $x^* \in X_\omega$ such that, $Fix(T^n) = Fix(T) = \{x^*\}$ for all $n \in \mathbb{N}$.

Proof. Define $\alpha, \eta : X_\omega \times X_\omega \to [0,+\infty)$ by

$$\alpha(x,y) = \omega_{\frac{1}{2}}(x,y) \quad \text{and} \quad \eta(x,y) = \frac{1}{2}\omega_1(x,y)$$

for all $x, y \in X_\omega$. Thus $\eta(x,y) \leq \alpha(x,y)$ for all $x, y \in X_\omega$. This ensures that conditions (i) and (iii) of Theorem 2.3 hold true. Let $\{x_n\}$ be a sequence with $x_n \to x$ as $n \to +\infty$. If for some $m \in \mathbb{N}$, we have $Tx_m = T^2x_m$, then Tx_m is a fixed point of T and we have nothing to prove. Hence we may assume that $Tx_n \neq T^2x_n$ for all $n \in \mathbb{N}$. Since $\frac{1}{2}\omega_1(Tx_n, T^2x_n) \leq \omega_{\frac{1}{2}}(Tx_n, T^2x_n)$ for all $n \in \mathbb{N}$, by (3.3), we get

$$\Theta\big(\omega_\lambda(T^2x_n, T^3x_n)\big) \leq [\Theta\big(\omega_\lambda(Tx_n, T^2x_n)\big)]^k < \Theta\big(\omega_\lambda(Tx_n, T^2x_n)\big)$$

and so by (Θ1), we deduce

$$\omega_\lambda(T^2x_n, T^3x_n) < \omega_\lambda(Tx_n, T^2x_n). \tag{3.4}$$

If for some $m \in \mathbb{N}$, we have

$$\eta(Tx_m, T^2x_m) > \alpha(Tx_m, x) \quad \text{and} \quad \eta(T^2x_m, T^3x_m) > \alpha(T^2x_m, x),$$

then

$$\frac{1}{2}\omega_1(Tx_m, T^2x_m) > \omega_{\frac{1}{2}}(Tx_m, x) \quad \text{and} \quad \frac{1}{2}\omega_1(T^2x_m, T^3x_m) > \omega_{\frac{1}{2}}(T^2x_m, x).$$

Therefore

$$\begin{aligned} \omega_1(Tx_m, T^2x_m) &\leq \omega_{\frac{1}{2}}(Tx_m, x) + \omega_{\frac{1}{2}}(T^2x_m, x) \\ &< \tfrac{1}{2}\omega_1(Tx_m, T^2x_m) + \tfrac{1}{2}\omega_1(T^2x_m, T^3x_m) \\ &\leq \tfrac{1}{2}\omega_1(Tx_m, T^2x_m) + \tfrac{1}{2}\omega_1(Tx_m, T^2x_m) = \omega_1(Tx_m, T^2x_m), \end{aligned}$$

which is a contradiction. Hence, either

$$\eta(Tx_n, T^2x_n) \leq \alpha(Tx_n, x) \quad \text{or} \quad \eta(T^2x_n, T^3x_n) \leq \alpha(T^2x_n, x)$$

holds for all $n \in \mathbb{N}$. Thus, the condition (iv) of Theorem 2.3 holds. Let $\eta(x, Tx) \leq \alpha(x, y)$, that is, $\frac{1}{2}\omega_1(x, Tx) \leq \omega_{\frac{1}{2}}(x, y)$. If $\omega_\lambda(Tx, Ty) > 0$, from (3.3), we get that

$$\Theta\big(\omega_\lambda(Tx, Ty)\big) \leq \Big[\Theta\big(\omega_\lambda(x, y)\big)\Big]^r.$$

Hence, all conditions of Theorem 2.3 hold and so there exist $x^* \in X_\omega$ such that $Fix(T^n) = Fix(T) = \{x^*\}$ for all $n \in \mathbb{N}$. □

4. Some results on Property P in Fuzzy Metric Spaces

For the sake of completeness, we briefly recall some basic concepts used in this section.

Definition 4.1. *A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if it satisfies the following assertions:*

(TN1) *$*$ is commutative and associative;*
(TN2) *$*$ is continuous;*
(TN3) *$a * 1 = a$ for all $a \in [0,1]$;*
(TN3) *$a * b \leq c * d$ when $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0,1]$.*

Definition 4.2. *(George and Veeramani [8]) A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a nonempty set, $*$ a continuous t-norm and M is a fuzzy set on $X \times X \times (0, +\infty)$ satisfying the following conditions, for all $x, y, z \in X$ and $t, s > 0$:*

(FM1) *$M(x, y, t) > 0$ for all $t > 0$;*
(FM2) *$M(x, y, t) = 1$ if and only if $x = y$;*
(FM3) *$M(x, y, t) = M(y, x, t)$;*
(FM4) *$M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;*
(FM5) *$M(x, y, \cdot) : (0, +\infty) \to (0, 1]$ is left continuous.*

Then the triple $(X, M, *)$ is called a fuzzy metric space.

Definition 4.3 ([6])**.** *Let $(X, M, *)$ be a fuzzy metric space. The fuzzy metric M is called triangular whenever*

$$\frac{1}{M(x, y, t)} - 1 \leq \frac{1}{M(x, z, t)} - 1 + \frac{1}{M(z, y, t)} - 1$$

for all $x, y, z \in X$ and all $t > 0$.

The result that follows highlights that to each triangular fuzzy metric can be associated a modular metric.

Lemma 4.4 ([16])**.** *Let $(X, M, *)$ be a triangular fuzzy metric space. The function $\omega : (0, +\infty) \times X \times X \to [0, +\infty]$ defined by*

$$\omega_\lambda(x, y) = \frac{1}{M(x, y, \lambda)} - 1 \tag{4.1}$$

for all $x, y \in X$ and all $\lambda > 0$ is a modular metric on X.

A fuzzy metric M on X is said to be regular if the following weaker version of (FM2) is satisfied

$$x = y \quad \text{if and only if} \quad M(x, y, t) = 1 \quad \text{for some } t > 0.$$

In the definition that follows, we give some notions of continuity.

Definition 4.5. *Let $(X, M, *)$ be a fuzzy metric and $T : X \to X$ be a self-mapping. Assume that $\alpha, \eta : X \times X \to [0, +\infty)$ are two functions. We say:*

(i) *T is an $M^1\alpha\eta$-continuous mapping, if $\lim\limits_{n\to+\infty} M(x_n, x, 1) = 1$ with $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ implies $\lim\limits_{n\to+\infty} M(Tx_n, Tx, 1) = 1$;*
(ii) *T is an $M^1\alpha$-continuous mapping, if $\lim\limits_{n\to+\infty} M(x_n, x, 1) = 1$ with $\alpha(x_n, x_{n+1}) \geq 1$ implies $\lim\limits_{n\to+\infty} M(Tx_n, Tx, 1) = 1$;*
(iii) *T is an M^1-continuous mapping, if $\lim\limits_{n\to+\infty} M(x_n, x, 1) = 1$ implies $\lim\limits_{n\to+\infty} M(Tx_n, Tx, 1) = 1$.*

Theorem 4.6. *Let $(X, M, *)$ be a complete fuzzy metric space with M triangular and regular and let $T : X \to X$ be a self-mapping. Assume that there exist two functions $\alpha, \eta : X \times X \to [0, +\infty)$ and a function $\Theta \in \Delta_\Theta$ such that the following assertions hold:*

(i) *T is an α-admissible mapping with respect to η;*
(ii) *there exists $r \in [0, 1)$ such that for $x, y \in X$ with $\eta(x, Tx) \leq \alpha(x, y)$ and $M(Tx, Ty, t) < 1$ and all $t > 0$, we have*

$$\Theta\big(\tfrac{1}{M(Tx,Ty,t)} - 1\big) \leq \Big[\Theta\big(\tfrac{1}{M(x,y,t)} - 1\big)\Big]^r; \tag{4.2}$$

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;*
(iv) *T is $M^1\alpha\eta$-continuous;*
(v) *there exists $z \in X$ such that $\lim_{t\to+\infty} M(Tx, z, t) = 1$ whenever $\lim_{t\to+\infty} M(x, z, t) = 1$.*

Then T has a fixed point. Moreover, T has the property P if for all $n \in \mathbb{N}$ and for all $w \in Fix(T^n)$, we have $\alpha(T^j w, T^{j+1} w) \geq \eta(T^j w, T^{j+1} w)$ for each $j \in \{1, \dots, n-1\}$. Further, $Fix(T^n) = Fix(T) = \{x^*\}$ for all $n \in \mathbb{N}$, when $\alpha(x, y) \geq \eta(x, x)$ for all $x, y \in Fix(T)$.

Proof. Let X_ω be the modular space (around z) induced by the modular ω defined by (4.1). Since the mapping T, considered as a self-mapping on X_ω, satisfies the hypotheses of Theorem 2.2, we get the conclusion. □

A similar remark to the one above and Theorem 2.3 yield the following result.

Theorem 4.7. *Let $(X, M, *)$ be a complete fuzzy metric space with M triangular and regular and let $T : X \to X$ be a self-mapping. Assume that there exist two functions $\alpha, \eta : X \times X \to [0, +\infty)$ and a function $\Theta \in \Delta_\Theta$ such that the following assertions hold:*

(i) *T is an α-admissible mapping with respect to η;*
(ii) *there exists $r \in [0, 1)$ such that for $x, y \in X$ with $\eta(x, Tx) \leq \alpha(x, y)$ and $M(Tx, Ty, t) < 1$ and all $t > 0$, we have*

$$\Theta\big(\tfrac{1}{M(Tx,Ty,t)} - 1\big) \leq \left[\Theta\big(\tfrac{1}{M(x,y,t)} - 1\big)\right]^r; \tag{4.3}$$

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;*
(iv) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ with $x_n \to x$ as $n \to +\infty$, then either*

$$\eta(Tx_n, T^2x_n) \leq \alpha(Tx_n, x) \quad or \quad \eta(T^2x_n, T^3x_n) \leq \alpha(T^2x_n, x)$$

holds for all $n \in \mathbb{N}$;
(v) *there exists $z \in X$ such that $\lim_{t\to+\infty} M(Tx, z, t) = 1$ whenever $\lim_{t\to+\infty} M(x, z, t) = 1$.*

Then T has a fixed point. Moreover, T has the property P if for all $n \in \mathbb{N}$ and for all $w \in Fix(T^n)$, we have $\alpha(T^j w, T^{j+1}w) \geq \eta(T^j w, T^{j+1}w)$ for each $j \in \{1, \dots, n-1\}$. Further, $Fix(T^n) = Fix(T) = \{x^*\}$ for all $n \in \mathbb{N}$, when $\alpha(x, y) \geq \eta(x, x)$ for all $x, y \in Fix(T)$.

Remark 2. *From the above obtained theorems, we may derive certain new results in partially ordered modular metric and fuzzy metric spaces(see* [12, 13, 14, 15]*) for details.*

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