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ORTHOGONALITY PRESERVING MAPPINGS IN KREIN SPACES

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Abstract. Let $(\mathcal{K}_1, [\cdot,\cdot]_1)$ and $(\mathcal{K}_2, [\cdot,\cdot]_2)$ be two Krein spaces, and let $T : \mathcal{K}_1 \to \mathcal{K}_2$ be a nonzero linear mapping such that $T(\mathcal{K}_1^+) = \mathcal{K}_2^+$ and $T(\mathcal{K}_1^-) = \mathcal{K}_2^-$, where $\mathcal{K}_i = \mathcal{K}_i^+[\oplus]\mathcal{K}_i^-$ is the canonical decomposition of $\mathcal{K}_i$ for $i = 1, 2$. We show that T preserves orthogonality if and only if there exists $\lambda > 0$ such that $\|Tx\| = \lambda\|x\|$ for each $x \in \mathcal{K}_1$. We also define four types of orthogonality in the framework of Krein spaces and show that the usual orthogonality is equivalent with them.

1. Introduction and preliminaries

Vector spaces equipped with indefinite inner products were used for the first time in the quantum field theory in physics and mechanic. They were later systematically studied by Pontrjagin [11], Krein, and others.

A Krein space, as an indefinite generalization of a Hilbert space, was formally defined by Ginzburg [7] and was applied in the quantum field theory by Jakóbczyk [8] and others. We present the standard terminology and some basic results of orthogonality on Krein spaces. We refer the reader to [3, 4, 6, 10] for more information on the basic theory of Krein spaces.

Orthogonality is one of the most important concepts in the theory of inner product spaces. In an inner product space two vectors x and y are orthogonal when $\langle x, y\rangle = 0$. Many mathematicians generalized the notions of orthogonality for normed spaces [1, 8, 9, 12]:

(i) The Birkhoff–James orthogonality: $\perp_B$: $x \perp_B y$ if $\|x\| \leq \|x + \lambda y\|$ for all scalars λ.
(ii) The Phythagorean orthogonality: $\perp_P$: $x \perp_P y$ if $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.
(iii) The Isosceles orthogonality: $\perp_I$: $x \perp_I y$ if $\|x + y\| = \|x - y\|$.
(iv) The Roberts orthogonality: $\perp_R$: $x \perp_R y$ if $\|x + \lambda y\| = \|x - \lambda y\|$ for all scalars λ.
(v) The Bisectrix orthogonality: $\perp_W$: $x \perp_W y$ if $\Big\|\|y\|x + \|x\|y\Big\| = \sqrt{2}\|x\|\|y\|$.

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Recall that, for inner product spaces, the above notions of orthogonality are equivalent to the usual orthogonality given by the inner product. Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are two inner product spaces. A mapping $T : X \to Y$ preserves orthogonality, if $x \perp y$ implies that $Tx \perp Ty$ for all $x, y \in \mathcal{X}$. Chmieliński [5] proved that a linear mapping T preserves orthogonality if and only if T is an isometry multiplied by a positive constant.

In this paper, we introduce four types of orthogonality in the setting of Krein spaces and show that the usual orthogonality, given by the indefinite inner product, is equivalent with them in the real Krein spaces. We also study the class of linear mappings that preserve the orthogonality between Krein spaces.

2. Orthogonality in Krein spaces

We begin this section with the definition of a Krein space and then state some of its significant properties.

Definition 2.1. *Suppose that $\mathcal{K}$ is a linear vector space equipped with a map $[\cdot,\cdot] : \mathcal{K} \times \mathcal{K} \to \mathbb{C}$ such that*

$$[x, y] = \overline{[y, x]},$$
$$[\alpha x + \beta y, z] = \alpha[x, z] + \beta[y, z]$$

for each $x, y, z \in \mathcal{K}$ and $\alpha, \beta \in \mathbb{C}$. Then $(\mathcal{K},\ [\cdot,\cdot])$ is called an indefinite inner product space.

Recall that an indefinite inner product has all properties of the usual inner product except the positive definiteness.

Let $(\mathcal{K},\ [\cdot,\cdot])$ be an indefinite inner product space and let $x, y \in \mathcal{K}$. We say that x is orthogonal to y, denoted by $x[\perp]y$, when $[x, y] = 0$. If $\mathcal{L}$ is a subspace of $\mathcal{K}$, then $\mathcal{L}^{[\perp]} \equiv \{x \in \mathcal{K} : [x, y] = 0 \text{ for all } y \in \mathcal{L}\}$ is called the orthogonal complement of $\mathcal{L}$, and $x_0 \in \mathcal{L}$ is an isotropic element of $\mathcal{L}$ if $x_0 \neq 0$ and $x_0[\perp]\mathcal{L}$. Also $\mathcal{L}^0 \equiv \mathcal{L} \bigcap \mathcal{L}^{[\perp]}$ is the isotropic part of $\mathcal{L}$. If $\mathcal{L}^0 = \{0\}$, then $\mathcal{L}$ is called nondegenerate.

Definition 2.2. *The indefinite inner product space $(\mathcal{K}, [\cdot,\cdot])$ is called a Krein space, if the vector space $\mathcal{K}$ can be expressed as an orthogonal direct sum $\mathcal{K} = \mathcal{K}^+[\oplus]\mathcal{K}^-$, where $(\mathcal{K}^+, [\cdot,\cdot])$ and $(\mathcal{K}^-, -[\cdot,\cdot])$ are Hilbert spaces. Any such representation is called a canonical decomposition of $\mathcal{K}$.*

In fact $\mathcal{K}^+ \subseteq \{x \in \mathcal{K} : [x, x] > 0\} \cup \{0\}$, and $\mathcal{K}^- \subseteq \{x \in \mathcal{K} : [x, x] < 0\} \cup \{0\}$.

Suppose that $(\mathcal{K}, [\cdot,\cdot])$ is a Krein space with a canonical decomposition $\mathcal{K} = \mathcal{K}^+[\oplus]\mathcal{K}^-$ and that $P^+ : \mathcal{K} \to \mathcal{K}^+$ and $P^- : \mathcal{K} \to \mathcal{K}^-$ are two mutually orthogonal projection operators generated by canonical decomposition above such that $P^+ + P^- = I$, where I is the identity operator on $\mathcal{K}$. Then $x = x^+ + x^-$, for each $x \in \mathcal{K}$, where $x^+ = P^+x$ and $x^- = P^-x$.

The linear operator $J : \mathcal{K} \to \mathcal{K}$, defined by $J = P^+ - P^-$, is called the canonical symmetry operator of the Krein space $\mathcal{K}$. Thus J is a bounded self-adjoint operator such that $J^2 = I$ and $J^{-1} = J^*$.

By using the canonical decomposition $\mathcal{K} = \mathcal{K}^+[\oplus]\mathcal{K}^-$ on the Krein space $(\mathcal{K}, [\cdot,\cdot])$, we can define an inner product as follows:

$$\langle x, y\rangle = [x^+, y^+] - [x^-, y^-], \tag{2.1}$$

where $x = x^+ + x^-$ and $y = y^+ + y^-$ are elements in $\mathcal{K}$.
Then $(\mathcal{K}, \langle\cdot,\cdot\rangle)$ is a Hilbert space with respect to the norm $\|x\|^2 = \langle x, x\rangle$. In fact, $\langle x, y\rangle = [x, y]$, for vectors $x, y \in \mathcal{K}^+$, and $\langle x, y\rangle = -[x, y]$ for $x, y \in \mathcal{K}^-$.

Relation (2.1) between the indefinite inner product $[\cdot,\cdot]$ and the definite inner product $\langle\cdot,\cdot\rangle$ on $\mathcal{K}$ implies that

$$\langle x, y\rangle = [Jx, y],$$
$$[x, y] = \langle Jx, y\rangle,$$

for each $x, y \in \mathcal{K}$, and $J(x^+ + x^-) = x^+ - x^-$. Also, $\|x\|^2 = \langle x, x\rangle = [Jx, x]$. A straightforward computation shows that

$$[x, y] = [x^+, y^+] + [x^-, y^-], \tag{2.2}$$
$$[x, y] = \langle x^+, y^+\rangle - \langle x^-, y^-\rangle, \quad [x, x] = \|x^+\|^2 - \|x^-\|^2.$$

Let $(\mathcal{K}, [\cdot,\cdot])$ be a Krein space, and let $x, y \in \mathcal{K}$. Then the conditions (2.1) and (2.2) imply that $[x, y] = 0$ and $\langle x, y\rangle = 0$ if and only if $[x^+, y^+] = 0$ and $[x^-, y^-] = 0$. Now we show that the condition $\langle x, y\rangle = 0$ cannot be omitted.

Example 2.3. *Let us consider the indefinite inner product space* $(\mathbb{R}^3, [\cdot,\cdot])$*, where*

$$[x, y] = -3x_1y_1 - 2x_2y_2 + 4x_3y_3$$

for each $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in \mathbb{R}^3$. *Then* $(\mathbb{R}^3, [\cdot,\cdot])$ *is a Krein space and admits the canonical decomposition* $\mathbb{R}^3 = \mathbb{R}^{3^+}[\oplus]\mathbb{R}^{3^-}$*, where* $\mathbb{R}^{3^+} = \{(0, 0, z) : z \in \mathbb{R}\}$ *and* $\mathbb{R}^{3^-} = \{(x, y, 0) : \ x, y \in \mathbb{R}\}$. *Let*

$$x = (1, 3, 1) = (0, 0, 1) + (1, 3, 0),$$
$$y = (2, 1, 3) = (0, 0, 3) + (2, 1, 0).$$

Then

$$[x^+, y^+] = [(0, 0, 1), (0, 0, 3)] = 12,$$
$$[x^-, y^-] = [(1, 3, 0), (2, 1, 0)] = -12,$$
$$[x, y] = [x^+, y^+] + [x^-, y^-] = 0.$$

On the other hand,

$$\langle x, y\rangle = [x^+, y^+] - [x^-, y^-] = 24 \neq 0.$$

Proposition 2.4. [2] *Let* $(X, \langle\cdot,\cdot\rangle)$ *be a real inner product space, and let* x, y *be two vectors in* X. *Then the following orthogonality types are equivalent:*
(i) $\langle x, y\rangle = 0$.
(ii) $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.
(iii) $\|x + y\| = \|x - y\|$.
(iv) $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in \mathbb{R}$.

If x, y are two orthogonal elements in the usual sense of a Krein space $\mathcal{K}$, then we show that they satisfy the following statements.

Theorem 2.5. *Let* $(\mathcal{K}, [\cdot,\cdot])$ *be a Krein space, and let* $x, y \in \mathcal{K}$ *be such that* $[x, y] = 0$. *Then the following properties hold:*
(i) $\|x + \lambda y\|^2 - \|x\|^2 = 4\mathrm{Re}(\overline{\lambda}[x^+, y^+]) + |\lambda^2|\|y\|^2$ for all $\lambda \in \mathbb{C}$.
(ii) $\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 4\mathrm{Re}[x^+, y^+]$.
(iii) $\|x + y\|^2 - \|x - y\|^2 = 8\mathrm{Re}[x^+, y^+]$.
(iv) $\big\|\|y\|x + \|x\|y\big\|^2 = 2\|y\|^2\|x\|^2 + 4\|y\|\|x\|\mathrm{Re}[x^+, y^+]$.

Proof. Let J be the canonical symmetry operator associated with $\mathcal{K}$.

(i) Fix an arbitrary element $\lambda \in \mathbb{C}$. We have

$$\begin{aligned}\|x+\lambda y\|^2 - \|x\|^2 &= [J(x+\lambda y), x+\lambda y] - [Jx, x] \\ &= [Jx,x] + \overline{\lambda}[Jx,y] + \lambda[Jy,x] + |\lambda|^2[Jy,y] - [Jx,x] \\ &= 2\mathrm{Re}(\overline{\lambda}[Jx,y]) + |\lambda|^2[Jy,y] \\ &= 4\mathrm{Re}(\overline{\lambda}[x^+,y^+]) + |\lambda|^2\|\|y\|^2.\end{aligned}$$

(ii) The assumption implies that

$$[x,y] = [x^+ + x^-, y^+ + y^-] = [x^+,y^+] + [x^-,y^-] = 0;$$

so $[x^+,y^+] = -[x^-,y^-]$.

On the other hand,

$$\begin{aligned}\|x+y\|^2 &= \big[J(x+y), x+y\big] = [Jx+Jy, x+y] = [x^+ - x^- + y^+ - y^-, x^+ + x^- + y^+ + y^-] \\ &= [x^+,x^+] + [x^+,y^+] - [x^-,x^-] - [x^-,y^-] + [y^+,x^+] + [y^+,y^+] - [y^-,x^-] - [y^-,y^-] \\ &= \|x\|^2 + \|y\|^2 + 4\mathrm{Re}[x^+,y^+]\end{aligned}$$

(iii) Property (ii) implies
$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 4\mathrm{Re}[x^+,y^+]$ and $\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 4\mathrm{Re}[x^+,y^+]$;
Thus $\|x+y\|^2 - \|x-y\|^2 = 8\mathrm{Re}[x^+,y^+]$.

(iv) The basic concepts imply

$$\begin{aligned}\big\|\|y\|x + \|x\|y\big\|^2 &= [J(\|y\|x + \|x\|y), \|y\|x + \|x\|y] \\ &= [\|y\|Jx + \|x\|Jy, \|y\|x + \|x\|y] \\ &= \|y\|^2[Jx,x] + \|y\|\|x\|[Jx,y] + \|x\|\|y\|[Jy,x] + \|x\|^2[Jy,y] \\ &= 2\|y\|^2\|x\|^2 + 2\mathrm{Re}\|y\|\|x\|[Jx,y] \\ &= 2\|y\|^2\|x\|^2 + 4\mathrm{Re}\|y\|\|x\|[x^+,y^+].\end{aligned}$$

□

Note that the expression $4\mathrm{Re}(\overline{\lambda}[x^+,y^+]) + |\lambda|^2\|\|y\|^2$ is not positive in general. To see this, put $x = (10,10,-10)$, $y = (-2,-5,4)$ and $\lambda = 1$ in Example 2.3. An easy computation shows that $[x^+,y^+] = -160$ and $\|y\|^2 = [Jy,y] = [y^+,y^+] - [y^-,y^-] = 126$. Thus $4[x^+,y^+] + \|y\|^2 = -640 + 126 = -514$.

Now, we define four types of orthogonality in the framework of Krein spaces, which are equivalent with the usual orthogonality in real Krein spaces.

Definition 2.6. *Let $(\mathcal{K}, [\cdot,\cdot])$ be a Krein space, let J be the canonical symmetry operator associated with $\mathcal{K}$, and let $x, y \in \mathcal{K}$. We say that*

(i) $x[\perp]y$ *if* $[x,y] = 0$
(ii) $x \perp_{P\mathcal{K}} y$ *if* $\|x^+ + y^+\|^2 - \|x^- + y^-\|^2 = \|x^+\|^2 - \|x^-\|^2 + \|y^+\|^2 - \|y^-\|^2$;
(iii) $x \perp_{B\mathcal{K}} y$ *if* $\|x + \lambda Jy\| \geq \|x\|$ *for all* $\lambda \in \mathbb{C}$;
(iv) $x \perp_{I\mathcal{K}} y$ *if* $\|x + Jy\| = \|x - Jy\|$;
(v) $x \perp_{W\mathcal{K}} y$ *if* $\big\|\|y^+\|x^- + \|x^-\|y^-\big\|^2 - \big\|\|y^+\|x^+ + \|x^-\|y^+\big\|^2 = \|x^-\|^2\|y^-\|^2 - \|x^+\|^2\|y^+\|^2$.

Theorem 2.7. *Let $(\mathcal{K}, [\cdot,\cdot])$ be a real Krein space, and let J be the canonical symmetry operator associated with $\mathcal{K}$, and let $x, y \in \mathcal{K}$. Then the following statements are equivalent.*

(i) $x[\perp]y$.
(ii) $x \perp_{P\mathcal{K}} y$.
(iii) $x \perp_{B\mathcal{K}} y$.
(iv) $x \perp_{I\mathcal{K}} y$.

Proof. Let $x, y \in \mathcal{K}$.
(i)$\Longrightarrow$ (ii): Let $x[\perp]y$. Then

$$\begin{aligned}
\|x^+ + y^+\|^2 - \|x^- + y^-\|^2 &= [J(x^+ + y^+), x^+ + y^+] - [J(x^- + y^-), x^- + y^-] \\
&= [Jx^+, x^+] + [Jy^+, y^+] + [J(x^+), y^+] + [Jy^+, x^+] \\
&\quad - [Jx^-, x^-] - [Jy^-, y^-] - [Jx^-, y^-] - [Jy^-, x^-] \\
&= \|x^+\|^2 + \|y^+\|^2 - \|x^-\|^2 - \|y^-\|^2 + 2[x^+, y^+] + 2[x^-, y^-] \\
&= 2[x, y] + \|x^+\|^2 + \|y^+\|^2 - \|x^-\|^2 - \|y^-\|^2 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(since } [x, y] = 0) \\
&= \|x^+\|^2 + \|y^+\|^2 - \|x^-\|^2 - \|y^-\|^2.
\end{aligned}$$

Thus $x \perp_{P\mathcal{K}} y$.
(ii)$\Longrightarrow$(i): It is clear.
(i)$\Longrightarrow$ (iii). Let $x[\perp]y$. Then

$$\begin{aligned}
\|x + \lambda Jy\|^2 - \|x\|^2 &= [J(x + \lambda Jy), x + \lambda Jy] - [Jx, x] \\
&= [Jx + \lambda y, x + \lambda Jy] - [Jx, x] \\
&= [Jx, x] + \lambda[Jx, Jy] + \lambda[x, y] + \lambda^2[y, Jy] - [Jx, x] \\
&= 2\lambda[x, y] + \lambda^2\|y\|^2 \qquad\qquad\qquad\qquad \text{(since } [x, y] = 0) \\
&= |\lambda|^2\|y\|^2 \geq 0.
\end{aligned}$$

Thus $x \perp_{B\mathcal{K}} y$.
(iii)$\Longrightarrow$ (i): Now assume that $x \perp_{B\mathcal{K}} y$.

$$\begin{aligned}
\|x\|^2 \leq \|x + \lambda Jy\|^2 &= [J(x + \lambda Jy), (x + \lambda Jy] \\
&= \|x\|^2 + \lambda[x, y] + \lambda[x, y] + \lambda^2[Jy, y].
\end{aligned}$$

Then $0 \leq 2\lambda[x, y] + \lambda^2\|y\|^2$. Put $\lambda = -\frac{[x,y]}{\|y\|^2}$, we get $[x, y] = 0$.
(i)$\Longrightarrow$ (iv): Since $[x, y] = 0$, so

$$\begin{aligned}
\|x + Jy\|^2 - \|x - Jy\|^2 &= [J(x + Jy), x + Jy] - [J(x - Jy), x - Jy] \\
&= [Jx, x] + [Jx, Jy] + [y, x] + [y, Jy] \\
&\quad - [Jx, x] + [Jx, Jy] + [y, x] - [y, Jy] \\
&= 4[x, y] = 0.
\end{aligned}$$

(iv)$\Longrightarrow$ (i): It is clear.

$\square$

Note that, under the same assumptions as above theorem, if $x[\perp]y$, then

$$\begin{aligned}\big\|\|y^+\|x^- + \|x^-\|y^-\big\|^2 - \big\|\|y^+\|x^+ + \|x^-\|y^+\big\|^2 &= [J(\|y^+\|x^- + \|x^-\|y^-), (\|y^+\|x^- + \|x^-\|y^-)] \\ &\quad - [J(\|y^+\|x^+ + \|x^-\|y^+), (\|y^+\|x^+ + \|x^-\|y^+)] \\ &= [-\|y^+\|x^- - \|x^-\|y^-, \|y^+\|x^- + \|x^-\|y^-] \\ &\quad - [\|y^+\|x^+ + \|x^-\|y^+, \|y^+\|x^+ + \|x^-\|y^+] \\ &= \|x^-\|^2\|y^-\|^2 - \|y^+\|^2\|x^+\|^2 \\ &\quad + \|x^-\|\|y^+\|\Big(- [x^-, y^-] - [y^-, x^-] - [x^+, y^+] - [y^+, x^+]\Big) \\ &= \|x^-\|^2\|y^-\|^2 - \|y^+\|^2\|x^+\|^2 - 2\|x^-\|\|y^+\|[x, y] \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(since } [x, y] = 0) \\ &= \|x^-\|^2\|y^-\|^2 - \|y^+\|^2\|x^+\|^2.\end{aligned}$$

Thus $x \perp_{W\mathcal{K}} y$. On the other hand, if $x \perp_{W\mathcal{K}} y$ and $\|x^-\| \neq 0$ and $\|y^+\| \neq 0$, then with a similar argument as above, we conclude that $[x, y] = 0$, that is $x[\perp]y$.

Remark. *If $(\mathcal{K}, [\cdot, \cdot])$ is a Krein space on $\mathbb{C}$, then the assumption $[x, y] = 0$ implies properties (ii), (iii), (iv) of Theorem 2.7, but the converse is not true in general.*

3. Orthogonality preserving mappings in Krein spaces

In this section, we investigate the linear mappings, which preserve orthogonality in Krein spaces. The following lemma is essential for our discussion .

Lemma 3.1. *Let T be a linear mapping from the Krein space $(\mathcal{K}_1, [\cdot, \cdot]_1)$ to the Krein space $(\mathcal{K}_2, [\cdot, \cdot]_2)$ with the canonical symmetry operators J_1 and J_2, respectively. If $T(\mathcal{K}_1^+) = \mathcal{K}_2^+$ and $T(\mathcal{K}_1^-) = \mathcal{K}_2^-$, then $TJ_1 = J_2T$.*

Proof. Let $x \in \mathcal{K}_1$. Then $x = x^+ + x^-$ and

$$T(x^+) + T(x^-) = T(x^+ + x^-) = T(x) = T(x)^+ + T(x)^-.$$

Since $\mathcal{K}_2$ is nondegenerate, we get $T(x^+) = (Tx)^+$ and $T(x^-) = (Tx)^-$. Therefore

$$\begin{aligned}TJ_1x &= TJ_1(x^+ + x^-) = T(x^+ - x^-) \\ &= Tx^+ - Tx^- = (Tx)^+ - (Tx)^- \\ &= J_2((Tx)^+ + (Tx)^-) = J_2Tx.\end{aligned}$$

Hence $TJ_1 = J_2T$. □

Theorem 3.2. *Let $(\mathcal{K}_1, [\cdot, \cdot]_1)$ and $(\mathcal{K}_2, [\cdot, \cdot]_2)$ be two real Krein spaces, and let $T : \mathcal{K}_1 \to \mathcal{K}_2$ be a nonzero linear mapping such that $T(\mathcal{K}_1^+) = \mathcal{K}_2^+$ and $T(\mathcal{K}_1^-) = \mathcal{K}_2^-$. Then T preserves orthogonality, if and only if there exists $\lambda > 0$ such that $\|Tx\| = \lambda\|x\|$ for each $x \in \mathcal{K}_1$.*

Proof. Suppose that T preserves orthogonality and $\dim \mathcal{K}_1 = 1$, the assertion follows from linearity of T. Now, If $\dim \mathcal{K}_1 \geq 2$ and there exist $x, y \in X - \{0\}$ such that $\|Tx\| = \alpha\|x\|$ and $\|Ty\| = \beta\|y\|$ for some $\alpha < \beta$, then x and y must be linearly independent. Now we show that x and y cannot be orthogonal. In contrary, let $x[\perp]_1y$. Put $u = \frac{x}{\|x\|}$ and $v = \frac{y}{\|y\|}$. Then we get

$$[u, v]_1 = \Big[\frac{x}{\|x\|}, \frac{y}{\|y\|}\Big]_1 = 0;$$

so $u[\perp]_1 v$. A straightforward computation shows that $(J_1u+v)[\perp]_1(u-J_1v)$, whence $Tu[\perp]_2Tv$ and $T(J_1u+v)[\perp]_2T(u-J_1v)$. On the other hand,

$$\begin{aligned}\big[T(J_1u+v),T(u-J_1v)\big]_2 &= \big[TJ_1u+Tv,Tu-TJ_1v\big]_2\\ &= \big[J_2Tu+Tv,Tu-J_2Tv\big]_2\\ &= \|Tu\|^2-\|Tv\|^2\\ &= \alpha^2-\beta^2\neq 0.\end{aligned}$$

That is a contradiction. So, we can assume that x and y are linearly independent and nonorthogonal. Put

$$z=J_1x-\frac{y}{[y,x]_1}\|x\|^2\neq 0,\quad w=\frac{z}{\|z\|}.$$

Obviously, $z[\perp]_1x$ and $w[\perp]_1u$. Also $J_1w+u[\perp]_1w-J_1u$, whence $Tz[\perp]_2Tx$, and $Tw[\perp]_2Tu$ and $T(J_1w+u)[\perp]_2T(w-J_1v)$. From equalities

$$\|z\|^2+\|x\|^2=\|J_1x-z\|^2,\quad \|Tz\|^2+\|Tx\|^2=\|T(J_1x)-Tz\|^2=\|J_2(Tx)-Tz\|^2,$$

we have

$$\|z\|^2=\frac{\|x\|^4\|y\|^2}{|[x,y]_1|^2}-\|x\|^2$$

and

$$\begin{aligned}\|Tz\|^2 &= \frac{\|x\|^4\|Ty\|^2}{|[x,y]_1|^2}-\|Tx\|^2\\ &= \frac{\|x\|^4\beta^2\|y\|^2}{|[x,y]_1|^2}-\alpha^2\|x\|^2\\ &> \frac{\|x\|^4\alpha^2\|y\|^2}{|[x,y]_1|^2}-\alpha^2\|J_1x\|^2 \qquad (\text{since } \|x\|=\|J_1x\|)\\ &> \alpha^2\big(\frac{\|x\|^4\|y\|^2}{|[x,y]_1|^2}-\|J_1x\|^2\big)\\ &> \alpha^2\|z\|^2.\end{aligned}$$

That is, $\frac{\|Tz\|^2}{\|z\|^2}>\alpha^2$.
Moreover, we have

$$\begin{aligned}[T(J_1w+u),T(w-J_1u)]_2 &= [J_2Tw+Tu,Tw-J_2Tu]_2\\ &= [J_2Tw,Tw]_2-[J_2Tw,J_2Tu]_2+[Tu,Tw]_2-[Tu,J_2Tu]_2\\ &= \|Tw\|^2-\|Tu\|^2\\ &= \frac{\|Tz\|^2}{\|z\|^2}-\frac{\|Tx\|^2}{\|x\|^2}>\alpha^2-\alpha^2=0.\end{aligned}$$

It shows that $T(J_1w+u)$ and $T(w-J_1u)$ cannot be orthogonal. It is a contradiction again. Then $\alpha=\beta$ and $\|Tx\|=\alpha\|x\|$ for each $x\in\mathcal{K}_1$.

Conversely, let $x,y\in\mathcal{K}_1$ such that $x[\perp]_1y$, Theorem 2.7 implies that $\|x+J_1y\|=\|x-J_1y\|$. Then for each $\lambda>0$

$$\lambda\|x+J_1y\|=\lambda\|x-J_1y\|.$$

Hence

$$\|Tx+TJ_1y\|=\|Tx-TJ_1y\|.$$

Lemma 3.1 implies that $\|Tx + J_2Ty\| = \|Tx - J_2Ty\|$, so $Tx[\perp]_2Ty$. □

Proposition 3.3. *Let $(\mathcal{K}_1, [\cdot,\cdot]_1)$ and $(\mathcal{K}_2, [\cdot,\cdot]_2)$ be two real Krein spaces, and let $T : \mathcal{K}_1 \to \mathcal{K}_2$ be a nonzero linear mapping such that $T(\mathcal{K}_1^+) = \mathcal{K}_2^+$ and $T(\mathcal{K}_1^-) = \mathcal{K}_2^-$. If $\|x\| \le \|y\|$ implies that $\|Tx\| \le \|Ty\|$, then T preserves orthogonality.*

Proof. Let $x[\perp]_1y$. Theorem 2.7 implies that $\|x + \lambda J_1y\| \ge \|x\|$ for any $\lambda \in \mathbb{R}$. Thus $\|Tx + \lambda TJ_1y\| \ge \|Tx\|$. Then $\|Tx + \lambda J_2Ty\| \ge \|Tx\|$. From Theorem 2.7, $Tx[\perp]_2Ty$. □

In the following example, we show that the conditions $T(\mathcal{K}_1^+) = \mathcal{K}_2^+$ and $T(\mathcal{K}_1^-) = \mathcal{K}_2^-$ in Theorem 3.2 are necessary.

Example 3.4. *Consider the Krein spaces $(\mathbb{R}^2, [\cdot,\cdot]_1)$ and $(\mathbb{R}^3, [\cdot,\cdot]_2)$, where*

$$[(x_1, x_2), (y_1, y_2)]_1 = x_1y_1 - 2x_2y_2$$

and

$$[(x_1, x_2, x_3), (y_1, y_2, y_3)]_2 = x_1y_1 - 2x_2y_2 + x_3y_3.$$

Let $\mathbb{R}^2 = \mathbb{R}^{2^+}[\oplus]\mathbb{R}^{2^-}$ be the canonical decomposition $\mathbb{R}^2$, where $\mathbb{R}^{2^+} = \{(x, 0)|x \in \mathbb{R}\}$ and $\mathbb{R}^{2^-} = \{(0, y)|y \in \mathbb{R}\}$, and let $\mathbb{R}^3 = \mathbb{R}^{3^+}[\oplus]\mathbb{R}^{3^-}$ be the canonical decomposition $\mathbb{R}^3$, where $\mathbb{R}^{3^+} = \{(x, 0, z)|x, z \in \mathbb{R}\}$ and $\mathbb{R}^{3^-} = \{(0, y, 0)|y \in \mathbb{R}\}$. Then $J_1\big((x_1, x_2)\big) = (x_1, -x_2)$ and $J_2\big((x_1, x_2, x_3)\big) = (x_1, -x_2, x_3)$. Now we define $T : (\mathbb{R}^2, [\cdot,\cdot]_1) \to (\mathbb{R}^3, [\cdot,\cdot]_2)$ by $T\big((x_1, x_2)\big) = (x_1 - x_2, x_2, x_1 + x_2)$. Then

$$\begin{aligned}
\|Tx\|^2 &= [J_2(Tx), Tx]_2 \\
&= [J_2(x_1 - x_2, x_2, x_1 + x_2), ((x_1 - x_2), x_2, x_1 + x_2)]_2 \\
&= [((x_1 - x_2, -x_2, x_1 + x_2), (x_1 - x_2, x_2, x_1 + x_2)]_2 \\
&= (x_1 - x_2)^2 + 2x_2^2 + (x_1 + x_2)^2 \\
&= 2x_1^2 + 4x_2^2 = 2(x_1^2 + 2x_2^2).
\end{aligned}$$

On the other hand,

$$\|(x_1, x_2)\|^2 = [J_1(x_1, x_2), (x_1, x_2)]_1 = [(x_1, -x_2), (x_1, x_2)]_1 = x_1^2 + 2x_2^2.$$

Therefore $\|Tx\| = \sqrt{2}\|x\|$. If $x = (6, 12)$ and $y = (4, 1)$, we observe that $(0, 12) \in \mathbb{R}^{2^-}$ but $T(0, 12) = (-12, 12, 12) \notin \mathbb{R}^{3^-}$. On the other hand $[(6, 12), (4, 1)]_1 = 0$, that is, $x[\perp]_1y$, and

$$[Tx, Ty]_2 = [(6 - 12, 12, 12 + 6), (4 - 1, 1, 1 + 4)]_2 = 48 \ne 0,$$

we conclude that T does not preserve orthogonality.

Lemma 3.5. *Let $(\mathcal{K}, [\cdot,\cdot])$ be a Krein space. Then*

$$\begin{aligned}
[x, y] = {} & \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2) - \frac{1}{2}(\|x^- + y^-\|^2 - \|x^- - y^-\|^2) \\
& + \frac{i}{4}(\|x + iy\|^2 - \|x - iy\|^2) - \frac{i}{2}(\|x^- + iy^-\|^2 - \|x^- - iy^-\|^2).
\end{aligned}$$

Proof. By a straightforward computation, we get

$$\begin{aligned}\|x+y\|^2-\|x-y\|^2 &= 2[Jx,y]+2[Jy,x]\\ &= 2[x^+,y^+]+2[y^+,x^+]-2[x^-,y^-]-2[y^-,x^-]\\ &= 4\mathrm{Re}([x^+,y^+]-4\mathrm{Re}([x^-,y^-]),\end{aligned}$$

similarly, we have

$$\|x^-+y^-\|^2-\|x^--y^-\|^2=-4\mathrm{Re}[x^-,y^-]$$

and

$$\begin{aligned}\|x+iy\|^2-\|x-iy\|^2 &= 2[Jx,iy]+2[J(iy),x]\\ &= -2i[x^+-x^-,y^++y^-]+2i[y^+-y^-,x^++x^-]\\ &= 2i([x^-,y^-]-[x^+,y^+]+[y^+,x^+]-[y^-,x^-])\\ &= 4i\mathrm{Im}[x^-,y^-]-4i\mathrm{Im}[x^+,y^+],\end{aligned}$$

whence

$$\|x^-+iy^-\|^2-\|x^--iy^-\|^2=4i\mathrm{Im}[x^-,y^-].$$

Therefore

$$\begin{aligned}\frac{1}{4}(\|x+y\|^2-\|x-y\|^2)-\frac{1}{2}(\|x^-+y^-\|^2-\|x^--y^-\|^2)+\frac{i}{4}(\|x+iy\|^2-\|x-iy\|^2)&\\ -\frac{i}{2}(\|x^-+iy^-\|^2-\|x^--iy^-\|^2) &= \frac{1}{4}(4\mathrm{Re}[x^+,y^+]-4\mathrm{Re}[x^-,y^-])\\ &\quad -\frac{1}{2}(-4\mathrm{Re}[x^-,y^-])-\frac{i}{2}(4i\mathrm{Im}[x^-,y^-])\\ &\quad +\frac{i}{4}(4i\mathrm{Im}[x^-,y^-]-4i\mathrm{Im}[x^+,y^+])\\ &= \mathrm{Re}[x^+,y^+]+\mathrm{Im}[x^+,y^+]\\ &\quad +\mathrm{Re}[x^-,y^-]+\mathrm{Im}[x^-,y^-]\\ &= [x^+,y^+]+[x^-,y^-]=[x,y].\end{aligned}$$

□

Theorem 3.6. *Let $(\mathcal{K}_1,[\cdot,\cdot]_1)$ and $(\mathcal{K}_2,[\cdot,\cdot]_2)$ be two Krein spaces, and let $T:\mathcal{K}_1\to\mathcal{K}_2$ be a linear mapping. If there exists a $\lambda>0$ such that $\|Tx\|=\lambda\|x\|$ for each $x\in\mathcal{K}_1$ and $T(\mathcal{K}_1^+)=\mathcal{K}_2^+$ and $T(\mathcal{K}_1^-)=\mathcal{K}_2^-$, then*

$$[Tx,Ty]_2=\lambda^2[x,y]_1,\quad x,y\in\mathcal{K}_1.$$

Proof. Let $x, y \in \mathcal{K}_1$. Then by Lemma 3.5, we have

$$\begin{aligned}
\frac{[x,y]_1}{\|x\|\ \|y\|} &= \frac{\lambda^2[x,y]_1}{\lambda^2\|x\|\ \|y\|} \\
&= \frac{\lambda^2}{\|Tx\|\ \|Ty\|}\big(\frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) - \frac{1}{2}(\|x^- + y^-\|^2 - \|x^- - iy^-\|^2\big) \\
&\quad + \frac{i}{4}(\|x+iy\|^2 - \|x-iy\|^2) - \frac{i}{2}(\|x^- + iy^-\|^2 - \|x^- - iy^-\|^2) \\
&= \frac{1}{\|Tx\|\ \|Ty\|}\Big(\frac{1}{4}(\|Tx+Ty\|^2 - \|Tx-Ty\|^2)\Big) \\
&\quad - \frac{1}{2}(\|(Tx)^- + (Ty)^-\|^2 - \|(Tx)^- - (Ty)^-\|^2) \\
&\quad + \frac{i}{4}(\|Tx+iTy\|^2 - \|Tx-iTy\|^2) - \frac{i}{2}(\|(Tx)^- + i(Ty)^-\|^2 \\
&\quad - \|((Tx)^- - i(Ty)^-\|^2) \\
&= \frac{[Tx,Ty]_2}{\|Tx\|\ \|Ty\|} = \frac{[Tx,Ty]_2}{\lambda^2\|x\|\ \|y\|}.
\end{aligned}$$

That is, $[Tx, Ty]_2 = \lambda^2[x, y]_1$. □

Example 3.7. *Consider the Krein spaces* $(\mathbb{R}^2, [\cdot,\cdot]_1)$ *and* $(\mathbb{R}^3, [\cdot,\cdot]_2)$*, where*

$$[(x_1, x_2), (y_1, y_2)]_1 = x_1y_1 - x_2y_2$$

and

$$[(x_1, x_2, x_3), (y_1, y_2, y_3)]_2 = x_1y_1 - 2x_2y_2 + x_3y_3.$$

Let $\mathbb{R}^2 = \mathbb{R}^{2^+}[\oplus]\mathbb{R}^{2^-}$ *be the canonical decomposition* $\mathbb{R}^2$*, where* $\mathbb{R}^{2^+} = \{(x, 0)|x \in \mathbb{R}\}$ *and* $\mathbb{R}^{2^-} = \{(0, y)|y \in \mathbb{R}\}$*, and let* $\mathbb{R}^3 = \mathbb{R}^{3^+}[\oplus]\mathbb{R}^{3^-}$ *be the canonical decomposition* $\mathbb{R}^3$*, where* $\mathbb{R}^{3^+} = \{(x, 0, z)|x, z \in \mathbb{R}\}$ *and* $\mathbb{R}^{3^-} = \{(0, y, 0)|y \in \mathbb{R}\}$*.Then* $J_1\big((x_1, x_2)\big) = (x_1, -x_2)$ *and* $J_2\big((x_1, x_2, x_3)\big) = (x_1, -x_2, x_3)$*.*
If we define $T : (\mathbb{R}^2, [\cdot,\cdot]_1) \to (\mathbb{R}^3, [\cdot,\cdot]_2)$ *by* $T\big((x_1, x_2)\big) = (x_1, x_2, x_1)$*, then*

$$\begin{aligned}
\|T(x_1, x_2)\|^2 = \|(x_1, x_2, x_1)\|^2 &= [J(x_1, x_2, x_1), (x_1, x_2, x_1)] \\
&= [(x_1, -x_2, x_1), (x_1, x_2, x_1)] \\
&= x_1^2 + 2x_2^2 + x_1^2 = 2x_1^2 + 2x_2^2 \\
&= 2(x_1^2 + x_2^2).
\end{aligned}$$

On the other hand, $\|(x_1, x_2)\|^2 = x_1^2 + x_2^2$*. Therefore,* $\|T(x_1, x_2)\| = \sqrt{2}\|(x_1, x_2)\|$*,* $T(\mathbb{R}^{2^+}) = \mathbb{R}^{3^+}$*, and* $T(\mathbb{R}^{2^-}) = \mathbb{R}^{3^-}$*. In addition,*

$$\begin{aligned}
\frac{[Tx,Ty]_2}{\|Tx\|\|Ty\|} &= \frac{x_1y_1 - 2x_2y_2 + x_1y_1}{\sqrt{2x_1^2 + 2x_2^2}\sqrt{2y_1^2 + 2y_2^2}} \\
&= \frac{2(x_1y_1 - x_2y_2)}{\sqrt{2}\sqrt{x_1^2 + x_2^2} \cdot \sqrt{2}\sqrt{y_1^2 + y_2^2}} \\
&= \frac{2(x_1y_1 - x_2y_2)}{2(\sqrt{x_1^2 + x_2^2} \cdot \sqrt{y_1^2 + y_2^2})} \\
&= \frac{[x,y]_1}{\|x\|\ \|y\|}.
\end{aligned}$$

That is, $x[\perp]_1 y$ *if and only if* $Tx[\perp]_2 Ty$.

Remark. *Let* $(\mathcal{K}_1, [\cdot,\cdot]_1)$ *and* $(\mathcal{K}_2, [\cdot,\cdot]_2)$ *be two real Krein spaces, and let* $T : \mathcal{K}_1 \to \mathcal{K}_2$ *be a linear mapping such that* $T(\mathcal{K}_1^+) = \mathcal{K}_2^+$ *and* $T(\mathcal{K}_1^-) = \mathcal{K}_2^-$. *If there exists a* $\lambda > 0$ *such that* $\|Tx\| = \lambda\|x\|$ *for each* $x \in \mathcal{K}_1$, *then Theorems 3.6 implies that* $x[\perp]_1 y$ *if and only if* $Tx[\perp]_2 Ty$ *and from Theorem 2.7,* $[\perp]_1$ *is equivalent with* $\perp_{P\mathcal{K}}$, $\perp_{B\mathcal{K}}$, $\perp_{I\mathcal{K}}$. *Thus* T *preserves the other orthogonalities.*

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References

[1] C. Alsina, J. Sikorska and M. Santos Tomás, *Norm derivatives and characterizations of inner product spaces*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2010.

[2] D. Amir, *Characterization of inner product spaces*, Birkhäuser Verlage, Basel-Boston-Stuttgart, 1986.

[3] T. Ando, *Linear operators on Krein space*, Hokkaido University, Sapporo, Japan, 1979.

[4] T. Ya. Azizov and I. S. Iokhvidov, *Linear operators in spaces with an indefinite metric*, Pure and Applied Mathematics (New York), 1986.

[5] J. Chmieliński, *Linear mappings approximately preserving orthogonality*, J. Math. Anal. Appl. **304** (2005), 158–169.

[6] M. Dehghani, S. M. S. Modarres and M. S. Moslehian, *Positive block matrices on Hilbert and Krein* C^**-modules*, Surv. Math. Appl. **8** (2013), 23–34.

[7] Ju. Ginzburg, *On* J*-contractive operator functions*, Dokl. Akad. Nauk SSSR, **117** (1957), 171–173.

[8] L. Jakóbczyk, *Borchers algebra formulation of an indefinite inner product quantum field theory*, J. Math. Phys. **25** (1984), no. 3, 617–622.

[9] M. Mirzavaziri and M. S. Moslehian, *Orthogonal coxstant mappings in isoceles orthogonal spaces*, Kragujevac J. Math. **29** (2006), 133–140.

[10] M. S. Moslehian and M. Dehghani, *Operator convexity in Krein spaces*, New York J. Math. **20** (2014), 133–144.

[11] L. Pontrjagin, *Hermitian operators in spaces with indefinite metric*, (Russian) Bull. Acad. Sci. URSS. Sr. Math. **8** (1994), 243–280.

[12] A. Zamani and M. S. Moslehian and M. Frank, *Angle preserving mappings*, Z. Anal. Anwend. **34** (2015), no. 4, 485–500.

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