Journal of Mathematical Analysis
ISSN: 2217-3412, URL: www.ilirias.com/jma
Volume 10 Issue 2 (2019), Pages 12-22.

ON SOME PROPERTIES OF LACUNARY STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES IN FUZZY N-NORMED SPACES

MUHAMMED RECAI TÜRKMEN

Abstract. In this study, the application of lacunary statistical convergence of double sequences, previously described in fuzzy normed spaces, has been redefined using fuzzy n-norm. First of all, the concept of lacunary summable has been introduced and then the definition of lacunary statistical convergence and the basic theorems related to this convergence have been introduced for double sequences in fuzzy n-normed spaces. Then the relations between lacunary summable and lacunary statistical convergence have examined and some theorems are given together with the proofs. Furthermore, the conditions of lacunary statistical Cauchy are given for double sequences in fuzzy n-normed spaces. Finally, the theorem that gives the relation between lacunary statistical convergence and lacunary statistical Cauchy has given in fuzzy n-normed spaces.

1. Introduction and Background

The statistical convergence was derived from the convergence of real sequences by Fast [6] and Schoenberg [29]. Mursaleen and Edely [15] extended this study to the double sequences. Fridy and Orhan [9] studied lacunary statistical convergence and then Çakan and Altay [4] pointed out multidimensional analogues of their study.

Matloka [13] was the first scholar who introduced the convergence of a sequence of fuzzy numbers and he showed evidence some basic theorems. In next years, Nanda [16] made studies the sequences of fuzzy numbers again and Şençimen and Pehlivan [31] introduced the notions of statistically convergent sequence and statistically Cauchy sequence in a fuzzy normed linear space. Mohiuddine et al. [14] studied same concepts for double sequences. Recently, on the one hand, Türkmen and Çınar [34] studied lacunary statistical convergence. On the other hand; Türkmen and Dündar [36] scrutinized same concepts for double sequences and Türkmen [33] reinterpreted these works in fuzzy n-normed spaces.

Fuzzy sets are considered with respect to a nonempty base set X of elements of interest. Zadeh [37] said that a fuzzy subset of X is a nonempty subset $\{(x, u(x)) :$

2010 *Mathematics Subject Classification.* 40C99, 40A35, 40G15, 46S40.
Key words and phrases. Statistical convergence, double sequences, lacunary sequence, fuzzy n-normed space.

Submitted November 15, 2018. Published March 25, 2019.
Communicated by Mikail Et.

$x \in X\}$ of $X \times [0,1]$ for some function $u : X \to [0,1]$ and the function u itself is often used for the fuzzy set.

If a fuzzy set u on $\mathbb{R}$ has the certain condition then u is called a fuzzy number. Also, we denote all fuzzy numbers as $L(\mathbb{R})$ and we have written $L^*(\mathbb{R})$ by the set of all non-negative fuzzy numbers.

Now, we recall the basic definitions and concepts (see [1, 2, 3, 5, 6, 7, 8, 10, 11, 12, 14, 15, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 30, 32, 35]).

For $u \in L(\mathbb{R})$, the α level set of u is defined by

$$[u]_\alpha = \begin{cases} \{x \in \mathbb{R} : u(x) \geq \alpha\}, & \text{if } \alpha \in (0,1] \\ supp\, u, & \text{if } \alpha = 0. \end{cases}$$

A partial order $\preceq$ on $L(\mathbb{R})$ is defined by $u \preceq v$ if $u_\alpha^- \leq v_\alpha^-$ and $u_\alpha^+ \leq v_\alpha^+$ for all $\alpha \in [0,1]$.

Arithmetic operations for α-level sets of fuzzy numbers are defined as follows: $u, v \in L(\mathbb{R})$ and $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$ and $[v]_\alpha = [v_\alpha^-, v_\alpha^+]$, $\alpha \in (0,1]$. Then, $[u \oplus v]_\alpha = [u_\alpha^- + v_\alpha^-, u_\alpha^+ + v_\alpha^+]$, $[u \ominus v]_\alpha = [u_\alpha^- - v_\alpha^+, u_\alpha^+ - v_\alpha^-]$, $[u \odot v]_\alpha = [u_\alpha^-.v_\alpha^-, u_\alpha^+.v_\alpha^+]$ and $\left[\tilde{1} \oslash u\right]_\alpha = \left[\frac{1}{u_\alpha^+}, \frac{1}{u_\alpha^-}\right]$, $u_\alpha^- > 0$.

For $u, v \in L(\mathbb{R})$, the supremum metric on $L(\mathbb{R})$ defined as

$$D(u,v) = \sup_{0 \leq \alpha \leq 1} \max\left\{\left|u_\alpha^- - v_\alpha^-\right|, \left|u_\alpha^+ - v_\alpha^+\right|\right\}.$$

If for every $\varepsilon > 0$ there exists a positive integer k_0 such that $D(x_k, x_0) < \varepsilon$ for $k > k_0$ then the sequence $x = (x_k)$ of fuzzy numbers convergent to the fuzzy number x_0, and also the sequence $x = (x_k)$ of fuzzy numbers convergens to levelwise to x_0 if and only if $\lim_{k\to\infty} [x_k]_\alpha = [x_0]_\alpha^-$ and $\lim_{k\to\infty} [x_k]_\alpha = [x_0]_\alpha^+$, where $[x_k]_\alpha = \left[(x_k)_\alpha^-, (x_k)_\alpha^+\right]$ and $[x_0]_\alpha = \left[(x_0)_\alpha^-, (x_0)_\alpha^+\right]$, for every $\alpha \in (0,1)$.

Let X be a vector space over $\mathbb{R}$, $\|.\| : X \to L^*(\mathbb{R})$ and the mappings $L; R$ (respectively, left norm and right norm) : $[0,1] \times [0,1] \to [0,1]$ be symetric, nondecreasing in both arguments and satisfy $L(0,0) = 0$ and $R(1,1) = 1$.

The quadruple $(X, \|.\|, L, R)$ is called fuzzy normed linear space (briefly$(X, \|.\|)$ FNS) and $\|.\|$ a fuzzy norm if the following axioms are satisfied:

(1) $\|x\| = \widetilde{0}$ if and only if $x = 0$,
(2) $\|rx\| = |r| \|x\|$ for $x \in X$, $r \in \mathbb{R}$,
(3) For all $x, y \in X$
  (a) $\|x+y\|(s+t) \geq L(\|x\|(s), \|y\|(t))$, whenever $s \leq \|x\|_1^-$, $t \leq \|y\|_1^-$ and $s + t \leq \|x+y\|_1^-$,
  (b) $\|x+y\|(s+t) \leq R(\|x\|(s), \|y\|(t))$, whenever $s \geq \|x\|_1^-$, $t \geq \|y\|_1^-$ and $s + t \geq \|x+y\|_1^-$.

Let $(X, \|.\|)$ be an FNS. A sequence $(x_n)_{n=1}^\infty$ in X is convergent to $x \in X$ with respect to the fuzzy norm on X and we denote by $x_n \overset{FN}{\to} x$, provided that $(D) - \lim_{n\to\infty} \|x_n - x\| = \widetilde{0}$; i.e., for every $\varepsilon > 0$, there is an $N(\varepsilon) \in \mathbb{N}$ such that $D\left(\|x_n - x\|, \widetilde{0}\right) < \varepsilon$ for all $n \geq N(\varepsilon)$. This means that for every $\varepsilon > 0$, there is an $N(\varepsilon) \in \mathbb{N}$ such that for all $n \geq N(\varepsilon)$, $\sup_{\alpha\in[0,1]} \|x_n - x\|_\alpha^+ = \|x_n - x\|_0^+ < \varepsilon$.

Let $(X, \|.\|)$ be an FNS. A sequence (x_k) in X is statistically convergent to $L \in X$ with respect to the fuzzy norm on X and we denote by $x_n \overset{FS}{\to} x$, provided

that for each $\varepsilon > 0$, we have $\delta\left(\left\{k \in \mathbb{N} : D\left(\|x_k - L\|, \tilde{0}\right) \geq \varepsilon\right\}\right) = 0$. This implies that for each $\varepsilon > 0$, the set

$$K(\varepsilon) = \left\{k \in \mathbb{N} : \|x_k - L\|_0^+ \geq \varepsilon\right\}$$

has natural density zero; namely, for each $\varepsilon > 0$, $\|x_k - L\|_0^+ < \varepsilon$ for almost all k.

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$.

A sequence $x = (x_k)$ in X is said to be lacunary statistically convergent or FS_θ-convergent to $L \in X$ with respect to fuzzy norm on X if for each $\varepsilon > 0$

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\left\{k \in I_r : \|x_k - L\|_0^+ \geq \varepsilon\right\}\right| = 0$$

where $|A|$ denotes the number of elements of the set $A \subseteq \mathbb{N}$. In this case, we write $x_k \xrightarrow{FS_\theta} L$ or $x_k \to L\,(FS_\theta)$ or $FS_\theta - \lim_{k\to\infty} x_k = L$.

A double sequence $x = (x_{jk})$ is said to be Pringsheim's convergent (or P-convergent) if for given $\varepsilon > 0$ there exists an integer N such that $|x_{jk} - l| < \varepsilon$, whenever $j, k > N$. We shall write this as $\lim\limits_{j,k\to\infty} x_{jk} = l$, where j and k tending to infinity independent of each other.

A double sequence $x = (x_{jk})$ is said to be bounded if there exists a positive real number M such that for all $k, j \in \mathbb{N}$, $|x_{jk}| < M$, that is, $\|x\|_\infty = \sup\limits_{k,j} |x_{jk}| < \infty$. We let the set of all bounded double sequences by l_∞.

Let $K \subset \mathbb{N} \times \mathbb{N}$. Let K_{mn} be the number of $(j,k) \in K$ such that $j \leq m$, $k \leq n$. If the sequence $\left\{\frac{K_{mn}}{m.n}\right\}$ has a limit in Pringsheim's sense then we say that K has double natural density and is denoted by

$$\delta_2(K) = \lim_{m,n\to\infty} \frac{K_{mn}}{m.n}.$$

A double sequence $x = (x_{jk})$ is said to be statistically convergent to the number l if for each $\varepsilon > 0$, the set $\{(j,k) : j \leq m \text{ and } k \leq n,\ |x_{jk} - l| \geq \varepsilon\}$ has double natural density zero. In this case, we write $st_2 - \lim\limits_{j,k} x_{jk} = l$.

Let $(X, \|.\|)$ be an FNS. Then a double sequence (x_{jk}) is said to be convergent to $x \in X$ with respect to the fuzzy norm on X if for every $\varepsilon > 0$ there exist a number $N = N(\varepsilon)$ such that

$$\|x_{jk} - x\|_0^+ < \varepsilon, \text{ for all } j, k \geq N.$$

In this case, we write $x_{jk} \xrightarrow{FN_2} x$.

Let $(X, \|.\|)$ be an FNS. A double sequence (x_{jk}) is said to be statistically convergent to $x \in X$ with respect to the fuzzy norm on X if for every $\varepsilon > 0$,

$$\delta_2\left(\left\{(j,k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk} - x\|_0^+ \geq \varepsilon\right\}\right) = 0.$$

Namely, for each $\varepsilon > 0$, $\|x_{jk} - x\|_0^+ < \varepsilon$ for almost all j, k. In this case, we write $FS_2 - \lim \|x_{jk} - x\|_0^+ = 0$ or $x_{jk} \xrightarrow{FS_2} x$.

Let $(X, \|.\|)$ be an FNS. Then a double sequence (x_{jk}) is said to be statistically Cauchy with respect to the fuzzy norm on X if for every $\varepsilon > 0$, there exist $N(\varepsilon)$

and $M(\varepsilon)$ such that for all $j,p \geq N$ and $k,q \geq M$,

$$\delta_2\left(\left\{(j,k) \in \mathbb{N}\times\mathbb{N} : j \leq n \text{ and } k \leq m,\ \|x_{jk} - x_{pq}\|_0^+ \geq \varepsilon\right\}\right) = 0.$$

The double sequence $\theta_2 = \{(k_r, j_u)\}$ is called double lacunary sequence if there exist two increasing sequence of integers such that $k_0 = 0,\ h_r = k_r - k_{r-1} \to \infty$ and $j_0 = 0, \bar{h}_u = j_u - j_{u-1} \to \infty$, as $r,u \to \infty$.

We use following notations in the sequel: $k_{ru} = k_r j_u,\ h_{ru} = h_r \bar{h}_u$, $I_{ru} = \{(k,j) : k_{r-1} < k \leq k_r \text{ and } j_{u-1} < j \leq j_u\},\ q_r = \frac{k_r}{k_{r-1}}$ and $\bar{q}_u = \frac{j_u}{j_{u-1}}$.

Let θ_2 be a double lacunary sequence. The double sequence $x = (x_{kj})$ is S''_{θ_2}-convergent to L provided that for every $\varepsilon > 0$,

$$P - \lim_{r,u\to\infty} \frac{1}{h_{ru}} \left|\{(k,j) \in I_{ru} : |x_{kj} - L| \geq \varepsilon\}\right| = 0.$$

In this case, write $S''_{\theta_2} - \lim x = L$ or $x_{kj} \longrightarrow L\left(S''_{\theta_2}\right)$.

A double sequence $x = (x_{mn})$ in X is said to be lacunary statistically convergent or FS_{θ_2}-convergent to $L \in X$ with respect to fuzzy norm on X if for each $\varepsilon > 0$,

$$\lim_{r,u\to\infty} \frac{1}{h_{ru}} \left|\left\{(m,n) \in I_{ru} : \|x_{mn} - L\|_0^+) \geq \varepsilon\right\}\right| = 0.$$

In this case, we write $x_{mn} \to L\,(FS_{\theta_2})$ or $FS_{\theta_2} - \lim_{m,n\to\infty} x_{mn} = L$ or $x_{mn} \overset{FS_{\theta_2}}{\longrightarrow} L$

Let X be a real linear space of dimension d, where $2 \leq d < \infty$. Let $\|\cdot,\cdot,...,\cdot\| : X^n \longrightarrow L^*(\mathbb{R})$ and the mappings $L; R$ (respectively, left norm and right norm) $: [0,1]\times[0,1] \to [0,1]$ be symetric, nondecreasing in both arguments and satisfy $L(0,0) = 0$ and $R(1,1) = 1$ then the quadruple $(X, \|\cdot,\cdot,...,\cdot\|, L, R)$ is called fuzzy n-normed linear space (briefly $(X, \|\cdot,\cdot,...,\cdot\|)$ $FnNS$) and $\|\cdot,\cdot,...,\cdot\|$ a fuzzy n-norm if the following axioms are satisfied for every $y, x_1, x_2, ..., x_n \in X$ and $s,t \in \mathbb{R}$

fnN_1 : $\|x_1, x_2, ..., x_n\| = \widetilde{0}$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent vectors,

$fnN_2 : \|x_1, x_2, ..., x_n\|$ is invariant under any permutation of $x_1, x_2, ..., x_n$,

$fnN_3 : \|\alpha x_1, x_2, ..., x_n\| = |\alpha|\,\|x_1, x_2, ..., x_n\|$ for all $\alpha \in \mathbb{R}$,

$fnN_4 : \|x_1 + y, x_2, ..., x_n\|(s+t) \geq L(\|x_1, x_2, ..., x_n\|(s), \|y, x_2, ..., x_n\|(t))$ whenever $s \leq \|x_1, x_2, ..., x_n\|_1^-$, $t \leq \|y, x_2, ..., x_n\|_1^-$ and $s + t \leq \|x_1 + y, x_2, ..., x_n\|_1^-$,

$fnN_5 : \|x_1 + y, x_2, ..., x_n\|(s+t) \leq R(\|x_1, x_2, ..., x_n\|(s), \|y, x_2, ..., x_n\|(t))$ whenever $s \geq \|x_1, x_2, ..., x_n\|_1^-$, $t \geq \|y, x_2, ..., x_n\|_1^-$ and $s + t \geq \|x_1 + y, x_2, ..., x_n\|_1^-$, where $[\|x_1, x_2, ..., x_n\|]_\alpha = \left[\|x_1, x_2, ..., x_n\|_\alpha^-, \|x_1, x_2, ..., x_n\|_\alpha^+\right]$ for $x_1, x_2, ..., x_n \in X, 0 \leq \alpha \leq 1$ and $\inf_{\alpha\in[0,1]} \|x_1, x_2, ..., x_n\|_\alpha^- > 0$. Hence the norm $\|\cdot,\cdot,...,\cdot\|$ is called fuzzy n-norm on X and pair $(X, \|\cdot,\cdot,...,\cdot\|)$ is called fuzzy n-normed space.

Let $(X, \|\cdot,\cdot,...,\cdot\|)$ be fuzzy n-normed space. A sequence $\{x_k\}$ in X is said to be statistically convergent to an element $x \in X$ with respect to the fuzzy n-norm on X if for every $\varepsilon > 0$ and for every $z_2, z_3, ..., z_n \neq 0$, $z_2, z_3, ..., z_n \in X$, we have $\delta\left(\left\{k \in \mathbb{N} : D\left(\|x_k - x, z_2, z_3, ..., z_n\|, \widetilde{0}\right) \geq \varepsilon\right\}\right) = 0$.

2. Main Results

In this section, we introduce the concepts of lacunary summable, lacunary statistically convergence and lacunary statistically Cauchy sequence in fuzzy n-normed spaces. Also, we investigate some properties and relationships between these concepts.

Throughout the paper, we consider $(X, \|\cdot, ..., \cdot\|)$ be an FnNS and $\theta_2 = \{(k_r, j_u)\}$ be a double lacunary sequence. And also we will get $z_2, z_3, ..., z_n \in X$.

Definition 2.1. *A double sequence $x = (x_{mn})_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ in X is said to be lacunary summable with respect to fuzzy n-norm on X if there is an $L \in X$ such that*

$$\lim_{r,u\to\infty} \frac{1}{h_{ru}} \left(\sum_{(m,n)\in I_{ru}} D\left(\|x_{mn} - L, z_2, z_3, ..., z_n\|, \tilde{0}\right) \right) = 0.$$

In this case, we write $x_{mn} \to L\left((N_{\theta_2})_{FnN}\right)$ or $x_{mn} \overset{(N_{\theta_2})_{FnN}}{\longrightarrow} L$ and $(N_{\theta_2})_{FnN} = \{(x_{mn}) : \lim_{r,u\to\infty} \frac{1}{h_{ru}} (\sum_{(m,n)\in I_{ru}} D(\|x_{mn} - L, z_2, z_3, ..., z_n\|, \tilde{0})) = 0$, for some $L\}$

Definition 2.2. *A double sequence $x = (x_{mn})$ in X is said to be lacunary statistically convergent or FnS_{θ_2}-convergent to $L \in X$ with respect to fuzzy n-norm on X if for each $\varepsilon > 0$*

$$\lim_{r,u\to\infty} \frac{1}{h_{ru}} \left|\left\{(m,n) \in I_{ru} : D\left(\|x_{mn} - L, z_2, z_3, ..., z_n\|, \tilde{0}\right) \geq \varepsilon\right\}\right| = 0. \tag{2.1}$$

In this case, we write $FnS_{\theta_2} - \lim_{m,n\to\infty} x_{mn} = L$ or $x_{mn} \to L\left(FnS_{\theta_2}\right)$ or $x_{mn} \overset{FnS_{\theta_2}}{\longrightarrow} L$. This implies that, for each $\varepsilon > 0$, the set

$$K(\varepsilon) = \left\{(m,n) \in I_{ru} : \|x_{mn} - L, z_2, z_3, ..., z_n\|_0^+ \geq \varepsilon\right\}$$

has natural density zero, namely, for each $\varepsilon > 0$, $\|x_{mn} - L, z_2, z_3, ..., z_n\|_0^+ < \varepsilon$ for almost all (m,n). In terms of neighborhoods, we have $x_{mn} \overset{FnS_{\theta_2}}{\longrightarrow} L$ if for every $\varepsilon > 0$,

$$\delta_2\left(\{(m,n) \in I_{ru} : x_{mn} \notin \mathcal{N}_L(\varepsilon, 0)\}\right) = 0,$$

that is, for each $\varepsilon > 0$, $(x_{mn}) \in \mathcal{N}_L(\varepsilon, 0)$ for almost all (m,n).

A useful interpretation of the above definition is the following;

$$x_{mn} \overset{FnS_{\theta_2}}{\longrightarrow} L \Leftrightarrow FnS_{\theta_2} - \lim \|x_{mn} - L, z_2, z_3, ..., z_n\|_0^+ = 0.$$

Note that $FnS_{\theta_2} - \lim \|x_{mn} - L, z_2, z_3, ..., z_n\|_0^+ = 0$ implies that $FnS_{\theta_2} - \lim \|x_{mn} - L, z_2, z_3, ..., z_n\|_\alpha^- = FnS_{\theta_2} - \lim \|x_{mn} - L, z_2, z_3, ..., z_n\|_\alpha^+ = 0$, for each $\alpha \in [0,1]$, since $0 \leq \|x_{mn} - L, z_2, z_3, ..., z_n\|_\alpha^- \leq \|x_{mn} - L, z_2, z_3, ..., z_n\|_\alpha^+ \leq \|x_{mn} - L, z_2, z_3, ..., z_n\|_0^+$ holds for every $m, n \in \mathbb{N}$ and for each $\alpha \in [0,1]$.

The set of all lacunary statistically convergent double sequence with respect to fuzzy n-norm on X will be denoted by
$FnS_{\theta_2} = \{x : \text{for some } L,\ FnS_{\theta_2} - \lim x = L\}$.

Theorem 2.1. *We have the following statements for every $x = (x_{mn})$ double sequences:*

(i) $x_{mn} \to L\left((N_{\theta_2})_{FnN}\right) \Rightarrow x_{mn} \to L\left(FnS_{\theta_2}\right)$.

(ii) $(N_{\theta_2})_{FnN}$ is a proper subset of FnS_{θ_2}.

Proof. (i) If $x_{mn} \to L\left((N_{\theta_2})_{FnN}\right)$, then for given $\varepsilon > 0$

$$\begin{aligned} & \sum_{(m,n)\in I_{ru}} D\left(\|x_{mn} - L, z_2, z_3, ..., z_n\|, \tilde{0}\right) \\ \geq & \sum_{\substack{(m,n)\in I_{ru} \\ D\left(\|x_{mn}-L,z_2,z_3,...,z_n\|,\tilde{0}\right)>\varepsilon}} D\left(\|x_{mn} - L, z_2, z_3, ..., z_n\|, \tilde{0}\right) \\ \geq & \ \varepsilon.\left|\left\{(m,n) \in I_{ru} : D\left(\|x_{mn} - L, z_2, z_3, ..., z_n\|, \tilde{0}\right) \geq \varepsilon\right\}\right|. \end{aligned}$$

Therefore, we have

$$\lim_{r,u\to\infty} \frac{1}{h_{ru}} \left|\left\{(m,n) \in I_{ru} : D\left(\|x_{mn}-L,z_2,z_3,...,z_n\|,\tilde{0}\right)\geq\varepsilon\right\}\right| = 0.$$

This implies that $x_{mn} \to L\left(FnS_{\theta_2}\right)$.

(ii) In order to indicate that the inclusion $(N_{\theta_2})_{FnN} \subseteq FnS_{\theta_2}$ in (i) is proper, let a double lacunary sequence θ_2 be given and define a sequence $x = (x_{mn})$ as follows:

$$x_{mn} = \begin{cases} (m,n), \text{ if } k_{r-1} < m < k_{r-1} + \left[\sqrt{h_r}\right], j_{u-1} < n < j_{u-1} + \left[\sqrt{h_u}\right], \\ (0,0) \quad \text{, otherwise.} \end{cases}$$

for r,u=1,2,... . Note that, $x = (x_{mn})$ is not bounded. We have, for every $\varepsilon > 0$ and for each $x \in X$,

$$\begin{aligned} & \frac{1}{h_r\bar{h}_u} \left|\left\{(m,n) \in I_{ru} : D\left(\|x_{mn} - 0, z_2, z_3, ..., z_n\|, \tilde{0}\right) \geq \varepsilon\right\}\right| \\ = & \frac{\left[\sqrt{h_r}\right]\left[\sqrt{\bar{h}_u}\right]}{h_r\bar{h}_u} \to 0, \quad \text{as} \quad r, u \to \infty. \end{aligned}$$

That is, $x_{mn} \to 0\left(FnS_{\theta_2}\right)$. On the other hand

$$\begin{aligned} & \frac{1}{h_r\bar{h}_u} \sum_{(m,n)\in I_{ru}} D\left(\|x_{mn} - 0, z_2, z_3, ..., z_n\|, \tilde{0}\right) \\ = & \frac{1}{h_r\bar{h}_u} \sum_{(m,n)\in I_{ru}} \|x_{mn}, z_2, z_3, ..., z_n\|_0^+ \\ = & \frac{1}{h_r\bar{h}_u} \cdot \frac{\left[\sqrt{h_r}\right].\left(\left[\sqrt{h_r}\right]+1\right)\left[\sqrt{\bar{h}_u}\right].\left(\left[\sqrt{\bar{h}_u}\right]+1\right)}{4} \\ \to & \frac{1}{4} \neq 0. \end{aligned}$$

Hence, $x_{mn} \nrightarrow 0\left((N_{\theta_2})_{FnN}\right)$. □

Theorem 2.2. *Let (x_{mn}) be a double sequence. Then, $(x_{mn}) \in l_\infty$ and $x_{mn} \to L\left(FnS_{\theta_2}\right) \Rightarrow x_{mn} \to L\left((N_{\theta_2})_{FnN}\right)$.*

Proof. Suppose that $x \in l_\infty$ and $x_{mn} \to L\,(FnS_{\theta_2})$. Then, we say that

$$D\left(\|x_{mn} - L, z_2, z_3, ..., z_n\|, \tilde{0}\right) < M,$$

for all (m,n). Given $\varepsilon > 0$, we get

$$\begin{aligned}
& \frac{1}{h_{ru}} \sum_{(m,n)\in I_{ru}} D\left(\|x_{mn} - L, z_2, z_3, ..., z_n\|, \tilde{0}\right) \\
= & \frac{1}{h_{ru}} \sum_{\substack{(m,n)\in I_{ru} \\ D\left(\|x_{mn}-L,z_2,z_3,...,z_n\|,\tilde{0}\right)\geq\varepsilon}} D\left(\|x_{mn} - L, z_2, z_3, ..., z_n\|, \tilde{0}\right) \\
+ & \frac{1}{h_{ru}} \sum_{\substack{(m,n)\in I_{ru} \\ D\left(\|x_{mn}-L,z_2,z_3,...,z_n\|,\tilde{0}\right)<\varepsilon}} D\left(\|x_{mn} - L, z_2, z_3, ..., z_n\|, \tilde{0}\right) \\
\leq & \frac{M}{h_{ru}} \cdot \left|\left\{(m,n) \in I_{ru} : D\left(\|x_{mn} - L, z_2, z_3, ..., z_n\|, \tilde{0}\right) \geq \varepsilon\right\}\right| + \varepsilon.
\end{aligned}$$

Hence, $x_{mn} \to L\,((N_{\theta_2})_{FnN})$. □

Theorem 2.3. *Let θ_2 be a double lacunary sequence. Then, $FnS_{\theta_2} \cap l_\infty = (N_{\theta_2})_{FnN} \cap l_\infty$.*

Proof. This follows from consequences Theorem 2.1 and Theorem 2.2. □

Theorem 2.4. *Let (x_{mn}) be a double sequence. If $\liminf_r q_r > 1$ and $\liminf_u \bar{q}_u > 1$, then $FnS_2 - \lim_{m,n\to\infty} x_{mn} = L$ implies $FnS_{\theta_2} - \lim_{m,n\to\infty} x_{mn} = L$.*

Proof. Suppose that $\liminf_r q_r > 1$ and $\liminf_u \bar{q}_u > 1$. Then, there exist $\delta > 0$, $\tau > 0$ such that $q_r > 1 + \delta$ for sufficiently large r and $\bar{q}_u > 1 + \tau$ for sufficiently large u which implies that $\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}$ and $\frac{\bar{h}_u}{j_u} \geq \frac{\tau}{1+\tau}$. If $x_{mn} \to L\,(FnS_2)$, then for every $\varepsilon > 0$ and sufficiently large r and u, we have

$$\begin{aligned}
& \frac{1}{k_{ru}} \left|\left\{m \leq k_r \text{ and } n \leq j_u : D\left(\|x_{mn} - L, z_2, z_3, ..., z_n\|, \tilde{0}\right) \geq \varepsilon\right\}\right| \\
\geq & \frac{1}{k_{ru}} \left|\left\{(m,n) \in I_{ru} : D\left(\|x_{mn} - L, z_2, z_3, ..., z_n\|, \tilde{0}\right) \geq \varepsilon\right\}\right| \\
\geq & \left(\tfrac{\delta}{1+\delta}\right)\left(\tfrac{\tau}{1+\tau}\right)\tfrac{1}{h_{ru}} \left|\left\{(m,n) \in I_{ru} : D\left(\|x_{mn} - L, z_2, z_3, ..., z_n\|, \tilde{0}\right) \geq \varepsilon\right\}\right|.
\end{aligned}$$

So this result indicates that the double sequence $x = (x_{mn})$ is lacunary statistically convergence with respect to fuzzy n-norm. □

Theorem 2.5. *Let (x_{mn}) be a double sequence. If $\limsup_r q_r < \infty$ and $\liminf_u \bar{q}_u < \infty$, then $FnS_{\theta_2} - \lim_{m,n\to\infty} x_{mn} = L$ implies $FnS_2 - \lim_{m,n\to\infty} x_{mn} = L$.*

Proof. If $\limsup_r q_r < \infty$ and $\liminf_u \bar{q}_u < \infty$, then there is an $M, N > 0$ such that $q_r < M$ and $\bar{q}_u < N$, for all r, u. Suppose that $x_{mn} \to L\,(FnS_{\theta_2})$ and let

$$FnN_{ru} = \left|\left\{(m,n) \in I_{ru} : D\left(\|x_{mn} - L, z_2, z_3, ..., z_n\|, \tilde{0}\right) \geq \varepsilon\right\}\right|.$$

By (2.1), given $\varepsilon > 0$, there is an $r_0, u_0 \in \mathbb{N}$ such that $\frac{FnN_{ru}}{h_{ru}} < \varepsilon$ for all $r > r_0$, $u > u_0$.

Now, let $H = \max\{FnN_{ru} : 1 \leq r \leq r_0, 1 \leq u \leq u_0\}$ and let t and v be any integers satisfying $k_{r-1} < t \leq k_r$ and $j_{u-1} < v \leq j_u$. Then, we write

$$\begin{aligned}
& \frac{1}{tv}\left|\left\{k \leq t, j \leq v : D\left(\|x_{kj} - L, z_2, z_3, ..., z_n\|, \tilde{0}\right) \geq \varepsilon\right\}\right| \\
\leq\ & \frac{1}{k_{r-1}j_{u-1}}\left|\left\{k \leq k_r, j \leq j_u : D\left(\|x_{kj} - L, z_2, z_3, ..., z_n\|, \tilde{0}\right) \geq \varepsilon\right\}\right| \\
=\ & \frac{1}{k_{r-1}j_{u-1}}\{FnN_{11}+FnN_{12}+FnN_{21}+FnN_{22}+...+FnN_{r_0u_0}+...+FnN_{ru}\} \\
\leq\ & \tfrac{H}{k_{r-1}j_{u-1}}r_0u_0+\tfrac{1}{k_{r-1}j_{u-1}}\{h_{r_0}\bar{h}_{u_0+1}\tfrac{FnN_{r_0,u_0+1}}{h_{r_0}\bar{h}_{u_0+1}}+h_{r_0+1}\bar{h}_{u_0}\tfrac{FnN_{r_0+1,u_0}}{h_{r_0+1}\bar{h}_{u_0}}+...+h_r\bar{h}_u\tfrac{FnN_{ru}}{h_r\bar{h}_u}\} \\
\leq\ & \frac{r_0u_0H}{k_{r-1}j_{u-1}} + \frac{1}{k_{r-1}j_{u-1}}(\sup_{\substack{r>r_0\\u>u_0}}\tfrac{FN_{ru}}{h_r\bar{h}_u})\{h_{r_0}\bar{h}_{u_0+1}+h_{r_0+1}\bar{h}_{u_0}+...+h_r\bar{h}_u\} \\
\leq\ & \frac{r_0u_0H}{k_{r-1}j_{u-1}} + \varepsilon\frac{(k_r - k_{r_0})(j_u - j_{u_0})}{k_{r-1}j_{u-1}} \\
\leq\ & \frac{r_0u_0H}{k_{r-1}j_{u-1}} + \varepsilon q_r\bar{q}_u \\
\leq\ & \frac{r_0u_0H}{k_{r-1}j_{u-1}} + \varepsilon MN.
\end{aligned}$$

Hence, we have $FnS_2 - \lim x = L$. □

Theorem 2.6. *Let θ_2 be a double lacunary sequence. If $1 < \lim_r \inf q_r \leq \lim_r \sup q_r < \infty$ and $1 < \lim_u \inf \bar{q}_u \leq \lim_u \sup \bar{q}_u < \infty$, then $FnS_2 = FnS_{\theta_2}$.*

Proof. This follows from Theorem 2.4 and Theorem 2.5. □

Theorem 2.7. *Let $(X, \|\cdot, ..., \cdot\|)$ be an FnNS and θ_2 be a double lacunary sequence. (x_{mn}) and (y_{mn}) be double sequences in X such that $x_{mn} \to L_1\,(FnS_{\theta_2})$ and $y_{mn} \to L_2\,(FnS_{\theta_2})$, where $L_1, L_2 \in X$. Then, we have the following;*

(i) $(x_{mn} + y_{mn}) \to L_1 + L_2\,(FnS_{\theta_2})$,

(ii) $\beta x_{mn} \to \beta L_1\,(FnS_{\theta_2})$, $(\beta \in \mathbb{R})$,

(iii) $FnS_{\theta_2} - \lim \|x_{mn}, z_2, z_3, ..., z_n\| = \|L_1, z_2, z_3, ..., z_n\|$.

Proof. (i) Assume that $x_{mn} \to L_1\,(FnS_{\theta_2})$ and $y_{mn} \to L_2\,(FnS_{\theta_2})$. Since $\|\cdot, \cdot, ..., \cdot\|_0^+$ is a n-norm in the usual sense, we get

$$\begin{aligned}
&\|(x_{mn} + y_{mn}) - (L_1 + L_2), z_2, z_3, ..., z_n\|_0^+ \leq \\
&\qquad \|x_{mn} - L_1, z_2, z_3, ..., z_n\|_0^+ + \|y_{mn} - L_2, z_2, z_3, ..., z_n\|_0^+
\end{aligned} \tag{2.2}$$

for all $(m, n) \in I_{ru}$. Now, let us write

$$\begin{aligned}
K(\varepsilon) &= \left\{(m, n) \in I_{ru} : \|(x_{mn} + y_{mn}) - (L_1 + L_2), z_2, z_3, ..., z_n\|_0^+ \geq \varepsilon\right\}, \\
K_1(\varepsilon) &= \left\{(m, n) \in I_{ru} : \|x_{mn} - L_1, z_2, z_3, ..., z_n\|_0^+ \geq \frac{\varepsilon}{2}\right\} \text{ and} \\
K_2(\varepsilon) &= \left\{(m, n) \in I_{ru} : \|y_{mn} - L_2, z_2, z_3, ..., z_n\|_0^+ \geq \frac{\varepsilon}{2}\right\}.
\end{aligned}$$

From (2.2) that $K(\varepsilon) \subseteq K_1(\varepsilon) \cup K_2(\varepsilon)$. Now, by assumption, we have $\delta_2(K_1(\varepsilon)) = \delta_2(K_2(\varepsilon)) = 0$. This yields $\delta_2(K(\varepsilon)) = 0$ which completes the proof.

(ii) It is obvious.

(iii) Since $\|\cdot, \cdot, ..., \cdot\|_\alpha^-$ and $\|\cdot, \cdot, ..., \cdot\|_\alpha^+$ are n-norms in the usual sense, we have the inequalities

$0 \leq \left| \|x_{mn}, z_2, z_3, ..., z_n\|_\alpha^- - \|L_1, z_2, z_3, ..., z_n\|_\alpha^- \right| \leq \|x_{mn} - L_1, z_2, z_3, ..., z_n\|_\alpha^-$ and $0 \leq \left| \|x_{mn}, z_2, z_3, ..., z_n\|_\alpha^+ - \|L_1, z_2, z_3, ..., z_n\|_\alpha^+ \right| \leq \|x_{mn} - L_1, z_2, z_3, ..., z_n\|_\alpha^+$ for all $\alpha \in [0,1]$ and $(m,n) \in I_{ru}$.
Hence,

$$\begin{aligned} 0 &\leq \max \left\{ \begin{array}{c} \left| \|x_{mn}, z_2, z_3, ..., z_n\|_\alpha^- - \|L_1, z_2, z_3, ..., z_n\|_\alpha^- \right|, \\ \left| \|x_{mn}, z_2, z_3, ..., z_n\|_\alpha^+ - \|L_1, z_2, z_3, ..., z_n\|_\alpha^+ \right| \end{array} \right\} \\ &\leq \|x_{mn} - L_1, z_2, z_3, ..., z_n\|_\alpha^+ \end{aligned}$$

for all $\alpha \in [0,1]$ and $(m,n) \in I_{ru}$. Taking supremum over $\alpha \in [0,1]$, we get $0 \leq D\left(\|x_{mn}, z_2, z_3, ..., z_n\|, \|L_1, z_2, z_3, ..., z_n\|\right) \leq \|x_{mn} - L_1, z_2, z_3, ..., z_n\|_0^+$.
Hence, we have $FnS_{\theta_2} - \lim \|x_{mn}, z_2, z_3, ..., z_n\| = \|L_1, z_2, z_3, ..., z_n\|$. □

Definition 2.3. *Let θ_2 be a double lacunary sequence. The double sequence $x = (x_{mn})$ is said to be an FnS_{θ_2}-Cauchy double sequence if there exists a double subsequence $\overline{x} = \{x_{\overline{m}_r \overline{n}_u}\}$ of x such that $(\overline{m}_r, \overline{n}_u) \in I_{ru}$ for each (r,u), $FnS_{\theta_2} - \lim_{r,u} x_{\overline{m}_r \overline{n}_u} = L$ and for every $\varepsilon > 0$,*

$$\lim_{r,u} \frac{1}{h_{ru}} \left| \left\{ (m,n) \in I_{ru} : D\left(\|x_{mn} - x_{\overline{m}_r \overline{n}_u}, z_2, z_3, ..., z_n\|, \widetilde{0} \right) \geq \varepsilon \right\} \right| = 0.$$

Theorem 2.8. *The double sequence $x = (x_{mn})$ is lacunary statistically convergent with respect to fuzzy n-norm on X if and only if $x = (x_{mn})$ is lacunary statistical Cauchy sequence.*

Proof. Let $(x_{mn}) \to L\,(FnS_{\theta_2})$ and

$$K^{t,v} = \left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : D\left(\|x_{mn} - L, z_2, z_3, ..., z_n\|, \widetilde{0} \right) < \frac{1}{tv} \right\}$$

for each $(t,v) \in \mathbb{N} \times \mathbb{N}$. We obtain the following

$$K^{t+1,v+1} \subseteq K^{t,v} \quad \text{and} \quad \frac{|K^{t,v} \cap I_{ru}|}{h_{ru}} \to 1, \quad \text{as} \quad r,u \to \infty.$$

This implies that there exist k_1 and l_1 such that $r \geq k_1$ and $u \geq l_1$ and $\frac{|K^{1,1} \cap I_{ru}|}{h_{ru}} > 0$, that is, $K^{1,1} \cap I_{ru} \neq \emptyset$. We next choose $k_2 > k_1$ and $l_2 > l_1$ such that $r > k_2$ and $u > l_2$ implies that $K^{2,2} \cap I_{ru} \neq \emptyset$. Thus, for each pair (r,u) such that $k_1 \leq r < k_2$ and $l_1 \leq u < l_2$. We select $(\overline{m}_r, \overline{n}_u) \in I_{ru}$ such that $(\overline{m}_r, \overline{n}_u) \in K^{r,u} \cap I_{ru}$ that is

$$D\left(\|x_{\overline{m}_r \overline{n}_u} - L, z_2, z_3, ..., z_n\|, \widetilde{0} \right) < 1.$$

In general, we choose $k_{t+1} > k_t$ and $l_{v+1} > l_v$ such that $r > k_{t+1}$ and $u > l_{v+1}$. This implies $K^{t+1,v+1} \cap I_{ru} \neq \emptyset$. Thus, for all (r,u) such that for $k_t \leq r < k_{t+1}$ and $l_v \leq u < l_{v+1}$ choose $(\overline{m_r}, \overline{n_u}) \in I_{ru}$, that is,

$$D\left(\|x_{mn} - L, z_2, z_3, ..., z_n\|, \widetilde{0} \right) < \frac{1}{tv}.$$

Thus, $(\overline{m}_r, \overline{n}_u) \in I_{ru}$ for each pair (r,u) and

$$D\left(\|x_{\overline{m}_r \overline{n}_u} - L, z_2, z_3, ..., z_n\|, \widetilde{0} \right) < \frac{1}{tv}$$

implies $FS_{\theta_2} - \lim_{r,u} x_{\overline{m}_r \overline{n}_u} = L$. Also, for each $\varepsilon > 0$

$$\begin{aligned} & \frac{1}{h_{ru}} \left| \left\{ (m,n) \in I_{ru} : D\left(\left\| x_{mn} - x_{\overline{m}_r \overline{n}_u}, z_2, z_3, ..., z_n \right\|, \widetilde{0} \right) \geq \varepsilon \right\} \right| \\ \leq \quad & \frac{1}{h_{ru}} \left| \left\{ (m,n) \in I_{ru} : D\left(\left\| x_{mn} - L, z_2, z_3, ..., z_n \right\|, \widetilde{0} \right) \geq \frac{\varepsilon}{2} \right\} \right| \\ & + \frac{1}{h_{ru}} \left| \left\{ (m,n) \in I_{ru} : D\left(\left\| x_{\overline{m}_r \overline{n}_u} - L, z_2, z_3, ..., z_n \right\|, \widetilde{0} \right) \geq \frac{\varepsilon}{2} \right\} \right| \end{aligned}$$

Since $x_{mn} \to L\left(FnS_{\theta_2}\right)$ and $FnS_{\theta_2} - \lim_{r,u} x_{\overline{m}_r \overline{n}_u} = L$, it follows that x is an FnS_{θ_2}-Cauchy double sequence.

Now suppose that x is an FnS_{θ_2}-Cauchy double sequence. Then

$$\begin{aligned} & \left| \left\{ (m,n) \in I_{ru} : D\left(\left\| x_{mn} - L, z_2, z_3, ..., z_n \right\|, \widetilde{0} \right) \geq \varepsilon \right\} \right| \\ \leq & \left| \left\{ (m,n) \in I_{ru} : D\left(\left\| x_{mn} - x_{\overline{m}_r \overline{n}_u}, z_2, z_3, ..., z_n \right\|, \widetilde{0} \right) \geq \frac{\varepsilon}{2} \right\} \right| \\ & + \left| \left\{ (m,n) \in I_{ru} : D\left(\left\| x_{\overline{m}_r \overline{n}_u} - L, z_2, z_3, ..., z_n \right\|, \widetilde{0} \right) \geq \frac{\varepsilon}{2} \right\} \right| \end{aligned}$$

Therefore, $x_{mn} \to L\left(FnS_{\theta_2}\right)$. Thus the theorem is proven. □

3. Conclusion

In this study, the definitions of lacunary summable, lacunary statistical convergence, and lacunary statistical Cauchy sequence for double sequences were given in fuzzy n-normed spaces.

We have also shown that the relationship between lacunary statistical convergence and lacunary statistical Cauchy sequence is similar in fuzzy n-normed spaces as in classical n-normed spaces.

In further studies, the ideal convergence of double sequences can be defined and examined in fuzzy n-normed spaces.

References

[1] H. Altınok, Y. Altın, and M. Et, *Lacunary almost statistical convergence of fuzzy numbers*, Thai Journal of Mathematics, 2(2) (2004), 265–274.

[2] T. Bag and S. Samanta, *Fixed point theorems in felbin's type fuzzy normed linear spaces*, J. Fuzzy Math., 16(1) (2008), 243–260.

[3] J. Connor, *The statistical and strong p-cesaro convergence of sequences*, Analysis, 8 (1988), 47–63.

[4] C. Çakan and B. Altay, *Statistically boundedness and statistical core of double sequences*, J Math Anal Appl, 317 (2006), 690–697.

[5] M. Çınar, M. Et, *Generalized weighted statistical convergence of double sequences and applications*, Filomat 30 (2016), 3, 753-762.

[6] H. Fast, *Sur la convergence statistique.*, Colloq. Math., 10 (1951), 142–149.

[7] C. Felbin, *Finite-dimensional fuzzy normed linear space.*, Fuzzy Sets and Systems, 48(2) (1992), 239–248.

[8] J. A. Fridy, *On statistical convergence*, Analysis, 5 (1985), 301–313.

[9] J. A. Fridy and C. Orhan, *Lacunary statistical convergence*, Pasific Journal of Mathematics, 160(1) (1993), 43–51.

[10] ———, *Lacunary statistical summability*, Jour Math.Anal.Appl, 173(2) (1993), 497–504.

[11] B. Hazarika, *Lacunary ideal convergence of multiple sequences*, Journal of the Egyptian Mathematical Society, 24 (2016), 54–59.

[12] A. Katsaras, *Fuzzy topological vector spaces*, Fuzzy sets and systems, 12 (1984), 143–154.

[13] M. Matloka, *Sequences of fuzzy numbers*, Busefal, 28 (1986), 28–37.

[14] S. Mohiuddine, H. Şevli, and M. Cancan, *Statistical convergence of double sequences in fuzzy normed spaces*, Filomat, 26(4) (2012), 673–681.
[15] M. Mursaleen and O. Edely, *Statistical convergence of double sequences*, J.Math.Anal.Appl., 288 (2003), 223–231.
[16] S. Nanda, *On sequences of fuzzy numbers*, Fuzzy Sets Systems, 33 (1989), 123–126.
[17] A. Narayan and S. Vijayabalaji, *Fuzzy n-normed linear space*, International journal of mathematics and mathematical sciences, 24 (2005), 3963–3977.
[18] F. Nuray, *Lacunary statistical convergence of sequences of fuzzy numbers*, Fuzzy Sets and Systems, 99 (1998), 353–355.
[19] F. Nuray and W. Ruckle, *Generalized statistical convergence and convergence free spaces*, J. Math. Anal. Appl., 245 (2000), 513–527.
[20] F. Nuray and E. Savaş, *Statistical convergence of sequences of fuzzy numbers*, Math. Slovaca, 45(3) (1995), 269–273.
[21] R. F. Patterson and E. Savaş, *Lacunary statistical convergence of double sequences*, Mathematical Communications, 10 (2005), 55–61.
[22] A. Pringsheim, *Zur theorie der zweifach unendlichen zahlenfolgen*, Math. Ann., 53 (1900), 289–321.
[23] D. Rath and B. Tripaty, *On statistically convergence and statistically cauchy sequences*, Indian J. Pure Appl. Math., 25(4) (1994), 381–386.
[24] B. Reddy, *Statistical convergence in n-normed spaces.*, International Mathematical Forum, 5(24) (2010), 1185–1193.
[25] B. Reddy and M. Srinivas, *Statistical convergence in fuzzy n-normed spaces*, International Journal of Pure and Applied Mathematics, 104(1) (2015), 29–42.
[26] T. Šalát, *On statistically convergent sequences of real numbers*, Math. Slovaca, 30 (1980), 139–150.
[27] E. Savaş, *On statistical convergent sequences of fuzzy numbers*, Inform. Sci., 277-282 (2001), p. 137.
[28] E. Savaş and M. Mursaleen, *On statistically convergent double sequences of fuzzy numbers*, Inform. Sci., 162 (2004), 183–192.
[29] I. J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly, 66 (1959), 361–375.
[30] H. Steinhaus, *Sur la convergence ordinaire et la convergence asymptotique*, Colloq. Math., 2 (1951), 73–74.
[31] C. Şençimen and S. Pehlivan, *Statistical convergence in fuzzy normed linear spaces*, Fuzzy Sets and Systems, 159 (2008), 361–370.
[32] M. R. Türkmen, *On Lacunary $\mathcal{I}_2$-Convergence Of Double Sequences And Some Properties In Fuzzy n-Normed Space* , AKU J. Sci.Eng, 18(3) (2018), 868–877.
[33] ———, *On lacunary statistical convergence and some properties in fuzzy n-normed spaces*, i-manager's Journal on Mathematics, 7(3) (2018), 1–9.
[34] M. R. Türkmen and M. Çınar, *Lacunary statistical convergence in fuzzy normed linear spaces*, Applied and Computational Mathematics, 6(5) (2017), 233–237.
[35] ———, *$\lambda-$statistical convergence in fuzzy normed linear spaces*, Journal of Intelligent and Fuzzy Systems, 34(6) (2018), 4023–4030.
[36] M. R. Türkmen and E. Dündar, *On lacunary statistical convergence of double sequences and some properties in fuzzy normed spaces*, Journal of Intelligent and Fuzzy Systems, 36(2) (2019), 1–8.
[37] L. Zadeh, *Fuzzy sets*, Inform. Contr., 6(5) (1965), 29–44.

Muhammed Recai Türkmen
Department of Mathematics, Faculty of Education, Afyon Kocatepe University, 03200, Afyonkarahisar, Turkey
E-mail address: mrtmath@gmail.com