EXPLICIT SOLUTION TO THE OPERATOR EQUATION
AXD + FX*B = C OVER HILBERT C*-MODULES

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Abstract. In this paper, by using operator matrices, we investigate the explicit solution of the operator equation

AXD + FX*B = C

in the general setting of the adjointable operators between Hilbert C*-modules. This solution is expressed in terms of the Moore-Penrose inverses of the operators A, B, D and F. The obtained results extend and generalize some known operator equations studied previously by a number of mathematicians.

1. Introduction and preliminaries

Matrix equations are very useful in engineering problems, information theory, linear system theory, linear estimation theory, numerical analysis, and other topics. The matrix equation AXD + FX*B = C has been widely used in system theory, such as eigenstructure assignment [7, 12, 16, 18, 19]. Other relevant applications include Lyapunov and Sylvester equations [2, 17]. The generalized Sylvester equation AV + BW = EVJ + R with unknown matrices V and W, has numerous applications in linear systems theory [5], and fault detection [9]. The equation TXS* − SX*T* = A was studied by Yuan for finite matrices [22]. When T equals an identity matrix or identity operator, this equation is reduced to XS* − SX* = A, which was studied by Braden [1] for finite matrices, later, by Djordjevic [4] for the Hilbert space operators. Cveticovic-Ilic [3] investigated the solvability of the operator equation AX + X*C = B for g-invertible Hilbert space operators.

Hilbert C*-modules are natural generalizations of finite-dimensional spaces, Hilbert spaces and C*-algebras. They were first introduced by Kaplansky [10] in 1953, and since then, they proved to be effective tool in the theory of C*-algebra, especially in the study of KK-groups, induced representations, locally compact quantum groups and non-commutative geometry [6, 9, 11]. Compared to the Hilbert space case, there exist some differences when we deal with operators in a general Hilbert C*-module; for instance, a closed topologically complemented submodule of a Hilbert
C*-module may not be orthogonally complemented. By [11, Theorem 3.2], we know that the former deficit can be mended if we restrict our attention to those adjointable operators whose ranges are closed. It is therefore meaningful to put forward a generalized version of the previous results about the equation

$$AXD + FX^*B = C. \tag{1.1}$$

for matrices and bounded linear operators between Hilbert spaces in the context of Hilbert C*-modules. For instance, Xu et al. [21] and Mohammadzadeh Karizaki et al. [13, 14, 15] studied the operator equation $TXS^* - SX^*T^* = A$ under closed range assumption for adjointable Hilbert C*-modules.

In this paper, applying operator matrices we provide a new approach to the study of the operator equation $AXD + FX^*B = C$ in the general setting of the adjointable operators between Hilbert C*-modules. Based on the Moore-Penrose inverses [20] of the associated operators, we propose the necessary and sufficient conditions for the existence of a solution to this equation, and obtain the general expression of the solution in the solvable case.

Throughout the paper, $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert modules over the same C*-algebra $\mathcal{A}$. We denote the set of all adjointable operators from $\mathcal{X}$ to $\mathcal{Y}$ by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ and by $\mathcal{L}(\mathcal{X})$, when $\mathcal{X} = \mathcal{Y}$. For $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, let $\text{ran}(T)$ and $\text{ker}(T)$ be the range of $T$ and the null space of $T$, respectively. The identity operator on $\mathcal{X}$ is denoted by $1_{\mathcal{X}}$ or $1$ if there is no ambiguity.

For the sake of convenience of the reader, here we collect a few facts which will be needed further on.

**Theorem 1.1.** [11, Theorem 3.2] Suppose that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range. Then

- $\text{ker}(T)$ is orthogonally complemented in $\mathcal{X}$, with complement $\text{ran}(T^*)$.
- $\text{ran}(T)$ is orthogonally complemented in $\mathcal{Y}$, with complement $\text{ker}(T^*)$.
- The map $T^* \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ has closed range.

Xu et al. [20] showed that an adjointable operator between two Hilbert $\mathcal{A}$-modules admits a bounded Moore-Penrose inverse if and only if it has closed range. The Moore-Penrose inverse $T^\dagger$ of $T$ is the unique element in $\mathcal{L}(\mathcal{Y}, \mathcal{X})$ which satisfies the following conditions:

$$TT^\dagger T = T, \quad T^\dagger TT^\dagger = T^\dagger, \quad (TT^\dagger)^* = TT^\dagger, \quad (T^\dagger T)^* = T^\dagger.$$

From these conditions we obtain that $(T^\dagger)^* = (T^*)^\dagger$, $TT^\dagger$ and $T^\dagger T$ are orthogonal projections, in the sense that they are self-adjoint idempotent operators. Furthermore, we have

$$\text{ran}(T) = \text{ran}(TT^\dagger), \quad \text{ran}(T^\dagger) = \text{ran}(T^\dagger T) = \text{ran}(T^*), \quad \text{ker}(T) = \text{ker}(T^\dagger T), \quad \text{ker}(T^\dagger) = \text{ker}(TT^\dagger) = \text{ker}(T^*).$$

**Remark.** Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert $\mathcal{A}$-modules, we use the notation $\mathcal{X} \oplus \mathcal{Y}$ to denote the direct sum of $\mathcal{X}$ and $\mathcal{Y}$, which is also a Hilbert $\mathcal{A}$-module whose $\mathcal{A}$-valued inner product is given by

$$\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle,$$

for $x_i \in \mathcal{X}$ and $y_i \in \mathcal{Y}$, $i = 1, 2$. 
**Theorem 1.2.** [15 Theorem 2.5.] Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be Hilbert $A$-modules, $S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $T \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$ be invertible operators and $A \in \mathcal{L}(\mathcal{Y})$. Then the following statements are equivalent:

(a) There exists a solution $X \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$ to the operator equation $TXS^* + SX^*T^* = A$.

(b) $A = A^*$.

If (a) or (b) is satisfied, then any solution to

$$TXS^* + SX^*T^* = A, \quad X \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$$

has the form

$$X = \frac{1}{2}T^{-1}A(S^*)^{-1} - T^{-1}Z(S^*)^{-1},$$

where $Z \in \mathcal{L}(\mathcal{Y})$ satisfies $Z^* = -Z$.

**Remark.** [15 Remark 2.1.] Let $T \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ and $S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ have closed ranges, and $A \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$. Then the equation

$$TXS = A, \quad X \in \mathcal{L}(\mathcal{Y}),$$

(1.2)

has a solution if and only if

$$TT^\dagger AS^\dagger = A.$$

In which case, any solution of (1.2) has the form

$$X = T^\dagger AS^\dagger.$$

We terminate this section with analogues versions of [15 Theorem 2.2.] and [14 Theorem 2.2.]. We will not reproduce here the corresponding proofs which can be easily obtained by slight modifications of the proofs given in these papers.

**Theorem 1.3.** Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be Hilbert $A$-modules. Suppose $S \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$ is an invertible operator and $T \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$ has closed range and $A \in \mathcal{L}(\mathcal{Y})$. Then the following statements are equivalent:

(a) There exists a solution $X \in \mathcal{L}(\mathcal{Z})$ to the operator equation $TXS^* + SX^*T^* = A$.

(b) $A = A^*$ and $(1 - TT^\dagger)A(1 - TT^\dagger) = 0$.

If (a) or (b) is satisfied, then any solution to $TXS^* + SX^*T^* = A$ has the form

$$X = \frac{1}{2}T^\dagger ATT^\dagger(S^*)^{-1} + T^\dagger ZTT^\dagger(S^*)^{-1} + T^\dagger A(1 - TT^\dagger)(S^*)^{-1} + (1 - T^\dagger T)Y(S^*)^{-1},$$

where $Z \in \mathcal{L}(\mathcal{Y})$ satisfies $T^*(Z + Z^*)T = 0$ and $Y \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ is arbitrary.

**Theorem 1.4.** Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be Hilbert $A$-modules, $A, S \in \mathcal{L}(\mathcal{X})$, $T \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ such that $S$ is self-adjoint, both $S$ and $T$ have closed ranges, and $AS^\dagger S = A$ and $T^\dagger S^\dagger S = T^\dagger$. Then the following statements are equivalent:

(a) There exists a solution $X \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ to the operator equation $TXS + SX^*T^* = A$.

(b) $A = A^*$ and $(1 - TT^\dagger)A(1 - TT^\dagger) = 0$.

If (a) or (b) is satisfied, then any solution to $TXS + SX^*T^* = A$ has the form

$$X = T^\dagger AS^\dagger - \frac{1}{2}T^\dagger ATT^\dagger S^\dagger + T^\dagger ZTT^\dagger S^\dagger + V - T^\dagger TVSS^\dagger,$$

where $Z \in \mathcal{L}(\mathcal{X})$ satisfies $T^*(Z + Z^*)T = 0$ and $V \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is arbitrary.
2. Main results

In this section, in the general setting of Hilbert $C^*$-module operators, we provide some new approaches for solving operator equation (1.1). Furthermore, the necessary and sufficient conditions for the existence of a solution of Eq. (1.1), were given and the set of solutions was completely described.

**Theorem 2.1.** Let $X, Y$ be Hilbert $A$-modules, $A, F \in \mathcal{L}(X, Y)$ and $B, D \in \mathcal{L}(Y, X)$ be invertible and $C \in \mathcal{L}(Y)$. If

$$A(B^*)^{-1}C^*(F^*)^{-1}D + F(D^*)^{-1}C^*(A^*)^{-1}B = 2C,$$  

(2.1)

then the operator equation (1.1) has a solution $X \in \mathcal{L}(X)$. In the case any solution of the Eq. (1.1) is represented by

$$X = \frac{1}{4}A^{-1}CD^{-1} + \frac{1}{4}(B^*)^{-1}C^*(F^*)^{-1} + (B^*)^{-1}K^*(F^*)^{-1} - A^{-1}KD^{-1},$$  

(2.2)

where $K \in \mathcal{L}(Y)$ satisfies $A(B^*)^{-1}K^*(F^*)^{-1}D = F(D^*)^{-1}K^*(A^*)^{-1}B$.

**Proof.** Suppose that (2.1) is satisfied. Taking $K = 0$ in (2.2), we have that the operator $X$ defined by (2.2) is the solution of the equation (2.1).

We will now show that if the equation (1.1) has a solution $X \in \mathcal{L}(X)$, then it must be of the form (2.2). Let $T = \begin{bmatrix} A & 0 \\ 0 & B^* \end{bmatrix}: X \oplus X \to Y \oplus Y$, $Y = \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix}$, $S = \begin{bmatrix} F & 0 \\ 0 & D^* \end{bmatrix}: X \oplus X \to Y \oplus Y$ and $N = \begin{bmatrix} 0 & C \\ C^* & 0 \end{bmatrix}: Y \oplus Y \to Y \oplus Y$. Obviously, $T$ and $S$ are invertible.

By $TYS^* + SY^*T^* = N$, we have

$$TYS^* + SY^*T^* = \begin{bmatrix} A & 0 \\ 0 & B^* \end{bmatrix} \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \begin{bmatrix} F^* & 0 \\ 0 & D \end{bmatrix}$$

$$+ \begin{bmatrix} F & 0 \\ 0 & D^* \end{bmatrix} \begin{bmatrix} 0 & X^* \\ X^* & 0 \end{bmatrix} \begin{bmatrix} A^* & 0 \\ 0 & B \end{bmatrix}$$

$$= \begin{bmatrix} 0 & C \\ C^* & 0 \end{bmatrix},$$

or equivalently

$$\begin{bmatrix} 0 & AXD + FX*B \\ B^*XF^* + D^*X^*A^* & 0 \end{bmatrix} = \begin{bmatrix} 0 & C \\ C^* & 0 \end{bmatrix}.$$

Therefore

$$AXD + FX*B = C.$$

By Theorem 1.2 it follows that $Y$ has the following representation

$$Y = \frac{1}{2}T^{-1}N(S^*)^{-1} - T^{-1}Z(S^*)^{-1},$$  

(2.3)
where $Z \in \mathcal{L}(Y \oplus Y)$ satisfies $Z^* = -Z$. Let $Z = \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix}$. From (2.3), we have

$$
\begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} A^{-1} & 0 \\ 0 & (B^*)^{-1} \end{bmatrix} \begin{bmatrix} 0 & C^* \\ C^* & 0 \end{bmatrix} \begin{bmatrix} (F^*)^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix} + \frac{1}{4} A^{-1} \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} \begin{bmatrix} (F^*)^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix}.
$$

It enforces that

$$
X = \frac{1}{2} A^{-1} CD^{-1} - A^{-1} Z_2 D^{-1} = \frac{1}{2} (B^*)^{-1} C^* (F^*)^{-1} - (B^*)^{-1} Z_3 (F^*)^{-1},
$$

and

$$
-A^{-1} Z_1 (F^*)^{-1} = -(B^*)^{-1} Z_4 D^{-1} = 0.
$$

From invertibility of the operators $A$, $B$, $D$ and $F$ it follows that $Z_1 = 0$ and $Z_4 = 0$. On the other hand, from $Z^* = -Z$, we get $Z_2 = -Z_3$. For $K = \frac{1}{2} Z_2$, we have

$$
X = \frac{1}{2} A^{-1} CD^{-1} - 2 A^{-1} KD^{-1} = \frac{1}{2} (B^*)^{-1} C^* (F^*)^{-1} + 2 (B^*)^{-1} K^* (F^*)^{-1},
$$

which implies that

$$
\frac{1}{2} A^{-1} CD^{-1} = \frac{1}{2} (B^*)^{-1} C^* (F^*)^{-1} + 2 (B^*)^{-1} K^* (F^*)^{-1} + 2 A^{-1} KD^{-1}.
$$

Replacing the second part of (2.4) in $FX^* B$ and the third part of it in $AXD$, it can be derived that

$$
A(B^*)^{-1} K^* (F^*)^{-1} D - F(D^*)^{-1} K^* (A^*)^{-1} B = \frac{1}{2} (AXD + FX^* B) - \frac{1}{4} [A(B^*)^{-1} C^* (F^*)^{-1} D + F(D^*)^{-1} C^* (A^*)^{-1} B].
$$

From this relation and the condition (2.1), it can be concluded that

$$
A(B^*)^{-1} K^* (F^*)^{-1} D = F(D^*)^{-1} K^* (A^*)^{-1} B.
$$

On the other hand, applying the third part of (2.4) and also using (2.5), we have

$$
X = A^{-1} CD^{-1} - A^{-1} FX^* BD^{-1}
= A^{-1} CD^{-1} - A^{-1} F \left( \frac{1}{2} F^{-1} CB^{-1} + 2 F^{-1} KB^{-1} \right) BD^{-1}
= \frac{1}{2} A^{-1} CD^{-1} - 2 A^{-1} KD^{-1}
= \frac{1}{4} A^{-1} CD^{-1} + \frac{1}{4} A^{-1} CD^{-1} - 2 A^{-1} KD^{-1}
= \frac{1}{4} A^{-1} CD^{-1} + \frac{1}{4} (B^*)^{-1} C^* (F^*)^{-1} + (B^*)^{-1} K^* (F^*)^{-1}
+ A^{-1} KD^{-1} - 2 A^{-1} KD^{-1}
= \frac{1}{4} A^{-1} CD^{-1} + \frac{1}{4} (B^*)^{-1} C^* (F^*)^{-1} + (B^*)^{-1} K^* (F^*)^{-1} - A^{-1} KD^{-1}.
$$

As a consequence of the above theorem we have the following result.
Corollary 2.2. Let $\mathcal{X}, \mathcal{Y}$ be Hilbert $A$-modules, $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $B \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ be invertible and $C \in \mathcal{L}(\mathcal{Y})$. If

$$A(B^*)^{-1}C^* + C^*(A^*)^{-1}B = 2C,$$

then the operator equation $AX + X^*B = C$ has a solution $X \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$. In the case any solution to

$$AX + X^*B = C, \quad X \in \mathcal{L}(\mathcal{Y}, \mathcal{X}),$$

is represented by

$$X = \frac{1}{4}A^{-1}C + \frac{1}{4}(B^*)^{-1}C^* + (B^*)^{-1}K^* - A^{-1}K,$$

where $K \in \mathcal{L}(\mathcal{Y})$ satisfies $(B^*)^{-1}K^*B^{-1} = A^{-1}K^*(A^*)^{-1}$.

Now, we will consider the more general case when operators $F$ and $D$ have closed ranges.

Theorem 2.3. Suppose $A, F \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $B, D \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ such that $A, B$ are invertible and $F, D$ have closed ranges, and $C \in \mathcal{L}(\mathcal{Y})$ such that

$$A(B^*)^{-1}D^1DC^*(F^*)^1D = -F(D^*|C^*FF|^1A^*)^{-1}B,$$

$$A(B^*)^{-1}C^*(F^*)^1D + F(D^*|C^*A^*)^{-1}B = C.$$

Then the following statements are equivalent:

(a) There exists a solution $X \in \mathcal{L}(\mathcal{X})$ to Eq. (1.1).

(b) $(1 - FF^1)C(1 - D^1D) = 0$.

If (a) or (b) is satisfied, then any solution to Eq. (1.1) has the form

$$X = \frac{1}{2}A^{-1}CD^1 + \frac{1}{2}(B^*)^{-1}C^*(F^*)^1 - \frac{1}{4}A^{-1}FF^1CD^1 - \frac{1}{4}(B^*)^{-1}D^1DC^*(F^*)^1$$

$$- \frac{1}{2}A^{-1}FF^1KD^1 + \frac{1}{2}(B^*)^{-1}D^1DK^*(F^*)^1$$

$$+ \frac{1}{2}(B^*)^{-1}V^*(1 - F^1F) - \frac{1}{2}A^{-1}W^*(1 - DD^1),$$

where $K \in \mathcal{L}(\mathcal{Y})$ and $V, W \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ satisfy

$$A(B^*)^{-1}V^*(1 - F^1F)D = F(1 - DD^1)W(A^*)^{-1}B,$$

$$A(B^*)^{-1}D^1DK^*(F^*)^1D = F(D^*|K^*FF|^1A^*)^{-1}B.$$

Proof. (a) $\Rightarrow$ (b): Suppose that Eq. (1.1) has a solution $X \in \mathcal{L}(\mathcal{X})$. Then

$$(1 - FF^1)C(1 - D^1D) = (1 - FF^1)(AXD + FX^*B)(1 - D^1D)$$

$$= (1 - FF^1)AXB(1 - D^1D) + (1 - FF^1)FX^*B(1 - D^1D) = 0.$$

(b) $\Rightarrow$ (a): Let $T = \begin{bmatrix} F & 0 \\ 0 & D^* \end{bmatrix}: \mathcal{X} \oplus \mathcal{X} \to \mathcal{Y} \oplus \mathcal{Y}$, $\tilde{X} = \begin{bmatrix} 0 & X^* \\ X^* & 0 \end{bmatrix}: \mathcal{X} \oplus \mathcal{X} \to \mathcal{X} \oplus \mathcal{X}$, and $S = \begin{bmatrix} A & 0 \\ 0 & B^* \end{bmatrix}: \mathcal{X} \oplus \mathcal{X} \to \mathcal{Y} \oplus \mathcal{Y}$.

Further, we consider the operator equation $T \tilde{X}S^* + S(\tilde{X})^*T^* = N$, where it can be written in the following form
\[ T\hat{X}S^* + S(\hat{X})^*T^* \]
\[ = \begin{bmatrix} F & 0 \\ 0 & D^* \end{bmatrix} \begin{bmatrix} X^* & 0 \\ 0 & X^* \end{bmatrix} \begin{bmatrix} A^* & 0 \\ 0 & B^* \end{bmatrix} + \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \begin{bmatrix} F^* & 0 \\ 0 & D \end{bmatrix} \]
\[ = \begin{bmatrix} 0 & FX^*B + AXD \\ D^*X^*A^* + B^*XF^* & 0 \end{bmatrix} \]
\[ = \begin{bmatrix} 0 & C \\ C^* & 0 \end{bmatrix}. \]

From this, we find that \( AXD + FX^*B = C \). On the other hand, by condition (b) we have
\[ (1 - TT^\dagger)N(1 - TT^\dagger) = \begin{bmatrix} 0 & (1 - FF^\dagger)C(1 - D^*(D^*)^\dagger) \\ (1 - D^*(D^*)^\dagger)C^*(1 - FF^\dagger) & 0 \end{bmatrix} \]
\[ = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \]

According to Theorem 1.3, the operator equation \( T\hat{X}S^* + S(\hat{X})^*T^* = N \), has the solution
\[ X = \frac{1}{2}T^\dagger NTT^\dagger(S^*)^{-1} + T^\dagger ZTT^\dagger(S^*)^{-1} + T^\dagger N(1 - TT^\dagger)(S^*)^{-1} + (1 - T^\dagger T)Y(S^*)^{-1}, \]
where \( Z \in \mathcal{L}(\mathcal{Y} \oplus \mathcal{Y}) \) satisfies \( T^*(Z + Z^*)T = 0 \) and \( Y \in \mathcal{L}(\mathcal{Y} \oplus \mathcal{Y}, \mathcal{X} \oplus \mathcal{X}) \) is arbitrary. Let us assume that \( Z = \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} \) and \( Y = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix} \). Then from the above solution we obtain
\[ \begin{bmatrix} 0 & X^* \\ X^* & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & F^\dagger CD^*(D^*)^\dagger B^{-1} \\ (D^*)^\dagger C^* FF^\dagger(A^*)^{-1} & 0 \end{bmatrix} \]
\[ + \begin{bmatrix} F^\dagger Z_1 FF^\dagger(A^*)^{-1} & F^\dagger Z_2 D^*(D^*)^\dagger B^{-1} \\ (D^*)^\dagger Z_3 FF^\dagger(A^*)^{-1} & (D^*)^\dagger Z_4 D^*(D^*)^\dagger B^{-1} \end{bmatrix} \]
\[ + \begin{bmatrix} 0 & F^\dagger C(1 - D^*(D^*)^\dagger)B^{-1} \\ (D^*)^\dagger C^*(1 - FF^\dagger)(A^*)^{-1} & 0 \end{bmatrix} \]
\[ + \begin{bmatrix} (1 - F^\dagger F)Y_1(A^*)^{-1} & (1 - F^\dagger F)Y_2 B^{-1} \\ (1 - (D^*)^\dagger D^*)Y_3(A^*)^{-1} & (1 - (D^*)^\dagger D^*)Y_4 B^{-1} \end{bmatrix}. \]

It then turns implies that
\[ F^\dagger Z_1 FF^\dagger(A^*)^{-1} + (1 - F^\dagger F)Y_1(A^*)^{-1} = 0. \quad (2.7) \]
\[ (D^*)^\dagger Z_4 D^*(D^*)^\dagger B^{-1} + (1 - (D^*)^\dagger D^*)Y_4 B^{-1} = 0. \quad (2.8) \]
\[ X^* = \frac{1}{2} F^\dagger CD^*(D^*)^\dagger B^{-1} + F^\dagger Z_2 D^*(D^*)^\dagger B^{-1} \]
\[ + F^\dagger C(1 - D^*(D^*)^\dagger)B^{-1} + (1 - F^\dagger F)Y_2 B^{-1}. \]
\[ X^* = \frac{1}{2} (D^*)^\dagger C^* FF^\dagger(A^*)^{-1} + (D^*)^\dagger Z_4 FF^\dagger(A^*)^{-1} \quad (2.9) \]
Also, by multiplication $F$ on the left and $A^*$ on the right to (2.7), we get $FF^\dagger Z_1 FF^\dagger = 0$.

Also, by multiplication $D^*$ on the left and $B$ on the right to (2.8), we obtain that $D^\dagger DZ_4 D^\dagger D = 0$. Further, utilizing the relation $T^\dagger (Z + Z^*)T = 0$, we observe that

$$F^\dagger Z_1 F = -F^\dagger Z_1^* F, \quad F^\dagger Z_2 D^* = -F^\dagger Z_3 D^*, \quad DZ_3 F = -DZ_3^* F, \quad DZ_4 D^* = -DZ_4^* D^*.$$  

Then the relation $(Z + Z^*)T \in \ker(T^\dagger) = \ker(T^\dagger)$ implies that $T^\dagger (Z + Z^*)T = 0$ and so it yields that

$$F^\dagger Z_1 F = -F^\dagger Z_1^* F, \quad F^\dagger Z_2 D^* = -F^\dagger Z_3 D^*, \quad (D^*)^\dagger Z_3 F = -(D^*)^\dagger Z_2^* F, \quad (D^*)^\dagger Z_4 D^* = -(D^*)^\dagger Z_4^* D^*.$$  

By taking adjoint from (2.9) and (2.10) and setting the two equal to each other, we get

$$\frac{1}{2}(B^*)^{-1} D^\dagger DC^*(F^*)^\dagger + (B^*)^{-1} D^\dagger DZ_3(F^*)^\dagger + (B^*)^{-1} Y_2^* (1 - F^\dagger F)$$

$$= \frac{1}{2} A^{-1}(F^*)^\dagger F^\dagger CD^\dagger + A^{-1}(F^*)^\dagger F^\dagger Z_3^* D^\dagger + A^{-1} CD^\dagger$$

$$+ A^{-1}(F^*)^\dagger F^\dagger CD^\dagger + A^{-1} Y_3^* (1 - (D^*)^\dagger D^*).$$

Take $K = Z_2$, $V = Y_2$ and $W = Y_3$ in the above relation. Using (2.11), it can be deduced that

$$A^{-1} CD^\dagger = \frac{1}{2}(B^*)^{-1} D^\dagger DC^*(F^*)^\dagger + (B^*)^{-1} D^\dagger D K^*(F^*)^\dagger$$

$$+ (B^*)^{-1} (1 - D^\dagger D) C^*(F^*)^\dagger + (B^*)^{-1} V^* (1 - F^\dagger F)$$

$$+ \frac{1}{2} A^{-1}(F^*)^\dagger F^\dagger CD^\dagger + A^{-1}(F^*)^\dagger F^\dagger KD^\dagger$$

$$- A^{-1} W^* (1 - (D^*)^\dagger D^*).$$

On the other hand, let $X$ be any solution of Eq. (1.1), then $AXD = C - FX^* B$.

Put (2.9) in the right side of this equation we get

$$AXD = C - \frac{1}{2} FF^\dagger CD^\dagger (D^*)^\dagger - FF^\dagger KD^* (D^*)^\dagger$$

$$- FF^\dagger C(1 - D^\dagger D^*)^\dagger - F(1 - FF^\dagger)V$$

$$= C - \frac{1}{2} FF^\dagger CD^\dagger D - FF^\dagger KD^\dagger D - FF^\dagger C(1 - D^\dagger D).$$

Set $M = C - \frac{1}{2} FF^\dagger CD^\dagger D - FF^\dagger KD^\dagger D - FF^\dagger C(1 - D^\dagger D)$. In view of the condition (b), it can be concluded that

$$AA^{-1} MD^\dagger D = CD^\dagger D - \frac{1}{2} FF^\dagger CD^\dagger D - FF^\dagger KD^\dagger D$$

$$= C - FF^\dagger C(1 - D^\dagger D) - \frac{1}{2} FF^\dagger CD^\dagger D - FF^\dagger KD^\dagger D$$

$$= M.$$
So, taking into account Remark 1 and (2.12) we observe that
\[ X = \frac{1}{2} A^\dagger CD^\dagger + \frac{1}{4} (B^*)^{-1} D^\dagger DC^*(F^*)^\dagger + \frac{1}{2} (B^*)^{-1} V^*(1 - F^\dagger F) \]
\[ + \frac{1}{4} A^{-1} F^\dagger CD^\dagger + \frac{1}{2} A^{-1} FF^\dagger KD^\dagger \]
\[ + \frac{1}{2} A^{-1} W^*(1 - DD^\dagger) - \frac{1}{2} A^{-1} FF^\dagger CD^\dagger - \frac{1}{2} A^{-1} FF^\dagger KD^\dagger \]
\[ = \frac{1}{2} A^\dagger CD^\dagger + \frac{1}{4} (B^*)^{-1} C^*(F^*)^\dagger - \frac{1}{4} A^{-1} FF^\dagger CD^\dagger \]
\[ - \frac{1}{4} (B^*)^{-1} D^\dagger DC^*(F^*)^\dagger - \frac{1}{2} A^{-1} FF^\dagger KD^\dagger \]
\[ + \frac{1}{2} (B^*)^{-1} D^\dagger DK^*(F^*)^\dagger + \frac{1}{2} (B^*)^{-1} V^*(1 - F^\dagger F) - \frac{1}{2} A^{-1} W^*(1 - DD^\dagger). \]

In the sequel, we shall establish the explicit solution to Eq. (1.1) when operators \( A, B, D \) and \( F \) have closed ranges.

**Theorem 2.4.** Let \( A, B, C \in \mathcal{L}(\mathcal{Y}), F \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \) and \( D \in \mathcal{L}(\mathcal{Y}, \mathcal{X}) \), such that \( A, B \) are self-adjoint, \( A, B, D, F \) have closed ranges. Moreover, suppose that \( CB^\dagger B = C, AA^\dagger = C, F^\dagger A^\dagger A = F^\dagger, BB^\dagger D^\dagger = D^\dagger, AB^\dagger D^\dagger DC^*(F^*)^\dagger D = -F(D^*)^\dagger C^* FF^\dagger A^\dagger B \).

Then the following statements are equivalent:
(a) There exists a solution \( X \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \) to Eq. (1.1).
(b) \( (1 - FF^\dagger)C(1 - D^\dagger D) = 0 \).

If (a) or (b) is satisfied, then any solution to Eq. (1.1) has the form
\[ X = \frac{1}{2} A^\dagger CD^\dagger + \frac{1}{2} B^\dagger C^*(F^*)^\dagger - \frac{1}{4} A^\dagger FF^\dagger CD^\dagger \]
\[ - \frac{1}{4} B^\dagger D^\dagger DC^*(F^*)^\dagger - \frac{1}{4} A^\dagger FF^\dagger KD^\dagger + \frac{1}{2} B^\dagger D^\dagger DK^*(F^*)^\dagger \]
\[ - \frac{1}{2} U^* + \frac{1}{2} A^\dagger AU^* DD^\dagger + \frac{1}{2} W^* - \frac{1}{2} B^\dagger BW^* F^\dagger, \]
where \( K \in \mathcal{L}(\mathcal{Y}) \) and \( U, W \in \mathcal{L}(\mathcal{Y}, \mathcal{X}) \) satisfy
\[ AB^\dagger D^\dagger DK^*(F^*)^\dagger D = F(D^*)^\dagger K^* FF^\dagger A^\dagger B, \]
\[ AW^* D - AB^\dagger BW^* F^\dagger D = FUB - FDD^\dagger UAA^\dagger B. \]

**Proof.** (a) ⇒ (b): The proof in this case is the same as the proof of Theorem 2.3.
(b) ⇒ (a): Let \( T = \begin{bmatrix} F & 0 \\ 0 & D^* \end{bmatrix} : \mathcal{X} \oplus \mathcal{X} \to \mathcal{Y} \oplus \mathcal{Y}, \hat{X} = \begin{bmatrix} 0 & X^* \\ X^* & 0 \end{bmatrix} : \mathcal{Y} \oplus \mathcal{Y} \to \mathcal{X} \oplus \mathcal{X} \).
From this we obtain \( X \oplus X, S = \begin{bmatrix} A & 0 \\ 0 & B^* \end{bmatrix} : \mathcal{Y} \oplus \mathcal{Y} \rightarrow \mathcal{Y} \oplus \mathcal{Y} \) and \( N = \begin{bmatrix} 0 & C \\ C^* & 0 \end{bmatrix} : \mathcal{Y} \oplus \mathcal{Y} \rightarrow \mathcal{Y} \oplus \mathcal{Y}. \)

By replacing these operator matrices in the operator equation \( T\tilde{X}S + S(\tilde{X})^*T^* = N, \) we get \( AXD + FX^*B = C. \) Further, from the condition (b) it can be derived that

\[
(1 - TT^\dagger)N(1 - TT^\dagger) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\]

According to Theorem 1.4 it follows that \( \hat{X} \) has the following representation

\[
\hat{X} = T^\dagger NS^\dagger - \frac{1}{2}T^\dagger NTT^\dagger S^\dagger + T^\dagger ZTT^\dagger S^\dagger + V - T^\dagger TVSS^\dagger,
\]

where \( Z \in \mathcal{L}(\mathcal{Y} \oplus \mathcal{Y}) \) satisfies \( T^*(Z + Z^*)T = 0 \) and \( V \in \mathcal{L}(\mathcal{Y} \oplus \mathcal{Y}, \mathcal{X} \oplus \mathcal{X}) \) is arbitrary. Let us assume that \( Z = \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} \) and \( V = \begin{bmatrix} V_1 & V_2 \\ V_3 & V_4 \end{bmatrix}. \) From the above solution we get

\[
\begin{bmatrix} 0 & X^* \\ X^* & 0 \end{bmatrix} = \begin{bmatrix} 0 & FC^\dagger B^\dagger \\ (D^*)^\dagger C^* A^\dagger & 0 \end{bmatrix}
- \frac{1}{2} \begin{bmatrix} 0 & F^\dagger CD^*(D^*)^\dagger B^\dagger \\ (D^*)^\dagger C^* F^\dagger A^\dagger & 0 \end{bmatrix}
+ \begin{bmatrix} F^\dagger Z_1 FF^\dagger A^\dagger & F^\dagger Z_2 D^*(D^*)^\dagger B^\dagger \\ (D^*)^\dagger Z_3 FF^\dagger A^\dagger & (D^*)^\dagger Z_4 D^*(D^*)^\dagger B^\dagger \end{bmatrix}
+ \begin{bmatrix} V_1 & V_2 \\ V_3 & V_4 \end{bmatrix} - \begin{bmatrix} F^\dagger FV_1 AA^\dagger & F^\dagger FV_2 BB^\dagger \\ (D^*)^\dagger D^* V_3 AA^\dagger & (D^*)^\dagger D^* V_4 BB^\dagger \end{bmatrix}.
\]

From this we obtain

\[
F^\dagger Z_1 FF^\dagger A^\dagger + V_1 - F^\dagger FV_1 AA^\dagger = 0, \quad (2.14)
\]
\[
(D^*)^\dagger Z_4 D^*(D^*)^\dagger B^\dagger + V_4 - (D^*)^\dagger D^* V_4 BB^\dagger = 0, \quad (2.15)
\]
\[
X^* = F^\dagger CB^\dagger - \frac{1}{2} F^\dagger CD^*(D^*)^\dagger B^\dagger + F^\dagger Z_2 D^*(D^*)^\dagger B^\dagger \quad (2.16)
+ V_2 - F^\dagger FV_2 BB^\dagger,
\]

\[
X^* = (D^*)^\dagger C^* A^\dagger - \frac{1}{2} (D^*)^\dagger C^* FF^\dagger A^\dagger + (D^*)^\dagger Z_3 FF^\dagger A^\dagger \quad (2.17)
+ V_3 - (D^*)^\dagger D^* V_3 AA^\dagger.
\]

Now, by multiplication \( F \) on the left and \( A \) on the right to (2.14) and our assumptions, we get

\[
FF^\dagger Z_1 FF^\dagger = 0.
\]

Also, by multiplication \( D^* \) on the left and \( B \) on the right to (2.15), we obtain

\[
D^\dagger DZ_4 D^\dagger D = 0.
\]
On the other hand, the relations $T^*(Z + Z^*)T = 0$ and $\ker(T^*) = \ker(T^\dagger)$ give us
\[(D^*)^\dagger Z_3 F = -(D^*)^\dagger Z_2^* F. \tag{2.18}\]

Regarding to (2.16) and (2.17) we observe that
\[B^\dagger C^* (F^*)^\dagger - \frac{1}{2} B^\dagger D^\dagger D C^* (F^*)^\dagger + B^\dagger D^\dagger D Z_2^* (F^*)^\dagger + V_2^* - B^\dagger B V_2^* F^* (F^*)^\dagger = A^\dagger C D^\dagger - \frac{1}{2} A^\dagger (F^*)^\dagger F^* C D^\dagger + A^\dagger (F^*)^\dagger F^* Z_2^* D^\dagger + V_3^* - A^\dagger A V_3^* D D^\dagger.\]

For $K = Z_2$, $W = V_2$, $U = V_3$ and also applying (2.18) we have
\[A^\dagger C D^\dagger = B^\dagger C^\ast (F^*)^\dagger - \frac{1}{2} B^\dagger D^\dagger D C^\ast (F^*)^\dagger + B^\dagger D^\dagger D K^\ast (F^*)^\dagger, \tag{2.19}\]
\[+ W^* - B^\dagger B W^* F^* (F^*)^\dagger + \frac{1}{2} A^\dagger (F^*)^\dagger F^* C D^\dagger + A^\dagger (F^*)^\dagger F^* K D^\dagger - U^* + A^\dagger A U^* D D^\dagger.\]

Next, let $X$ be any solution of Eq. (1.1). Replace $X^*$ in the right side of the equation $AXD = C - F X^* B$ with (2.16) and applying relations $C B^\dagger B = C$ and $(D^*)^\dagger B^\dagger B = (D^*)^\dagger$, it yields
\[AXD = C - F^\dagger \left( F^\dagger C B^\dagger - \frac{1}{2} F^\dagger C D^\dagger (D^*)^\dagger B^\dagger + V_2^* D^\dagger Z_2^* D^\dagger B^\dagger \right) B \]
\[+ V_2 - F^\dagger F V_2^* B B^\dagger B \]
\[= C - F F^\dagger C B^\dagger B + \frac{1}{2} F F^\dagger C D^\dagger (D^*)^\dagger B^\dagger B - F F^\dagger K D^\dagger (D^*)^\dagger B^\dagger B = C - F F^\dagger C + \frac{1}{2} F F^\dagger C D^\dagger D - F F^\dagger K D^\dagger D.\]

Set $M = C - F F^\dagger C + \frac{1}{2} F F^\dagger C D^\dagger D - F F^\dagger K D^\dagger D$. Then applying relations $A A^\dagger C = C$, $F^\dagger A^\dagger A = F^\dagger$ and regarding the condition (b) we deduce
\[AA^\dagger M D^\dagger D = AA^\dagger C D^\dagger D - AA^\dagger F F^\dagger C D^\dagger D \]
\[+ \frac{1}{2} AA^\dagger F F^\dagger C D^\dagger D D^\dagger D - AA^\dagger F F^\dagger K D^\dagger D D^\dagger D = C D^\dagger D - (F^*)^\dagger F^* C D^\dagger D + \frac{1}{2} (F^*)^\dagger F^* C D^\dagger (D^*)^\dagger - (F^*)^\dagger F^* K D^\dagger (D^*)^\dagger = C - F F^\dagger C + F F^\dagger C D^\dagger D - F F^\dagger C D^\dagger D + \frac{1}{2} F F^\dagger C D^\dagger D - F F^\dagger K D^\dagger D = M.\]
Finally, from Remark [1] and (2.19) we get

\[
X = A^1 M D^\dagger
= A^1 C D^\dagger - A^1 F F^\dagger C D^\dagger + \frac{1}{2} A^1 F F^\dagger C D^\dagger - A^1 F F^\dagger K D^\dagger
\]

\[
= \frac{1}{2} A^1 C D^\dagger + \frac{1}{2} B^\dagger C^*(F^*)^\dagger - \frac{1}{2} B^\dagger D^\dagger D C^*(F^*)^\dagger
\]

\[
+ \frac{1}{2} B^\dagger D^\dagger D K^*(F^*)^\dagger + \frac{1}{2} W^* - \frac{1}{2} B^\dagger B W^* F^*(F^*)^\dagger
\]

\[
+ \frac{1}{2} A^1(F^*)^\dagger F^* C D^\dagger + \frac{1}{2} A^1(F^*)^\dagger F^* K D^\dagger - \frac{1}{2} U^* + \frac{1}{2} A^1 A U^* D D^\dagger
\]

\[
- A^1 F F^\dagger C D^\dagger + \frac{1}{2} A^1 F F^\dagger C D^\dagger - A^1 F F^\dagger K D^\dagger
\]

\[
= \frac{1}{2} A^1 C D^\dagger + \frac{1}{2} B^\dagger C^*(F^*)^\dagger - \frac{1}{2} A^1 F F^\dagger C D^\dagger - \frac{1}{2} B^\dagger D^\dagger D C^*(F^*)^\dagger
\]

\[
- \frac{1}{2} A^1 F F^\dagger K D^\dagger + \frac{1}{2} B^\dagger D^\dagger D K^*(F^*)^\dagger
\]

\[
- \frac{1}{2} U^* + \frac{1}{2} A^1 A U^* D D^\dagger + \frac{1}{2} W^* - \frac{1}{2} B^\dagger B W^* F^* F^*.
\]

\[
\square
\]

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References

EXPLICIT SOLUTION TO THE OPERATOR EQUATION $AXD + FX^*B = C$  


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