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MULTIVALUED MIXED QUASI BIFUNCTION VARIATIONAL INEQUALITIES

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Abstract. In this paper, we introduce a new class of bifunction variational inequalities, which is called the multivalued mixed quasi bifunction variational inequalities. Using the auxiliary principle technique, we suggest and analyze a proximal point method for solving the multivalued mixed quasi bifunction variational inequalities. It is shown that the convergence of the proposed method requires only pseudomonotonicity, which is a weaker condition than monotonicity. Our results represent an improvement and refinement of previously known results. Since the multivalued mixed quasi bifunction variational inequalities include bifunction variational inequalities and related optimization problems as special cases, results proved in this paper continue to hold for these problems

1. Introduction

It is well known that he variational inequality theory, which was introduced and considered by Stampacchia [35], provides us with a unified, innovative and general framework to study a wide class of problems arising in finance, economics, network analysis, transportation, elasticity and optimization. and applied sciences. Variational inequalities have been generalized and extended in several directions using the novel and new techniques. An important and significant class of variational inequalities, is called the bifunction variational inequalities. Crespi et al [1,2,3,4], Fang and Hu [5], Lalitha and Mehra [11] and Noor [19,26] have studied various aspects of the bifunction variational inequalities. We would like to remark that the variational inequalities represent the optimality conditions of the convex functions. Noor [19] has shown that the minimum of the directionally differentiable convex function can be characterized by a class of variational inequality, which is called the directionally variational inequality. Using the technique of Noor [19], one show that the minimum of the sum of directionally differentiable convex function and the nondifferentiable bifunction function can be characterized by the mixed quasi bifunction variational inequalities.

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Inspired and motivated by the research and activities going in this fascinating area, we introduce and consider a new class of bifunction variational inequalities, which is called the multivalued mixed quasi bifunction variational inequality. This class is quite general and unifying ones and includes several classes of bifunction variational inequalities and variational inequalities as special cases. In recent years, several numerical techniques including projection, resolvent and auxiliary principle have been developed and analyzed for solving variational inequalities. We would like to point out that the projection-type methods and their invariant forms can not be used for solving the bifunction variational inequalities. To overcome this drawback, one usually uses the auxiliary principle technique, which is due to Glowinski, Lions and Tremolieres [10]. This technique has been used to suggest and analyze several methods for solving bifunction variational inequalities and variational inequalities. It has been shown that a substantial number of numerical methods can be obtained as special cases from this technique. In this paper, we again use the auxiliary principle technique to suggest and analyze an implicit method for solving the generalized mixed quasi bifunction variational inequalities. It is shown that the proposed proximal method converges for pseudomonotone operators, which is a weaker condition than monotonicity. Our results can be viewed as a significant extension and generalization of the previously known results for solving classical (bifunction) variational inequalities.

2. Basic Concepts

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|.\|$ respectively. Let $C(H)$ be a family of all nonempty compact subset of H. Let $T : H \longrightarrow C(H)$ be a multivalued operator. Let K be a nonempty closed convex set in H. Let $\varphi(.,.) : H \times H \to R \cup \{+\infty\}$ be a continuous bifunction. For a given bifunction $B(.,.) : K \times K \longrightarrow C(H)$, we consider the problem of finding $u \in K, \nu \in T(u)$ such that

$$B(\nu, v-u) + \varphi(v,u) - \varphi(u,u) \geq 0, \quad \forall v \in K, \tag{1}$$

which is called the *multivalued mixed quasi bifunction variational inequalities* .

If T is a single-valued operator, then problem (1) is equivalent to finding $u \in K$ such that

$$B(T(u), v-u) + \varphi(v,u) - \varphi(u,u) \geq 0, \quad \forall v \in K, \tag{2}$$

is called the mixed quasi bifunction variational inequalities. For the applications and numerical results of mixed quasi bifunction variational inequalities, see [19,23] and the references therein.

If $B(\nu, v-u) \equiv \langle \nu, v-u\rangle$, then problem (2.1) is equivalent to finding $u \in K, \nu \in T(u)$ such that

$$\langle \nu, v-u\rangle + \varphi(v,u) - \varphi(u,u) \geq 0, \quad \forall v \in K, \tag{3}$$

which is known as the multivalued mixed quasi variational inequality, see [1-24] for applications and numerical results. In brief, for suitable and appropriate choice of the operator and the spaces, one can obtain several known and new classes of variational inequalities and related optimization problems as special cases of

problem (1). This shows that problem (1)is quite general, flexible and unifying one. Furthermore, It is well-known that a wide class of obstacle, unilateral, contact, free, moving and equilibrium problems arising in mathematical, engineering, economics and finance can be studied in the unified and general framework of problems (1)-(3) and their special cases, see [1-35].

We also need the following concepts and results.

Lemma 2.1. $\forall u, v \in H$,

$$2\langle u, v\rangle = \|u+v\|^2 - \|u\|^2 - \|v\|^2. \tag{4}$$

Definition 2.1. The bifunction $B(.,.) : K \times K \longrightarrow H$ said to be *pseudomonotone* with respect to the bifunction $\varphi(.,.)$, iff

$$\begin{aligned} &B(\nu, v-u) + \varphi(v,u) - \varphi(u,u) \geq 0 \\ &\quad \Longrightarrow \\ &-B(\mu, u-v) + \varphi(v,u) - \varphi(u,u) \geq 0, \quad \forall u, v \in K, \quad \nu \in T(u), \mu \in T(v). \end{aligned}$$

Definition 2.2. The bifunction $\varphi(.,.)$ is said to be *skew-symmetric,* if,

$$\varphi(u,u) - \varphi(u,v) - \varphi(v,u) + \varphi(v,v) \geq 0, \quad \forall u, v \in H.$$

Clearly, if the bifunction $\varphi(.,.)$ is linear in both arguments, then,

$$\varphi(u,u) - \varphi(u,v) - \varphi(v,u) + \varphi(v,v) = \varphi(u-v, u-v) \geq 0, \quad \forall u, v \in H,$$

which shows that the bifunction $\varphi(.,.)$ is nonnegative.

It is worth mentioning that the points $(u,u), (u,v), (v,u), (v,v)$ make up a set of the four vertices of the square. In fact, the skew-symmetric bifunction $\varphi(.,.)$ can be written in the form

$$\frac{1}{2}\varphi(u,u) + \frac{1}{2}\varphi(v,v) \geq \frac{1}{2}\varphi(u,v) + \frac{1}{2}\varphi(v,u), \quad \forall u, v \in H.$$

This shows that the arithmetic average value of the skew-symmetric bifunction calculated at the north-east and south-west vertices of the square is greater than or equal to the arithmetic average value of the skew-symmetric bifunction computed at the north-west and south-west vertices of the same square. The skew-symmetric bifunction have the properties which can be considered an analogs of monotonicity of gradient and nonnegativity of a second derivative for the convex functions.

Definition 2.3. $\forall u_1, u_2 \in H, w_1 \in T(u_1), w_2 \in T(u_2)$ the operator $T : H \longrightarrow C(H)$ is said to be *M-Lipschitz continuous,* if there exists a constant $\delta > 0$ such that

$$M(T(u_1), T(u_2)) \leq \delta ||u_1 - u_2||,$$

where $M(.,.)$ is the Hausdorff metric on $C(H)$.

3. Main Results

We suggest and analyze a proximal method for multivalued mixed quasi bifunction variational inequalities (1) using the auxiliary principle technique of Glowinski, Lions and Tremolieres [10] as developed by Noor [13-26].

For a given $u \in K$ satisfying (1), we consider the auxiliary problem of finding a unique $w \in K, \eta \in T(w)$ such that

$$\rho B(\eta, v-w) + \langle w-u, v-w\rangle + \varphi(v,w) - \varphi(w,w) \geq 0, \quad \forall v \in K, \tag{5}$$

where $\rho > 0$ is a constant.

We note that if $w = u$, then clearly w is solution of (1). This observation enables us to suggest and analyze the following iterative method for solving (1).

Algorithm 3.1. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\rho B(\eta_{n+1}, v-u_{n+1}) + \langle u_{n+1}-u_n, v-u_{n+1}\rangle + \varphi(v,u_{n+1}) - \varphi(u_{n+1},u_{n+1}) \geq 0, \quad \forall v \in K, \tag{6}$$

$$\eta_n \in T(w_n) : \|\eta_{n+1}-\eta_n\| \leq M(T(w_{n+1}), T(w_n)), \tag{7}$$

which is known as the proximal method for solving multivalued mixed quasi bifunction variational inequalities (1).

If $B(\nu, v-u) = \langle \nu, v-u\rangle$, where $T : K \longrightarrow C(H)$ is a nonlinear multivalued operator, then Algorithm 3.1 reduce to:

Algorithm 3.2. For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho\nu_{n+1} + u_{n+1} - u_n, v-u_{n+1}\rangle + \varphi(v,u_{n+1}) - \varphi(u_{n+1},u_{n+1}) \geq 0, \quad \forall \quad v \in K$$

$$\eta_n \in T(w_n) : \|\eta_{n+1}-\eta_n\| \leq M(T(w_{n+1}), T(w_n)).$$

Algorithm 3.2 is known as the proximal point algorithm for solving multivalued mixed quasi variational inequalities (3). In a similar way, one can obtain several iterative methods for equilibrium problems and variational inequalities, see [8-20].

We now study the convergence analysis of Algorithm 3.1 and this is the main motivation of our next result.

Theorem 3.1. Let $B.,.)$ be pseudomonotone with respect to the bifunction $\varphi(.,.)$ and the bifunction $\varphi(.,.)$ be skew-symmetric. If $u \in K, \nu \in T(u)$ is a solution of (1) and u_{n+1} is an approximate solution obtained from Algorithm 3.1, then

$$\|u_{n+1}-u\|^2 \quad \leq \quad \|u_n-u\|^2 - \|u_{n+1}-u_n\|^2. \tag{8}$$

Proof. Let $u \in K, \nu \in (Tu)$ be a solution of (1). Then

$$B(\nu, v-u) + \varphi(v,u) - \varphi(u,u) \geq 0, \quad \forall v \in K,$$

which implies that

$$-B(\mu, u-v) + \varphi(v,u) - \varphi(u,u) \geq 0, \quad \forall v \in K, \quad \mu \in T(v), \tag{9}$$

since $B(.,.)$ is pseudomonotone with respect to the bifunction $\varphi(.,.)$.

Taking $v = u_{n+1}$ in (9), we have

$$-B(\mu_{n+1}, u - u_{n+1}) + \varphi(u_{n+1}, u) - \varphi(u, u) \geq 0. \tag{10}$$

Now taking $v = u$ in (6), we obtain

$$\rho B(\mu_{n+1}, u - u_{n+1}) + \langle u_{n+1} - u_n, u - u_{n+1} + \varphi(u, u_{n+1}) - \varphi(u_{n+1}, u_{n+1})\rangle \geq 0. \tag{11}$$

From (10) and (11), we have

$$\begin{aligned} \langle u_{n+1} - u_n, u - u_{n+1}\rangle &\geq -\rho B(\mu_{n+1}, u - u_{n+1}) + \varphi(u_{n+1}, u_{n+1}) - \varphi(u, u_{n+1}) \\ &\geq +\varphi(u_{n+1}, u_{n+1}) - \varphi(u, u_{n+1} - \varphi(u_{n+1}, u) + \varphi(u, u) \\ &\geq 0, \end{aligned} \tag{12}$$

since the bifunction $\varphi(.,.)$ is skew-symmetric.

Setting $u = u - u_{n+1}$ and $v = u_{n+1} - u_n$ in (24), we obtain

$$2\langle u_{n+1} - u_n, u - u_{n+1}\rangle = \|\bar{u} - u_n\|^2 - \|u - u_{n+1}\|^2 - \|u_n - u_{n+1}\|^2. \tag{13}$$

Combining (12) and (13), we have

$$\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2 - \|u_{n+1} - u_n\|^2,$$

the required result. □.

Theorem 3.2. Let H be a finite dimensional space. If u_{n+1} is the approximate solution obtained from Algorithm 3.1 and $u \in K, \nu \in T(u)$ is a solution of (1), then $\lim_{n\longrightarrow\infty} u_n = u$.

Proof. Let $\bar{u} \in K$ be a solution of (1). From (8), it follows that the sequence $\{\|\bar{u} - u_n\|\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Also from (8), we have

$$\sum_{n=0}^{\infty} \|u_{n+1} - u_n\|^2 \leq \|u_0 - u\|^2,$$

which implies that

$$\lim_{n\longrightarrow\infty} \|u_{n+1} - u_n\| = 0. \tag{14}$$

Let $\hat{u}$ be a cluster point of $\{u_n\}$ and the subsequence $\{u_{n_j}\}$ of the sequence $\{u_n\}$ converge to $\hat{u} \in H$. Replacing u_n by u_{n_j} in (6) and taking the limit $n_j \longrightarrow \infty$ and using (14), we have

$$B(\hat{\nu}, v - \hat{u}) + \varphi(v, \hat{u}) - \varphi(\hat{u}, \hat{u}) \geq 0, \quad \forall v \in K,$$

which implies that $\hat{u}$ solves the multivalued mixed quasi bifunction variational inequality (1) and

$$\|u_{n+1} - u_n\|^2 \leq \|u_n - u\|^2.$$

Thus it follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$ and

$$\lim_{n\longrightarrow\infty} u_n = \hat{u}.$$

It remains to show that $\nu \in T(u)$. From (7) and using the M-Lipschitz continuity of the multivalued operator T, we have

$$||\nu_n - \nu|| \leq M(T(u_n), T(u)) \leq \delta||u_n - u||,$$

which implies that $\nu_n \longrightarrow \nu$ as $n \longrightarrow \infty$. Now consider

$$\begin{aligned} d(\nu, T(u)) &\leq ||\nu - \nu_n|| + d(\nu, T(u)) \\ &\leq ||\nu - \nu_n|| + M(T(u_n), T(u)) \\ &\leq ||\nu - \nu_n|| + \delta||u_n - u|| \quad \longrightarrow 0 \text{ as } n \longrightarrow \infty, \end{aligned}$$

where $d(\nu, T(u)) = inf\{||\nu - z|| : z \in T(u)\}$. and $\delta > 0$ is the M-Lipschitz continuity constant of the operator T.. From the above inequality, it follows that $d(\nu, T(u)) = 0$. This implies that $\nu \in T(u)$, since $T(u) \in C(H)$. This completes the proof. $\square$.

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