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CHEBYSHEV TYPE INTEGRAL INEQUALITIES FOR GENERALIZED k-FRACTIONAL CONFORMABLE INTEGRALS

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Abstract. The investigation of inequalities involving the fractional differential and integral operators is considered to be important due to its renowned applications among researchers. This paper consigns to the generalizations of certain fractional integral inequalities. The classical Chebychev type inequalities are generalized by involving $k-$fractional conformable integrals ($kFCI$), which is the $k-$analogue of the fractional conformable integrals. We present the main results consisting of the inequalities using one and two fractional parameters $kFCI$ by taking into account the extended Chebyshev functional in the case of synchronous functions. Some certain interesting consequent results of the main inequalities are also depicted.

1. Introduction

For two integrable functions $f_1, f_2 : [a, b] \to \mathbb{R}$, define the functional, which is known as Chebychev's functional in the literature:

$$T(f_1, f_2; a, b) = \frac{1}{b-a}\int_a^b f_1(t)f_2(t)dt - \frac{1}{(b-a)^2}\int_a^b f_1(t)dt\int_a^b f_2(t)dt, \qquad (1)$$

provided that the involved integrals exist, f_1 and f_2 are synchronous on $[a, b]$, $(i.e. \left(f(x) - f(y)\right)\left(g(x) - g(y)\right) \geq 0, \text{ for any } x, y \in [a, b])$. Many studies are present in the literature involving (1), see for instance [6, 8, 23, 20].

The theory of fractional calculus has recently been given an ever-rising consideration due to its wide applications. In many branches of pure and applied mathematics, the fractional differential and integral operators are very helpful tools to perform the real number or complex number powers of the differentiation and integration. For a comprehensive description of fractional calculus operators as well as their properties and applications, we refer the readers to the research manuscripts by Miller and Ross [17] and Kiryakova [15]. It is fairly renowned that numerous diverse definitions of fractional integrals along with their applications can be sought in the literature. Each description has its own advantages and appropriate for applications to different problems in different subjects of sciences. Recently,

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Jarad et al. [11] contributed one more aspect to the study of fractional operators by introducing new fractional integral and derivative operators that are based on the standard fractional calculus iteration procedure on conformable derivatives introduced by Abdeljawad [1]. The topical elaboration of fractional calculus can be studied in the recent papers [2, 3, 10, 12, 21, 22, 28].

Inequalities involving fractional integrals are deemed to be crucial as they are valuable in the study of different differential and integral equations (see [18]). This discipline has drawn the attention of many mathematicians during the past several years. For inequalities involving generalized fractional operators we refer [4, 5, 7, 23, 24, 25, 26, 27]. A large number of the fractional integral operators are discussed in the literature for their applications in many fields of sciences.

The k-analogues of different classical and fractional operators have been considered about a decade ago by some researchers. We describe some k-analogues of classical operators existing in the literature. The theory of special k-functions was originated by Diaz and Pariguan in the form of Pochhammer k-symbol $(x)_{n,k}$, the gamma k-function Γ_k and the beta k-function B_k (see [9]):

$$(u)_{n,k} := u(u+k)(u+2k)...(u+(n-1)k), (n \in \mathbb{N}, k > 0), \tag{2}$$

and

$$\Gamma_k(u) = \lim_{n\to\infty} \frac{n!k^n (nk)^{\frac{u}{k}-1}}{(u)_{n,k}}, k > 0 \tag{3}$$

where $(u)_{n,k}$ is the Pochammer k-symbol for factorial function. The $k-$gamma function can also be shown explicitly as the Mellin transform of the exponential function $e^{-\frac{t^k}{k}}$ given by

$$\Gamma_k(u) = \int_0^{+\infty} t^{u-1} e^{\frac{-t^k}{k}} dt, x > 0.$$

Clearly,

$$\Gamma(u) = \lim_{k\to 1} \Gamma_k(u), \Gamma_k(u) = k^{\frac{u}{k}-1}\Gamma\left(\frac{u}{k}\right)$$

and

$$\Gamma_k(u+k) = u\Gamma_k(u).$$

Further, $k-$beta function denoted by $B_k(x,y)$ is defined as

$$B_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1} dt$$

such that $B_k(x,y) = \frac{1}{k}B\left(\frac{x}{k}, \frac{y}{k}\right)$ and $B_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}$.

Mubeen and Habibullah [19] used this special k-functions theory in fractional calculus for the first time in literature in the form of k-Riemann–Liouville integral. Recently, many researchers are presenting the new fractional differential and integral operators and their generalized forms by iteration procedure and by involving a new parameter $k > 0$. They also found the relationships of these generalized fractional operators with existing fractional and classical operators under the specific values of the parameters involved.

The key purpose of this manuscript is to introduce the fractional conformable integrals reported in [11] in the framework of $k > 0$ as well as its existence. We also generalize the Chebyshev-type integral inequalities given in [27] for two synchronous functions involving our newly introduced k-fractional conformable integrals ($kFCI$). We have presented the inequalities and related results involving one and two fractional parameters. The details of the work concerned to the inequalities, their applications and stability we refer the readers to [13, 14, 16, 29].

Abdeljawad [1] introduced the left and right fractional conformable derivatives for a differentiable function f in the form:

$$\mathfrak{I}^{\alpha}_{a^+} f(\mathcal{T}) = (\mathcal{T} - a)^{1-\alpha} f^{'}(\mathcal{T}), \tag{4}$$

$$\mathfrak{I}^{\alpha}_{b^-} f(\mathcal{T}) = (b - \mathcal{T})^{1-\alpha} f^{'}(\mathcal{T}), \tag{5}$$

The corresponding left and right fractional conformable integrals for $0 < \alpha < 1$, by

$$\mathcal{T}^{\alpha}_{a^+} f(x) = \int_a^x f(t) \frac{dt}{(t-a)^{1-\alpha}}, \tag{6}$$

$$\mathcal{T}^{\alpha}_{b^-} f(x) = \int_x^b f(t) \frac{dt}{(b-t)^{1-\alpha}}. \tag{7}$$

1.1. **Definition.** The left fractional conformable integral operator (FCI) of order $\beta \in \mathbb{C}, Re(\beta) > 0$, introduced by Jarad et al. [11] is obtained by iterating the left integral in relation (3) β times then by interchanging the order of integrals and result as

$${}^{\beta}\mathcal{T}^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(\beta)} \int_a^x \left(\frac{(x-a)^{\alpha} - (t-a)^{\alpha}}{\alpha} \right)^{\beta-1} f(t) \frac{dt}{(t-a)^{1-\alpha}}. \tag{8}$$

1.2. **Definition.** The right fractional conformable integral operator (FCI) of order $\beta \in \mathbb{C}, Re(\beta) > 0$, introduced by Jarad et al. [11] is is obtained by iterating the right integral in relation (4) β times then by interchanging the order of integrals and result as

$${}^{\beta}\mathcal{T}^{\alpha}_{b^-} f(x) = \frac{1}{\Gamma(\beta)} \int_x^b \left(\frac{(b-x)^{\alpha} - (b-t)^{\alpha}}{\alpha} \right)^{\beta-1} f(t) \frac{dt}{(b-t)^{1-\alpha}}. \tag{9}$$

where $\Gamma(\beta)$ is the Gamma function of β and defined as

$$\Gamma(\beta) = \int_0^{+\infty} e^{-u} u^{\beta-1} du. \tag{10}$$

2. Main results

2.1. **Generalized k-fractional Conformable Integrals.** In this section, we introduce the generalized left and right fractional conformable integrals in the framework of a new parameter $k>0$ which generalizes Riemann-Liouville, Hadamard, Katugampola and generalized fractional integrals as

2.1.1. **Definition.** Let f be a continuous function on a finite real interval $[a,b]$. Then generalized left $kFCI$ of order $\beta\in\mathbb{C}, Re(\beta)>0$ is defined as

$$ {}_{k}^{\beta}\mathcal{A}_{a+}^{\alpha} f(x)=\frac{1}{k\Gamma_k(\beta)}\int_a^x\left(\frac{(x-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1} f(t)\frac{dt}{(t-a)^{1-\alpha}}, x\in[a,b], \tag{11}$$

where Γ_k is the Euler $k-$gamma function, $k>0, \alpha\in\mathbb{R}\backslash\{0\}$.

2.1.2. **Definition.** Let f be a continuous function on a finite real interval $[a,b]$. Then generalized right $kFCI$ of order $\beta\in\mathbb{C}, Re(\beta)>0$ is defined as

$$ {}_{k}^{\beta}\mathcal{A}_{b-}^{\alpha} f(x)=\frac{1}{k\Gamma_k(\beta)}\int_x^b\left(\frac{(b-x)^\alpha-(b-t)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1} f(t)\frac{dt}{(b-t)^{1-\alpha}}, x\in[a,b], \tag{12}$$

where Γ_k is the Euler $k-$gamma function, $k>0, \alpha\in\mathbb{R}\backslash\{0\}$.
Now, we present that the integrals defined in (11) and (12) are well-defined.
2.1.1.**Theorem.** Let $g\in L_1[a,b], \alpha\in\mathbb{R}\backslash\{0\}$ and $k>0$. Then ${}_{k}^{\beta}\mathcal{A}_{a+}^{\alpha}g(x)\left({}_{k}^{\beta}\mathcal{A}_{b-}^{\alpha}g(x)\right)$ exists for any $x\in[a,b], Re(\beta)>0$.

Proof. Let $\Lambda^{'}:=[a,b]\times[a,b]$ and $P^{'}:\Lambda^{'}\to\mathbb{R}$
such that

$$P^{'}(x,t)=((x-a)^\alpha-(t-a)^\alpha)^{\frac{\beta}{k}-1}(t-a)^{\alpha-1}.$$

Clearly, it can be seen that

$$P^{'}=P_{+}^{'}+P_{-}^{'},$$

where

$$P_{+}^{'}(x,t):=\begin{cases}((x-a)^\alpha-(t-a)^\alpha)^{\frac{\beta}{k}-1}(t-a)^{\alpha-1}, & a\le t\le x\le b,\\ 0, & a\le x\le t\le b.\end{cases}$$

and

$$P_{-}^{'}(x,t):=\begin{cases}((t-a)^\alpha-(x-a)^\alpha)^{\frac{\beta}{k}-1}(x-a)^{\alpha-1}, & a\le t\le x\le b,\\ 0, & a\le x\le t\le b.\end{cases}$$

since $P^{'}$ is measurable on $\Lambda^{'}$, then it can be written as

$$\begin{aligned}\int_a^b P^{'}(x,t)dt = \int_a^x P^{'}(x,t)dt &= \int_a^x ((x-a)^\alpha - (t-a)^\alpha)^{\frac{\beta}{k}-1} (t-a)^{\alpha-1}dt \\ &= \frac{\alpha k}{\beta}(x-a)^{\frac{\alpha\beta}{k}}\end{aligned}$$

By using the double integral, we get

$$\begin{aligned}\int_a^b \left(\int_a^b P^{'}(x,t)|g(x)|dt\right) dx &= \int_a^b |g(x)| \left(\int_a^b P^{'}(x,t)dt\right) dx \\ &= \frac{\alpha k}{\beta}\int_a^b (x-a)^{\frac{\alpha\beta}{k}}|g(x)|dx \\ &\leq \frac{\alpha k}{\beta}(b-a)^{\frac{\alpha\beta}{k}}\int_a^b |g(x)|dx\end{aligned}$$

i.e.,

$$\begin{aligned}\int_a^b \left(\int_a^b P^{'}(x,t)|g(x)|dt\right) dx &= \int_a^b |g(x)| \left(\int_a^b P^{'}(x,t)dt\right) dx \\ &\leq \frac{\alpha k}{\beta}(b-a)^{\frac{\alpha\beta}{k}} \|g(x)\|_{L_1[a,b]} < \infty.\end{aligned}$$

So, the function $Q^{'} : \Lambda^{'} \to \mathbb{R}$ such that $Q^{'}(x,t) := P^{'}(x,t)g(x)$ is integrable over $\triangle^{'}$ by Tonelli's theorem. Hence, by Fubibni's theorem $\int_a^b P^{'}(x,t)g(x)dx$ is an integrable function over $[a,b]$, as a function of $t \in [a,b]$. i.e.${}_k^\beta\mathcal{A}_{a+}^\alpha g(x)$ exists.
The existence of the right k-fractional conformable integral ${}_k^\beta\mathcal{A}_{b-}^\alpha g(x)$ can be proved in the similar manner.

3. k-Fractional Conformable Integral Inequalities

This section refers to the proofs of Chebychev type inequalities involving the generalized k-fractional conformable integrals ${}_k^\beta\mathcal{A}_{a+}^\alpha$ defined in (11).

3.1. **Theorem.** Let f_1, f_2 be two synchronous on $[0,\infty]$, then for all $0 \leq a < x, \beta > 0, \gamma > 0$, the following inequalities for k-fractional conformable integrals ${}_k^\beta\mathcal{A}_{a+}^\alpha$ hold true:

$$\left({}_k^\beta\mathcal{A}_{a^+}^\alpha\right) f_1f_2(x) \geq \frac{1}{\left({}_k^\beta\mathcal{A}_{a^+}^\alpha\right)(1)} \left({}_k^\beta\mathcal{A}_{a^+}^\alpha\right) f_1(x) \left({}_k^\beta\mathcal{A}_{a^+}^\alpha\right) f_2(x) \tag{13}$$

$$\left({}_k^\beta\mathcal{A}_{a^+}^\alpha\right) f_1f_2(x) \left({}_k^\gamma\mathcal{A}_{a^+}^\alpha\right)(1) + \left({}_k^\gamma\mathcal{A}_{a^+}^\alpha\right) f_1f_2(x) \left({}_k^\beta\mathcal{A}_{a^+}^\alpha\right)(1)$$

$$\geq \left({}_{k}^{\beta}\mathcal{A}_{a^+}^{\alpha}\right) f_1(x) \left({}_{k}^{\gamma}\mathcal{A}_{a^+}^{\alpha}\right) f_2(x) + \left({}_{k}^{\beta}\mathcal{A}_{a^+}^{\alpha}\right) f_2(x) \left({}_{k}^{\gamma}\mathcal{A}_{a^+}^{\alpha}\right) f_1(x). \tag{14}$$

Proof. since the functions f_1 and f_2 are synchronous on $[0,\infty)$, then for all $\rho, \tau \geq 0$, we have

$$(f_1(\rho) - f_1(\tau))(f_2(\rho) - f_2(\tau)) \geq 0.$$

$$f_1(\rho)f_2(\rho) + f_1(\tau)f_2(\tau) \geq f_1(\rho)f_2(\tau) + f_1(\tau)f_2(\rho). \tag{15}$$

Multiplying both sides of (15) by $\frac{1}{k\Gamma_k(\beta)}\left(\frac{(x-a)^\alpha-(\rho-a)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}\frac{1}{(\rho-a)^{1-\alpha}}$, then integrating the resulting inequality w.r.t ρ from a to x, we get

$$\frac{1}{k\Gamma_k(\beta)}\int_a^x \left(\frac{(x-a)^\alpha-(\rho-a)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}\frac{f_1(\rho)f_2(\rho)d\rho}{(\rho-a)^{1-\alpha}}$$

$$+\frac{1}{k\Gamma_k(\beta)}\int_a^x \left(\frac{(x-a)^\alpha-(\rho-a)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}\frac{f_1(\tau)f_2(\tau)d\rho}{(\rho-a)^{1-\alpha}}$$

$$\geq \frac{1}{k\Gamma_k(\beta)}\int_a^x \left(\frac{(x-a)^\alpha-(\rho-a)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}\frac{f_1(\rho)f_2(\tau)d\rho}{(\rho-a)^{1-\alpha}}$$

$$+\frac{1}{k\Gamma_k(\beta)}\int_a^x \left(\frac{(x-a)^\alpha-(\rho-a)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}\frac{f_1(\tau)f_2(\rho)d\rho}{(\rho-a)^{1-\alpha}}. \tag{16}$$

i.e.

$$\left({}_{k}^{\beta}\mathcal{A}_{a^+}^{\alpha}\right) f_1f_2(x)+f_1(\tau)f_2(\tau)\left({}_{k}^{\beta}\mathcal{A}_{a^+}^{\alpha}\right)(1) \geq f_2(\tau)\left({}_{k}^{\beta}\mathcal{A}_{a^+}^{\alpha}\right) f_1(x)+f_1(\tau)\left({}_{k}^{\beta}\mathcal{A}_{a^+}^{\alpha}\right) f_2(x). \tag{17}$$

Multiplying both sides of (17) by $\frac{1}{k\Gamma_k(\beta)}\left(\frac{(x-a)^\alpha-(\tau-a)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}\frac{1}{(\tau-a)^{1-\alpha}}$, then integrating the resulting inequality w.r.t τ from a to x, we get

$$\left({}_{k}^{\beta}\mathcal{A}_{a^+}^{\alpha}\right) f_1f_2(x)\left({}_{k}^{\beta}\mathcal{A}_{a^+}^{\alpha}\right)(1) + \left({}_{k}^{\beta}\mathcal{A}_{a^+}^{\alpha}\right)(1)\left({}_{k}^{\beta}\mathcal{A}_{a^+}^{\alpha}\right) f_1f_2(x)$$

$$\geq \left({}_{k}^{\beta}\mathcal{A}_{a^+}^{\alpha}\right) f_1(x)\left({}_{k}^{\beta}\mathcal{A}_{a^+}^{\alpha}\right) f_2(x) + \left({}_{k}^{\beta}\mathcal{A}_{a^+}^{\alpha}\right) f_2(x)\left({}_{k}^{\beta}\mathcal{A}_{a^+}^{\alpha}\right) f_1(x).$$

i.e.

$$\left({}_{k}^{\beta}\mathcal{A}_{a^+}^{\alpha}\right) f_1f_2(x) \geq \frac{1}{\left({}_{k}^{\beta}\mathcal{A}_{a^+}^{\alpha}\right)(1)}\left({}_{k}^{\beta}\mathcal{A}_{a^+}^{\alpha}\right) f_1(x)\left({}_{k}^{\beta}\mathcal{A}_{a^+}^{\alpha}\right) f_2(x).$$

Hence inequality (13) is proved.

To prove the second result, multiplying both sides of (17) by $\frac{1}{k\Gamma_k(\gamma)}\left(\frac{(x-a)^\alpha-(\tau-a)^\alpha}{\alpha}\right)^{\frac{\gamma}{k}-1}\frac{1}{(\tau-a)^{1-\alpha}}$, then integrating the resulting inequality w.r.t τ from a to x, we get

$$\left({}_{k}^{\beta}\mathcal{A}_{a^+}^{\alpha}\right) f_1f_2(x)\left({}_{k}^{\gamma}\mathcal{A}_{a^+}^{\alpha}\right)(1) + \left({}_{k}^{\beta}\mathcal{A}_{a^+}^{\alpha}\right)(1)\left({}_{k}^{\gamma}\mathcal{A}_{a^+}^{\alpha}\right) f_1f_2(x)$$

$$\geq \left({}_{k}^{\beta}\mathcal{A}_{a^+}^{\alpha}\right) f_1(x)\left({}_{k}^{\gamma}\mathcal{A}_{a^+}^{\alpha}\right) f_2(x) + \left({}_{k}^{\gamma}\mathcal{A}_{a^+}^{\alpha}\right) f_2(x)\left({}_{k}^{\beta}\mathcal{A}_{a^+}^{\alpha}\right) f_1(x).$$

The proof is completed.

3.2. **Theorem.** Let f_1, f_2 be two synchronous on $[0,\infty]$, $f_3 \geq 0$ then for all $0 \leq a < x, \beta > 0, \gamma > 0$, the following inequalities for $k-$fractional conformable integrals ${}_k^\beta\mathcal{A}_{a+}^\alpha$ hold true:

$$\left({}_k^\beta\mathcal{A}_{a^+}^\alpha\right) f_1f_2f_3(x)\left({}_k^\gamma\mathcal{A}_{a^+}^\alpha\right)(1)+\left({}_k^\beta\mathcal{A}_{a^+}^\alpha\right)(1)\left({}_k^\gamma\mathcal{A}_{a^+}^\alpha\right) f_1f_2f_3(x) \geq \left({}_k^\beta\mathcal{A}_{a^+}^\alpha\right) f_1f_3(x)\left({}_k^\gamma\mathcal{A}_{a^+}^\alpha\right) f_2(x)$$
$$+\left({}_k^\beta\mathcal{A}_{a^+}^\alpha\right) f_2f_3(x)\left({}_k^\gamma\mathcal{A}_{a^+}^\alpha\right) f_1(x)-\left({}_k^\beta\mathcal{A}_{a^+}^\alpha\right) f_3(x)\left({}_k^\gamma\mathcal{A}_{a^+}^\alpha\right) f_1f_2(x)-\left({}_k^\beta\mathcal{A}_{a^+}^\alpha\right) f_1f_2(x)\left({}_k^\gamma\mathcal{A}_{a^+}^\alpha\right) f_3(x)$$
$$+\left({}_k^\beta\mathcal{A}_{a^+}^\alpha\right) f_1(x)\left({}_k^\gamma\mathcal{A}_{a^+}^\alpha\right) f_2f_3(x)+\left({}_k^\beta\mathcal{A}_{a^+}^\alpha\right) f_2(x)\left({}_k^\gamma\mathcal{A}_{a^+}^\alpha\right) f_1f_3(x). \qquad (18)$$

Proof. since the functions f_1 and f_2 are synchronous on $[0,\infty)$ and $f_3 \geq 0$, then for all $\rho, \tau \geq 0$, we have

$$(f_1(\rho)-f_1(\tau))(f_2(\rho)-f_2(\tau))(f_3(\rho)+f_3(\tau)) \geq 0.$$

Expanding the left hand side of above inequality, we get

$$f_1(\rho)f_2(\rho)f_3(\rho)+f_1(\tau)f_2(\tau)f_3(\tau) \geq f_1(\rho)f_2(\tau)f_3(\rho)+f_1(\tau)f_2(\rho)f_3(\rho)$$
$$-f_1(\tau)f_2(\tau)f_3(\rho)-f_1(\rho)f_2(\rho)f_3(\tau)+f_1(\rho)f_2(\tau)f_3(\tau)+f_1(\tau)f_2(\rho)f_3(\tau). \qquad (19)$$

Multiplying both sides of (19) by $\frac{1}{k\Gamma_k(\beta)}\left(\frac{(x-a)^\alpha-(\rho-a)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}\frac{1}{(\rho-a)^{1-\alpha}}$, then integrating the resulting inequality w.r.t ρ from a to x, we get

$$\frac{1}{k\Gamma_k(\beta)}\int_a^x\left(\frac{(x-a)^\alpha-(\rho-a)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}\frac{f_1(\rho)f_2(\rho)f_3(\rho)d\rho}{(\rho-a)^{1-\alpha}}$$
$$+\frac{1}{k\Gamma_k(\beta)}\int_a^x\left(\frac{(x-a)^\alpha-(\rho-a)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}\frac{f_1(\tau)f_2(\tau)f_3(\tau)d\rho}{(\rho-a)^{1-\alpha}}$$
$$\geq\frac{1}{k\Gamma_k(\beta)}\int_a^x\left(\frac{(x-a)^\alpha-(\rho-a)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}\frac{f_1(\rho)f_2(\tau)f_3(\rho)d\rho}{(\rho-a)^{1-\alpha}}$$
$$+\frac{1}{k\Gamma_k(\beta)}\int_a^x\left(\frac{(x-a)^\alpha-(\rho-a)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}\frac{f_1(\tau)f_2(\rho)f_3(\rho)d\rho}{(\rho-a)^{1-\alpha}}$$
$$-\frac{1}{k\Gamma_k(\beta)}\int_a^x\left(\frac{(x-a)^\alpha-(\rho-a)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}\frac{f_1(\tau)f_2(\tau)f_3(\rho)d\rho}{(\rho-a)^{1-\alpha}}$$
$$-\frac{1}{k\Gamma_k(\beta)}\int_a^x\left(\frac{(x-a)^\alpha-(\rho-a)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}\frac{f_1(\rho)f_2(\rho)f_3(\tau)d\rho}{(\rho-a)^{1-\alpha}}$$
$$+\frac{1}{k\Gamma_k(\beta)}\int_a^x\left(\frac{(x-a)^\alpha-(\rho-a)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}\frac{f_1(\rho)f_2(\tau)f_3(\tau)d\rho}{(\rho-a)^{1-\alpha}}$$
$$+\frac{1}{k\Gamma_k(\beta)}\int_a^x\left(\frac{(x-a)^\alpha-(\rho-a)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}\frac{f_1(\tau)f_2(\rho)f_3(\tau)d\rho}{(\rho-a)^{1-\alpha}}. \qquad (20)$$

i.e.

$$\left({}_k^\beta\mathcal{A}_{a^+}^\alpha\right) f_1f_2f_3(x)+f_1(\tau)f_2(\tau)f_3(\tau)\left({}_k^\beta\mathcal{A}_{a^+}^\alpha\right)(1) \geq f_2(\tau)\left({}_k^\beta\mathcal{A}_{a^+}^\alpha\right) f_1f_3(x)$$

$$+f_1(\tau)\left({}_k^{\beta}\mathcal{A}_{a^+}^{\alpha}\right)f_2f_3(x)-f_1(\tau)f_2(\tau)\left({}_k^{\beta}\mathcal{A}_{a^+}^{\alpha}\right)f_3(x)-f_3(\tau)\left({}_k^{\beta}\mathcal{A}_{a^+}^{\alpha}\right)f_1f_2(x)$$

$$+f_2(\tau)f_3(\tau)\left({}_k^{\beta}\mathcal{A}_{a^+}^{\alpha}\right)f_1(x)+f_1(\tau)f_3(\tau)\left({}_k^{\beta}\mathcal{A}_{a^+}^{\alpha}\right)f_2(x). \tag{21}$$

Multiplying both sides of (21) by $\frac{1}{k\Gamma_k(\beta)}\left(\frac{(x-a)^{\alpha}-(\tau-a)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}-1}\frac{1}{(\tau-a)^{1-\alpha}}$, then integrating the resulting inequality w.r.t τ from a to x, we get

$$\left({}_k^{\beta}\mathcal{A}_{a^+}^{\alpha}\right)f_1f_2f_3(x)\left({}_k^{\gamma}\mathcal{A}_{a^+}^{\alpha}\right)(1)+\left({}_k^{\beta}\mathcal{A}_{a^+}^{\alpha}\right)(1)\left({}_k^{\gamma}\mathcal{A}_{a^+}^{\alpha}\right)f_1f_2(x)\geq\left({}_k^{\beta}\mathcal{A}_{a^+}^{\alpha}\right)f_1f_3(x)\left({}_k^{\gamma}\mathcal{A}_{a^+}^{\alpha}\right)f_2(x)$$

$$+\left({}_k^{\beta}\mathcal{A}_{a^+}^{\alpha}\right)f_2f_3(x)\left({}_k^{\gamma}\mathcal{A}_{a^+}^{\alpha}\right)f_1(x)-\left({}_k^{\beta}\mathcal{A}_{a^+}^{\alpha}\right)f_3(x)\left({}_k^{\gamma}\mathcal{A}_{a^+}^{\alpha}\right)f_1f_2(x)-\left({}_k^{\beta}\mathcal{A}_{a^+}^{\alpha}\right)f_1f_2(x)\left({}_k^{\gamma}\mathcal{A}_{a^+}^{\alpha}\right)f_3(x)$$

$$+\left({}_k^{\beta}\mathcal{A}_{a^+}^{\alpha}\right)f_1(x)\left({}_k^{\gamma}\mathcal{A}_{a^+}^{\alpha}\right)f_2f_3(x)+\left({}_k^{\beta}\mathcal{A}_{a^+}^{\alpha}\right)f_2(x)\left({}_k^{\gamma}\mathcal{A}_{a^+}^{\alpha}\right)f_1f_3(x).$$

Hence inequality (18) is proved.

3.3. **Corollary.** Let f_1, f_2 be two synchronous on $[0,\infty]$, $f_3 \geq 0$ then for all $0 \leq a < x, \beta > 0$, the following inequalities for k-fractional conformable integrals ${}_k^{\beta}\mathcal{A}_{a+}^{\alpha}$ hold true:

$$\left({}_k^{\beta}\mathcal{A}_{a^+}^{\alpha}\right)f_1f_2f_3(x)\left({}_k^{\beta}\mathcal{A}_{a^+}^{\alpha}\right)(1)\geq\left({}_k^{\beta}\mathcal{A}_{a^+}^{\alpha}\right)f_1f_3(x)\left({}_k^{\beta}\mathcal{A}_{a^+}^{\alpha}\right)f_2(x)$$

$$+\left({}_k^{\beta}\mathcal{A}_{a^+}^{\alpha}\right)f_2f_3(x)\left({}_k^{\beta}\mathcal{A}_{a^+}^{\alpha}\right)f_1(x)-\left({}_k^{\beta}\mathcal{A}_{a^+}^{\alpha}\right)f_3(x)\left({}_k^{\beta}\mathcal{A}_{a^+}^{\alpha}\right)f_1f_2(x). \tag{22}$$

Proof. The proof can be made by replacing γ with β in Theorem 3.2, as (22) is the inequality involving only one fractional parameter.

3.4. **Theorem.** Let f_1, f_2 and f_3 be three monotonic functions defined on $[0,\infty]$ satisfying the following condition:

$$(f_1(\rho)-f_1(\tau))(f_2(\rho)-f_2(\tau))(f_3(\rho)-f_3(\tau))\geq 0,$$

for all $\rho, \tau \geq 0$, then for all $0 \leq a < x, \beta > 0, \gamma > 0$, the following inequalities for $k-$fractional conformable integrals ${}_k^{\beta}\mathcal{A}_{a+}^{\alpha}$ hold true:

$$\left({}_k^{\beta}\mathcal{A}_{a^+}^{\alpha}\right)f_1f_2f_3(x)\left({}_k^{\gamma}\mathcal{A}_{a^+}^{\alpha}\right)(1)-\left({}_k^{\beta}\mathcal{A}_{a^+}^{\alpha}\right)(1)\left({}_k^{\gamma}\mathcal{A}_{a^+}^{\alpha}\right)f_1f_2f_3(x)\geq\left({}_k^{\beta}\mathcal{A}_{a^+}^{\alpha}\right)f_1f_3(x)\left({}_k^{\gamma}\mathcal{A}_{a^+}^{\alpha}\right)f_2(x)$$

$$+\left({}_k^{\beta}\mathcal{A}_{a^+}^{\alpha}\right)f_2f_3(x)\left({}_k^{\gamma}\mathcal{A}_{a^+}^{\alpha}\right)f_1(x)-\left({}_k^{\beta}\mathcal{A}_{a^+}^{\alpha}\right)f_3(x)\left({}_k^{\gamma}\mathcal{A}_{a^+}^{\alpha}\right)f_1f_2(x)+\left({}_k^{\beta}\mathcal{A}_{a^+}^{\alpha}\right)f_1f_2(x)\left({}_k^{\gamma}\mathcal{A}_{a^+}^{\alpha}\right)f_3(x)$$

$$-\left({}_k^{\beta}\mathcal{A}_{a^+}^{\alpha}\right)f_1(x)\left({}_k^{\gamma}\mathcal{A}_{a^+}^{\alpha}\right)f_2f_3(x)-\left({}_k^{\beta}\mathcal{A}_{a^+}^{\alpha}\right)f_2(x)\left({}_k^{\gamma}\mathcal{A}_{a^+}^{\alpha}\right)f_1f_3(x). \tag{23}$$

Proof. The proof is similar to that given in Theorem 3.2.

3.5. **Theorem.** Let f_1 and f_2 be two functions on $[0,\infty]$, then for all $0 \leq a < x, \beta > 0, \gamma > 0$, the following inequalities for $k-$fractional conformable integrals ${}^{\beta}_{k}\mathcal{A}^{\alpha}_{a+}$ hold true:

$$\left({}^{\beta}_{k}\mathcal{A}^{\alpha}_{a+}\right) f_1^2(x) \left({}^{\gamma}_{k}\mathcal{A}^{\alpha}_{a+}\right)(1)+\left({}^{\beta}_{k}\mathcal{A}^{\alpha}_{a+}\right)(1)\left({}^{\gamma}_{k}\mathcal{A}^{\alpha}_{a+}\right) f_2^2(x) \geq 2\left({}^{\beta}_{k}\mathcal{A}^{\alpha}_{a+}\right) f_1(x)\left({}^{\gamma}_{k}\mathcal{A}^{\alpha}_{a+}\right) f_2(x). \tag{24}$$

$$\left({}^{\beta}_{k}\mathcal{A}^{\alpha}_{a+}\right) f_1^2(x) \left({}^{\gamma}_{k}\mathcal{A}^{\alpha}_{a+}\right) f_2^2(x)+\left({}^{\beta}_{k}\mathcal{A}^{\alpha}_{a+}\right) f_1^2(x)\left({}^{\gamma}_{k}\mathcal{A}^{\alpha}_{a+}\right) f_2^2(x) \geq 2\left({}^{\beta}_{k}\mathcal{A}^{\alpha}_{a+}\right) f_1 f_2(x)\left({}^{\gamma}_{k}\mathcal{A}^{\alpha}_{a+}\right) f_1 f_2(x). \tag{25}$$

Proof. since for all $\rho, \tau \geq 0$,

$$\left(f_1(\rho) - f_2(\tau)\right)^2 \geq 0,$$

then we have

$$f_1^2(\rho) + f_2^2(\tau) \geq 2 f_1(\rho) f_2(\tau). \tag{26}$$

Multiplying both sides of (26) by $\frac{1}{k\Gamma_k(\beta)}\left(\frac{(x-a)^\alpha-(\rho-a)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}\frac{1}{(\rho-a)^{1-\alpha}}$, then integrating the resulting inequality w.r.t ρ from a to x, we get

$$\frac{1}{k\Gamma_k(\beta)}\int_a^x \left(\frac{(x-a)^\alpha-(\rho-a)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}\frac{f_1^2(\rho)d\rho}{(\rho-a)^{1-\alpha}}$$

$$\geq \frac{1}{k\Gamma_k(\beta)}\int_a^x \left(\frac{(x-a)^\alpha-(\rho-a)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}\frac{2f_1(\rho)f_2(\tau)d\rho}{(\rho-a)^{1-\alpha}}. \tag{27}$$

i.e.

$$\left({}^{\beta}_{k}\mathcal{A}^{\alpha}_{a+}\right) f_1^2(x) + f_2^2(\tau)\left({}^{\beta}_{k}\mathcal{A}^{\alpha}_{a+}\right)(1) \geq 2 f_2(\tau)\left({}^{\beta}_{k}\mathcal{A}^{\alpha}_{a+}\right) f_1(x). \tag{28}$$

Multiplying both sides of (28) by $\frac{1}{k\Gamma_k(\beta)}\left(\frac{(x-a)^\alpha-(\tau-a)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}\frac{1}{(\tau-a)^{1-\alpha}}$, then integrating the resulting inequality w.r.t τ from a to x, we get

$$\left({}^{\beta}_{k}\mathcal{A}^{\alpha}_{a+}\right) f_1^2(x) \left({}^{\gamma}_{k}\mathcal{A}^{\alpha}_{a+}\right)(1)+\left({}^{\beta}_{k}\mathcal{A}^{\alpha}_{a+}\right)(1)\left({}^{\gamma}_{k}\mathcal{A}^{\alpha}_{a+}\right) f_2^2(x) \geq 2\left({}^{\beta}_{k}\mathcal{A}^{\alpha}_{a+}\right) f_1(x)\left({}^{\gamma}_{k}\mathcal{A}^{\alpha}_{a+}\right) f_2(x).$$

which completes the proof of first part. To obtain the second part, we have

$$\left(f_1(\rho)f_2(\tau) - f_1(\tau)f_2(\rho)\right)^2 \geq 0,$$

then we have

$$f_1^2(\rho)f_2^2(\tau) + f_1^2(\tau)f_2^2(\rho) \geq 2 f_1(\rho)f_1(\tau)f_2(\rho)f_2(\tau). \tag{29}$$

Multiplying both sides of (29) by $\frac{1}{k\Gamma_k(\beta)}\left(\frac{(x-a)^\alpha-(\rho-a)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}\frac{1}{(\rho-a)^{1-\alpha}}$, then integrating the resulting inequality w.r.t ρ from a to x, we get

$$\frac{1}{k\Gamma_k(\beta)}\int_a^x \left(\frac{(x-a)^\alpha-(\rho-a)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}\frac{f_1^2(\rho)f_2^2(\tau)d\rho}{(\rho-a)^{1-\alpha}}$$

$$+\frac{1}{k\Gamma_k(\beta)}\int_a^x \left(\frac{(x-a)^\alpha-(\rho-a)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}\frac{f_1^2(\tau)f_2^2(\rho)d\rho}{(\rho-a)^{1-\alpha}}$$

$$\geq 2\frac{1}{k\Gamma_k(\beta)}\int_a^x \left(\frac{(x-a)^\alpha-(\rho-a)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}\frac{f_1(\rho)f_2(\rho)f_1(\tau)f_2(\tau)d\rho}{(\rho-a)^{1-\alpha}}. \tag{30}$$

i.e.

$$f_2^2(\tau)\left({}_k^\beta\mathcal{A}_{a^+}^\alpha\right)f_1^2(x)+f_1^2(\tau)\left({}_k^\beta\mathcal{A}_{a^+}^\alpha\right)f_2^2(x)\geq 2f_1(\tau)f_2(\tau)\left({}_k^\beta\mathcal{A}_{a^+}^\alpha\right)f_1f_2(x). \quad (31)$$

Multiplying both sides of (31) by $\frac{1}{k\Gamma_k(\gamma)}\left(\frac{(x-a)^\alpha-(\tau-a)^\alpha}{\alpha}\right)^{\frac{\gamma}{k}-1}\frac{1}{(\tau-a)^{1-\alpha}}$, then integrating the resulting inequality w.r.t τ from a to x, we get (25).

3.6. **Corollary.** Let f_1 and f_2 be two functions on $[0,\infty]$, then for all $0\leq a<x,\beta>0,\gamma>0$, the following inequalities for $k-$fractional conformable integrals ${}_k^\beta\mathcal{A}_{a+}^\alpha$ hold true:

$$\left({}_k^\beta\mathcal{A}_{a^+}^\alpha\right)(1)\left[\left({}_k^\beta\mathcal{A}_{a^+}^\alpha\right)f_1^2(x)+\left({}_k^\beta\mathcal{A}_{a^+}^\alpha\right)f_2^2(x)\right]\geq 2\left({}_k^\beta\mathcal{A}_{a^+}^\alpha\right)f_1(x)\left({}_k^\beta\mathcal{A}_{a^+}^\alpha\right)f_2(x). \quad (32)$$

$$\left({}_k^\beta\mathcal{A}_{a^+}^\alpha\right)f_1^2(x)\left({}_k^\beta\mathcal{A}_{a^+}^\alpha\right)f_2^2(x)\geq\left[\left({}_k^\beta\mathcal{A}_{a^+}^\alpha\right)f_1f_2(x)\right]^2. \quad (33)$$

Proof. The proof of both inequalities can be made by replacing γ with β in both inequalities of previous theorem as (32) and (33) are the inequalities involving only one fractional parameter.

3.7. **Theorem.** Let $f:\mathbb{R}\to\mathbb{R}$ and defined by

$$\overline{f}(x)=\int_a^x\frac{f(t)dt}{(t-a)^{1-\alpha}};0\leq a<x,\alpha\in\mathbb{R}\setminus\{0\}$$

then for all $0<k\leq\beta$

$$\left({}_k^\beta\mathcal{A}_{a^+}^\alpha\right)f(x)=\frac{1}{k}\left({}_k^{\beta-k}\mathcal{A}_{a^+}^\alpha\right)\overline{f}(x). \quad (34)$$

Proof. By definition of generalized $k-$fractional conformable integrals, we have

$$\begin{aligned}
{}_k^\beta\mathcal{A}_{a^+}^\alpha\overline{f}(x) &= \frac{1}{k\Gamma_k(\beta)}\int_a^x\left(\frac{(x-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}\overline{f}(t)\frac{dt}{(t-a)^{1-\alpha}}\\
&= \frac{1}{k\Gamma_k(\beta)}\int_a^x\left(\frac{(x-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}\frac{1}{(t-a)^{1-\alpha}}\int_a^t\frac{f(u)du}{(u-a)^{1-\alpha}}\\
&= \frac{1}{k\Gamma_k(\beta)}\int_a^x\frac{f(u)}{(u-a)^{1-\alpha}}\int_u^x\left(\frac{(x-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}\frac{dt}{(t-a)^{1-\alpha}}du\\
&= \frac{1}{k\Gamma_k(\beta)}\int_a^x\left(\frac{(x-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^{\frac{\beta}{k}}\frac{f(u)}{(u-a)^{1-\alpha}}du\\
&= \frac{1}{\Gamma_k(\beta+k)}\int_a^x\left(\frac{(x-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^{\frac{\beta+k}{k}-1}\frac{f(u)}{(u-a)^{1-\alpha}}du\\
&= \left({}_k^{\beta+k}\mathcal{A}_{a^+}^\alpha\right)f(x).
\end{aligned}$$

which completes the proof.

To present the next result, we recall the generalized Cauchy-Buniakovsky-Schwarz inequality as follows:

3.8. **Lemma.** Let $f_1, f_2, f_3 : [a,b] \to (0,\infty)$ be three functions $0 \le a < b$, then

$$\left(\int_a^b f_2^m(t) f_3^r(t) f_1(t)dt\right)\left(\int_a^b f_2^n(t) f_3^s(t) f_1(t)dt\right) \ge \left(\int_a^b f_2^{\frac{m+n}{2}}(t) f_3^{\frac{r+s}{2}}(t) f_1(t)dt\right)^2. \tag{35}$$

where m, n, r, s are arbitrary real numbers.
Proof. We have

$$\int_a^b \left[\sqrt{f_2^m(t) f_3^r(t) f_1(t)}\sqrt{\int_a^b f_2^n(t) f_3^s(t) f_1(t)dt} - \sqrt{f_2^n(t) f_3^s(t) f_1(t)}\sqrt{\int_a^b f_2^m(t) f_3^r(t) f_1(t)dt}\right]^2 dt \ge 0,$$

$$\int_a^b \left[f_2^m(t) f_3^r(t) f_1(t) \int_a^b f_2^n(t) f_3^s(t) f_1(t)dt + f_2^n(t) f_3^s(t) f_1(t) \int_a^b f_2^m(t) f_3^r(t) f_1(t)dt\right]^2 dt$$

$$-2 f_2^{\frac{m+n}{2}}(t) f_3^{\frac{r+s}{2}}(t) \sqrt{\int_a^b f_2^m(t) f_3^r(t) f_1(t)dt}\sqrt{\int_a^b f_2^n(t) f_3^s(t) f_1(t)dt} \ge 0,$$

$$2\left(\int_a^b f_2^m(t) f_3^r(t) f_1(t)dt\right)\left(\int_a^b f_2^n(t) f_3^s(t) f_1(t)dt\right)$$

$$\ge 2\left(\int_a^b f_2^{\frac{m+n}{2}}(t) f_3^{\frac{r+s}{2}}(t) f_1(t)dt\right)\sqrt{\int_a^b f_2^m(t) f_3^r(t) f_1(t)dt}\sqrt{\int_a^b f_2^n(t) f_3^s(t) f_1(t)dt}$$

which can be written in the form of required inequality.

3.9. **Theorem.** Let $f \in L_1[a,b]$, then

$$\left(\left({}_k^{m\left(\frac{\beta}{k}-1\right)+1}\mathcal{A}_{a^+}^{\alpha}\right) f^r(x)\right)\left(\left({}_k^{n\left(\frac{\beta}{k}-1\right)+1}\mathcal{A}_{a^+}^{\alpha}\right) f^p(x)\right) \ge \left(\left({}_k^{\frac{m+n}{2}\left(\frac{\beta}{k}-1\right)+1}\mathcal{A}_{a^+}^{\alpha}\right) f^{\frac{r+p}{2}}(x)\right)^2. \tag{36}$$

Proof. By taking $f_2(t) = \left(\frac{(x-a)^\alpha - (t-a)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}$, $f_1(t) = \frac{(t-a)^{\alpha-1}}{k\Gamma_k(beta)}$ and $f_3(t) = f(t)$ in (35), we get

$$\left(\frac{1}{k\Gamma_k(\beta)}\int_a^x \left(\frac{(x-a)^\alpha - (t-a)^\alpha}{\alpha}\right)^{m\left(\frac{\beta}{k}-1\right)} \frac{f^r(t)dt}{(t-a)^{1-\alpha}}\right)$$

$$\left(\frac{1}{k\Gamma_k(\beta)}\int_a^x \left(\frac{(x-a)^\alpha - (t-a)^\alpha}{\alpha}\right)^{n\left(\frac{\beta}{k}-1\right)} \frac{f^p(t)dt}{(t-a)^{1-\alpha}}\right)$$

$$\ge \left(\frac{1}{k\Gamma_k(\beta)}\int_a^x \left(\frac{(x-a)^\alpha - (t-a)^\alpha}{\alpha}\right)^{\frac{m+n}{2}\left(\frac{\beta}{k}-1\right)} \frac{f^{\frac{r+p}{2}}(t)dt}{(t-a)^{1-\alpha}}\right)^2$$

which can be written as (36).

3.10. **Remark:** For $k = 1$ in (36), we get the following inequality:

$$\left(\left({}^{m(\beta-1)+1}\mathcal{A}^{\alpha}_{a^+}\right) f^r(x)\right)\left(\left({}^{n(\beta-1)+1}\mathcal{A}^{\alpha}_{a^+}\right) f^p(x)\right) \geq \left(\left({}^{\frac{m+n}{2}(\beta-1)+1}\mathcal{A}^{\alpha}_{a^+}\right) f^{\frac{r+p}{2}}(x)\right)^2.$$

3.11. **Remark:** If we take $k = 1$, the above results reduce to the Chebyshev inequalities involving fractional conformable integrals ${}^{\beta}\mathcal{A}^{\alpha}_{a^+}$.

3.12. **Remark:** The above inequalities and results can be obtained for the generalized $k-$fractional conformable integrals ${}^{\beta}_{k}\mathcal{A}^{\alpha}_{b-}$ defined in (12).

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Conflict of Interests.
The author(s) declare(s) that there is no conflict of interests regarding the publication of this article.

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