Journal of Inequalities and Special Functions
ISSN: 2217-4303, URL: http://ilirias.com/jiasf
Volume 9 Issue 3 (2018), Pages 34-47.

COMMON FIXED POINT THEOREMS FOR MATKOWSKI-TYPE NONLINEAR CONTRACTIONS IN ORDERED METRIC SPACES

RQEEB GUBRAN, IDREES A. KHAN AND MOHAMMAD IMDAD

Abstract. In this article, we prove some existence and uniqueness results for g-increasing mappings under φ-contraction condition in ordered metric spaces wherein φ is a comparison function (often called Matkowski-type nonlinear contraction). Our main results generalize several core results due to Ran and Reurings [25], Nieto and Rodríguez-López [22,23] and Agarwal *et.al* [1] beside several others. An application and some illustrative examples are furnished to highlight the realized improvements in our results.

1. Introduction and Preliminaries

The study of order-theoretic metric fixed point theory was initiated in 1986 by Turinici [28,29]. Thereafter, in 2004, Ran and Reurings [25] studied the existence of a fixed point for contraction type mappings in partially ordered metric space with some applications to matrix equations. Since then, several researchers proved a multitude of order-theoretic fixed point theorems and by now there exists an extensive literature on this theme. For the work of this kind, one can be referred to [1–3,9,10,13,22–24,30]. Here it can be pointed out that in order-theoretic metrical fixed point results the involved contraction condition is required to hold merely on those pairs of points which are comparable in the underlying partial ordering. Very recently, Alam *et al.* [4] employed compatibility of the underlying partial ordering along with some order-theoretic metrical notions namely: completeness, continuity, g-continuity and compatibility to prove their results.

Throughout this article, $\mathbb{R}$ and $\mathbb{N}_0$ respectively denote the set of all real numbers and set of whole numbers (i.e. set of all natural numbers including zero). As usual, I denotes the identity mapping defined on the underlying set. For brevity, we write fx rather than $f(x)$.

Now, we collect relevant basic notions, definitions and results needed in our subsequent discussion.

2000 *Mathematics Subject Classification.* 54H25, 47H10.
Key words and phrases. comparison function, fixed point; g-monotone mappings; φ-contractions; Matkowski function.

Submitted Maj 17, 2018. Published August 20, 2018.
Communicated by M. Mursaleen.

Definition 1.1. *[17, 18] Let (f,g) be a pair of self-mappings on a set X. An element $x \in X$ is called a coincidence point of the pair if $gx = fx = y$, for some $y \in X$. In this case, y is called a point of coincidence. Further, if $x = y$ then x is called a common fixed point.*

Definition 1.2. *[24] A triple $(X, d, \preceq)$ is called an ordered metric space if (X, d) is a metric space and $(X, \preceq)$ is an ordered set. Moreover, two elements $x, y \in X$ are said to be comparable if either $x \preceq y$ or $y \preceq x$. For brevity, we denote it by $x \prec\succ y$.*

Let $\{x_n\}$ be a sequence in an ordered metric space $(X, d, \preceq)$. Then, if $\{x_n\}$ is decreasing (resp. increasing, monotone) and converges to x, we denote it by $x_n \downarrow x$ (resp. $x_n \uparrow x$, $x_n \uparrow\downarrow x$).

Definition 1.3. *[8] Let (f,g) be a pair of self-mappings on an ordered metric space $(X, \preceq)$. Then*

(i) f is said to be g-increasing if $gx \preceq gy \Rightarrow fx \preceq fy$, for all $x, y \in X$,
(ii) f is said to be g-decreasing if $gx \preceq gy \Rightarrow fx \succeq fy$, for all $x, y \in X$,
(iii) f is said to be g-monotone if f is either g-increasing or g-decreasing.

On setting $g = I$, Definition 1.3(i), Definition 1.3(ii) and Definition 1.3(iii) reduce to the usual definition of increasing, decreasing and monotone mapping f on X respectively.

Definition 1.4. *A pair (f,g) of self-mappings on an ordered metric space $(X, d, \preceq)$ is said to be:*

(i) [27] weakly commuting if $d(g(fx), f(gx)) \leq d(gx, fx)$, for all $x \in X$.
(ii) [19] compatible if $\lim_{n\to\infty} d(g(fx_n), f(gx_n)) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n$.
(iii) [30] O-compatible if $\lim_{n\to\infty} d(g(fx_n), f(gx_n)) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\{gx_n\}$ is monotone and $\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n$.
(iv) [3] $\overline{\mathrm{O}}$-compatible (resp. $\underline{\mathrm{O}}$-compatible, O-compatible) if

$$\lim_{n\to\infty} d(g(fx_n), f(gx_n)) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\{gx_n\}$ and $\{fx_n\}$ are increasing (resp. decreasing, monotone) sequences with $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n$.
(v) [17] weakly compatible if $g(fx) = f(gx)$, for every coincidence point $x \in X$ of the pair (f,g).

Observe that, the notion of O-compatibility is slightly weaker than the notion of O-compatibility defined by Luong and Thuan [30]. Luong and Thuan assumed that only the sequence $\{gx_n\}$ is monotone but here both $\{gx_n\}$ and $\{fx_n\}$ assumed to be monotone ($cf.$ [3]).

Remark 1.1. *In an ordered metric space, weak commutativity $\Rightarrow$ compatibility $\Rightarrow$ O-compatibility $\Rightarrow$ O-compatibility $\Rightarrow$ $\overline{\mathrm{O}}$-compatibility (as well as $\underline{\mathrm{O}}$-compatibility) $\Rightarrow$ weak compatibility. However, in general, reverse implications are not true (see [4]).*

Definition 1.5. *[26] Let (f,g) be a pair of self-mappings on an ordered metric space $(X, d, \preceq)$. Then f is said to be g-continuous at $x \in X$ if for every sequence $\{x_n\} \subset X$, $gx_n \xrightarrow{d} gx \Rightarrow fx_n \xrightarrow{d} fx$. Moreover, f is called g-continuous if it is g-continuous at every point of X.*

Definition 1.6. *[3] Let (f, g) be a pair of self-mappings on an ordered metric space $(X, d, \preceq)$ and $x \in X$. Then f is called $(g, \overline{\mathrm{O}})$-continuous (resp. $(g, \underline{\mathrm{O}})$-continuous, (g, O)-continuous) at x if $fx_n \xrightarrow{d} fx$, for every sequence $\{x_n\} \subset X$ with $gx_n \uparrow gx$ (resp. $gx_n \downarrow gx$, $gx_n \uparrow\downarrow gx$). Moreover, f is called (g, O)-continuous (resp. $(g, \overline{\mathrm{O}})$-continuous, $(g, \underline{\mathrm{O}})$-continuous) if it is (g, O)-continuous (resp. $(g, \overline{\mathrm{O}})$-continuous, $(g, \underline{\mathrm{O}})$-continuous) at every point of X.*

Notice that, on setting $g = I$, (g, O)-continuity (resp. $(g, \overline{\mathrm{O}})$-continuity, $(g, \underline{\mathrm{O}})$-continuity) reduces to O-continuity (resp. $\overline{\mathrm{O}}$-continuity, $\underline{\mathrm{O}}$-continuity).

Remark 1.2. *In an ordered metric space, g-continuity $\Rightarrow$ (g, O)-continuity $\Rightarrow$ $(g, \overline{\mathrm{O}})$-continuity (as well as $(g, \underline{\mathrm{O}})$-continuity).*

Definition 1.7. *[3] An ordered metric space $(X, d, \preceq)$ is called $\overline{\mathrm{O}}$-complete (resp. $\underline{\mathrm{O}}$-complete, O-complete) if every increasing (resp. decreasing, monotone) Cauchy sequence in X converges to a point of X.*

Remark 1.3. *In an ordered metric space, completeness $\Rightarrow$ O-completeness $\Rightarrow$ $\overline{\mathrm{O}}$-completeness (as well as $\underline{\mathrm{O}}$-completeness).*

Definition 1.8. *[4] Let (f, g) be a pair of self-mappings on an ordered metric space $(X, d, \preceq)$. Then $(X, d, \preceq)$ is said to enjoy g-ICU property if g-image of every increasing convergent sequence $\{x_n\}$ in X is bounded above by g-image of its limit (as an upper bound), i.e.,*

$$x_n \uparrow x \Rightarrow gx_n \preceq gx \ \ \forall \ n \in \mathbb{N}_0,$$

Notice that, under the restriction $g = I$, the notion of g-ICU property gives rise ICU property.

Definition 1.9. *[3] Let (f, g) be a pair of self-mappings on an ordered metric space $(X, \preceq)$. Then*

- *(i) $(X, d, \preceq)$ is said to enjoy g-ICC property if every increasing convergent sequence $\{x_n\}$ in X has a subsequence $\{x_{n_k}\}$ such that every term of $\{gx_{n_k}\}$ is comparable with g-image of the limit of $\{x_n\}$.*
- *(ii) $(X, d, \preceq)$ is said to enjoy g-DCC property if every decreasing convergent sequence $\{x_n\}$ in X has a subsequence $\{x_{n_k}\}$ such that every term of $\{gx_{n_k}\}$ is comparable with g-image of the limit of $\{x_n\}$.*
- *(iii) $(X, d, \preceq)$ is said to enjoy g-MCC property if every monotone convergent sequence $\{x_n\}$ in X has a subsequence $\{x_{n_k}\}$ such that every term of $\{gx_{n_k}\}$ is comparable with g-image of the limit of $\{x_n\}$.*

Notice that, under the restriction $g = I$, the notions of g-ICC property, g-DCC property and g-MCC property respectively give rise ICC property, DCC property and MCC property.

Remark 1.4. *For an ordered metric space, g-ICU property $\Rightarrow$ g-ICC property.*

Definition 1.10. *[2] Let (f, g) be a pair of self-mappings on an ordered metric space $(X, \preceq)$. Then $(X, \preceq)$ is (f, g)-directed if for every pair $x, y \in X$, $\exists$ $z \in X$ such that $fx \prec\succ gz$ and $fy \prec\succ gz$. In cases $g = I$ (resp. $f = g = I$), $(X, \preceq)$ is called f-directed (resp. directed).*

Definition 1.11. *[2] Let $(X, \preceq)$ be an ordered set, $Y \subseteq X$ and $a, b \in Y$. A finite subset $\{e_1, e_2, ..., e_k\}$ of Y is called $\prec\succ$-chain between a and b in Y if*

(i) $k \geq 2$,

(ii) $e_1 = a$ and $e_k = b$,

(iii) $e_i \prec\succ e_{i+1}$ *for every* i $(1 \leq i \leq k-1)$.

We denote by $\mathrm{C}(a, b, \prec\succ, Y)$ the class of all $\prec\succ$-chains between a and b in Y. In particular for $Y = X$, we write $\mathrm{C}(a, b, \prec\succ)$ instead of $\mathrm{C}(a, b, \prec\succ, X)$.

In order to extend Banach Contraction Principle, several authors attempted to replace the contraction (nonnegative) constant by a mapping $\varphi : [0, \infty) \to [0, \infty)$ satisfying $\varphi(t) < t$ for all $t > 0$. Such contractions are often called nonlinear contractions. This trend was initiated by Browder [6] in 1968. Later, many authors generalized Browder's fixed point theorem by varying the properties of the function φ. One of the important generalizations of Browder's fixed point theorem is due to Boyd and Wong [5] where the authors assumed φ to satisfy: $\varphi(t) < t$ and $\limsup\limits_{r \to t^+} \varphi(r) < t$ for every $t > 0$. Turinici [29] called such φ as Boyd-Wong functions (in short BW-function).

On the similar lines, in 1975, Matkowski [21] initiated a special type of nonlinear contractions, wherein φ was referred as a comparison function. A function $\varphi : [0, \infty) \to [0, \infty)$ is called a Matkowski-type (comparison) if it is increasing and satisfying $\lim\limits_{n \to \infty} \varphi^n(t) = 0$ for all $t > 0$.

The following lemma highlights some basic proprieties of Matkowski-type functions.

Lemma 1.1. *[20, 21] Let φ be a Matkowski-type function then $\varphi(0) = 0$ and $\varphi(t) < t$ for all $t > 0$.*

Matkowski-type functions differ from BW-functions. To substantiate this fact, consider the functions $\varphi, \psi : [0, \infty) \to [0, \infty)$ defined by:

$$\varphi(t) = \begin{cases} 0, & \text{if } t = 0, \\ \frac{1}{n+1}, & \text{if } t \in (\frac{1}{n+1}, \frac{1}{n}], n = 1, 2, 3, ..., \\ 1, & \text{if } t > 1, \end{cases} \quad ; \ \psi(t) = \begin{cases} \frac{t}{5}, & \text{if } t < 2, \\ \frac{1}{t}, & \text{if } t \geq 2. \end{cases}$$

Then φ is a Matkowski-type function but not BW-function as it is not upper semi continuous from the right (see [21]). The same function is also used in the form of Example 3 in [15] and Example 2 in [16]. On the other hand, the non-increasing function ψ is BW-function but not a Matkowski-type. It is worth mentioning here that Ćirić [7] proved that for a BW-function φ, $\lim\limits_{n \to \infty} \varphi^n(t) = 0$ for all $t > 0$. Unfortunately, this is false as pointed out in Jachymski [15] (see also [12, 14]).

Now, we record some results which will be used later to prove our results.

Lemma 1.2. *[4] Let (f, g) be a pair of self-mappings defined on an ordered set $(X, \preceq)$. If f is g-monotone and $gx = gy$, then $fx = fy$.*

Lemma 1.3. *[4] Let (f, g) be a pair of weakly compatible self-mappings defined a non-empty set X. Then every point of coincidence of the pair (f, g) is also a coincidence point.*

Lemma 1.4. *[11] Let X be a nonempty set and f a self-mapping on X. Then, there exists a subset $Y \subseteq X$ such that $f(Y) = f(X)$ and $f : Y \to X$ is one-one.*

Agarwal *et al.* in [1] presented the following fixed point theorem for Matkowski-type nonlinear contractions in an ordered metric space.

Theorem 1.1 (Theorem 2.1 in [1])**.** *Let $(X, d, \preceq)$ be an ordered metric space and f a self-mapping on X. Suppose that the following conditions hold:*
(a) (X, d) is complete,
(b) f is increasing,
(c) either f is continuous or $(X, d, \preceq)$ enjoys ICU property,
(d) there exists $x_0 \in X$ such that $x_0 \preceq fx_0$,
(e) there exists a Matkowski-type function φ such that $d(fx, fy) \leq \varphi(d(x, y))$ for all $x, y \in X$ with $x \succeq y$.

Then f has a fixed point.

The aim of this paper is to extend Theorem 1.1 to a pair (f, g) of self-mappings on an ordered metric space X wherein either X or one of the subspaces $f(X)$ and $g(X)$ is complete. Some illustrative examples are also furnished to highlight the realized improvements in our results. An applications is also discussed.

2. Main Results

Now, we are equipped to prove our main result as follows:

Theorem 2.1. *Let $(X, d, \preceq)$ be an ordered metric space and (f, g) a pair of self-mappings on X. Suppose that the following conditions hold:*
(a) $f(X) \subseteq g(X)$,
(b) f is g-increasing,
(c) there exists $x_0 \in X$ such that $gx_0 \preceq fx_0$,
(d) there exists a Matkowski-type function φ such that

$$d(fx, fy) \leq \varphi(d(gx, gy)), \ \forall \ x, y \in X \text{ with } gx \prec\succ gy,$$

and either
(e) (e1) (X, d) is $\overline{\mathrm{O}}$-complete,
(e2) (f, g) is $\overline{\mathrm{O}}$-compatible pair,
(e3) g is $\overline{\mathrm{O}}$-continuous,
(e4) either f is $\overline{\mathrm{O}}$-continuous or $(X, d, \preceq)$ enjoys g-ICC property.

or alternately
(è) (è1) either $(g(X), d, \preceq)$ or $(f(X), d, \preceq)$ is $\overline{\mathrm{O}}$-complete,
(è2) either f is $(g,\overline{\mathrm{O}})$-continuous or f and g are continuous or $(g(X), d, \preceq)$ enjoys ICC property.

Then the pair (f, g) has a coincidence point.

Proof. The proof of this theorem is divided into two parts. The first part is levelled as Step 1 wherein we use conditions (a)-(d), while the second part is levelled as Step 2 wherein we use conditions embodied in (e) or alternatively, Step $2'$ where we use conditions contained in (e').

Step1: Let $x_0 \in X$ such that $gx_0 \preceq fx_0$ and consider the sequence $\{x_n\} \subseteq X$ defined by

$$fx_n = gx_{n+1} \text{ for all } n \in \mathbb{N}_0 \tag{2.1}$$

Since $gx_0 \preceq fx_0$ and $fx_0 = gx_1$, we have $gx_0 \preceq gx_1$. As f is $g-$increasing, we must have $fx_0 \preceq fx_1$. Continuing this process inductively, we define an increasing sequences $\{gx_n\}$ and $\{fx_n\}$ satisfying (2.1). Notice that, if $fx_{n+1} = fx_n$ for any $n \in \mathbb{N}_0$, then x_{n+1} is a coincidence point of the pair (f, g) and no need more. So, we may assume such equality does not occur for all $n \in \mathbb{N}_0$.

Now we show that $\{gx_n\}$ as well as $\{fx_n\}$ are Cauchy sequences. Since the terms of the increasing sequence $\{gx_n\}$ are comparable, then by (d) we have,

$$\begin{array}{rcl} d(fx_2, fx_1) & \leq & \varphi(d(gx_2, gx_1)) \\ & = & \varphi(d(fx_1, fx_0)). \end{array}$$

By induction

$$d(fx_{n+1}, fx_n) \leq \varphi^n(d(fx_1, fx_0) \to 0 \ as \ n \to \infty.$$

Let ϵ be fixed. Choose $n \in \mathbb{N}_0$ so that

$$d(fx_{n+1}, fx_n) < \epsilon - \varphi(\epsilon).$$

Now,

$$\begin{array}{rcl} d(fx_{n+2}, fx_n) & \leq & d(fx_{n+2}, fx_{n+1}) + d(fx_{n+1}, fx_n) \\ & < & \varphi(d(gx_{n+2}, gx_{n+1}) + \epsilon - \varphi(\epsilon) \\ & = & \varphi(d(fx_{n+1}, fx_n) + \epsilon - \varphi(\epsilon) \\ & \leq & \varphi(\epsilon - \varphi(\epsilon)) + \epsilon - \varphi(\epsilon) \\ & \leq & \varphi(\epsilon) + \epsilon - \varphi(\epsilon) \\ & = & \epsilon. \end{array}$$

Also,

$$\begin{array}{rcl} d(fx_{n+3}, fx_n) & \leq & d(fx_{n+3}, fx_{n+1}) + d(fx_{n+1}, fx_n) \\ & < & \varphi(d(gx_{n+3}, gx_{n+1})) + \epsilon - \varphi(\epsilon) \\ & = & \varphi(d(fx_{n+2}, fx_n)) + \epsilon - \varphi(\epsilon) \\ & \leq & \varphi(\epsilon) + \epsilon - \varphi(\epsilon) \\ & = & \epsilon. \end{array}$$

By induction

$$d(fx_{n+k}, fx_n) < \epsilon, \ for \ all \ k \in \mathbb{N}.$$

Thus, $\{fx_n\}$ is a Cauchy sequence and by (2.1) $\{gx_n\}$ is so.
Step2: Owing to $(e1)$ and (2.1), there exists some $z \in X$ such that

$$gx_n \uparrow z \ and \ fx_n \uparrow z. \tag{2.2}$$

In view of the $\overline{\text{O}}$-continuity of g, we have

$$\lim_{n\to\infty} g(gx_n) = \lim_{n\to\infty} g(fx_n) = gz. \tag{2.3}$$

Using the $\overline{\text{O}}$-compatibility of the pair (f, g) (i.e., condition (e2)), we have

$$\lim_{n\to\infty} d(g(fx_n), f(gx_n)) = 0. \tag{2.4}$$

Now, we show that z is a coincidence point of the pair (f, g) using assumption $(e4)$. To accomplish this, let f be $\overline{\text{O}}$-continuous. Then using (2.2), we have

$$\lim_{n\to\infty} f(gx_n) = f(\lim_{n\to\infty} gx_n) = fz. \tag{2.5}$$

By combining (2.3), (2.4) and (2.5), we conclude that $fz = gz$ and we are done.

Alternately, let $(X, d, \preceq)$ enjoy g- ICC property. Since $gx_n \uparrow z$, there exists a subsequence $\{gx_{n_k}\}$ of $\{gx_n\}$ such that

$$g(gx_{n_k}) \prec\succ gz \;\forall\; k \;\in \mathbb{N}_0. \tag{2.6}$$

By applying condition (d) on (2.6) and using Lemma 1.1 (distinguishing the cases whether $d(g(gx_{n_k}), gz)$ is zero or non-zero), we obtain

$$\begin{aligned} d(f(gx_{n_k}), fz) &\leq \varphi(d(g(gx_{n_k}), gz)), \\ &\leq d(g(gx_{n_k}), gz). \end{aligned} \tag{2.7}$$

By using (2.3), (2.4), (2.7) and the triangular inequality, we get

$$\begin{aligned} d(gz, fz) &\leq d(gz, g(fx_{n_k})) + d(g(fx_{n_k}), f(gx_{n_k})) + d(f(gx_{n_k}), fz) \\ &\leq d(gz, g(fx_{n_k})) + d(g(fx_{n_k}), f(gx_{n_k})) + d(g(gx_{n_k}), gz) \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus $z \in X$ is a coincidence point of the pair (f, g) and hence we are also through with this Step as well.

step2′: In step 1, we constructed an increasing Cauchy sequences $\{gx_n\}$ and $\{fx_n\}$. Assume that $(e'1)$ hold.

If $(g(X), d, \preceq)$ is $\overline{\text{O}}$-complete, then there exists $u \in X$ such that $gx_n \uparrow gu$. Alternately, if $(f(X), d \preceq)$ is $\overline{\text{O}}$-complete then there exists some $w \in X$ such that $fx_n \uparrow fw$. By assumption (a), there exists $u \in X$ with $fw = gu$ which along with (2.1) give,

$$gx_n \uparrow gu. \tag{2.8}$$

Now, by $(e'2)$ we show that u is a coincidence point of the pair (f, g). Firstly, suppose that f is $(g, \overline{\text{O}})$-continuous. From (2.8) we get, $fx_n \uparrow fu$ i.e., $gx_{n+1} \uparrow fu$. Now, the uniqueness of the limit implies $gu = fu$ and hence we are through.

Next, suppose that f and g are continuous mappings, then our proof can be completed on the lines of the proof of Theorem 1 in [4] wherein Lemma 1.4 is used.

Finally, suppose that $(g(X), d, \preceq)$ enjoys ICC property. Since $gx_n \uparrow gu$ there exists a subsequence $\{gx_{n_k}\}$ of $\{gx_n\}$ such that

$$gx_{n_k} \prec\succ gu \;,\; \forall\; k \;\in \mathbb{N}_0.$$

Using assumption (d), (2.1) and Lemma 1.1 we obtain

$$\begin{aligned} d(gx_{n_k+1}, fu) &= d(fx_{n_k}, fu) \\ &\leq \varphi(d(gx_{n_k}, gu)) \\ &\leq d(gx_{n_k}, gu) \end{aligned} \tag{2.9}$$

On using (2.8), (2.9) and continuity of d, we get

$$\begin{aligned} d(gu, fu) &= d(\lim_{n\to\infty} gx_{n_k+1}, fu) \\ &= \lim_{n\to\infty} d(gx_{n_k+1}, fu) \\ &\leq \lim_{n\to\infty} d(gx_{n_k}, gu) \\ &= 0 \end{aligned}$$

so that, $gu = fu$. Hence u is a coincidence point of the pair (f, g). □

Example 2.1. *Consider $X = (-1, 0]$ with usual metric and usual order. Then, $(X, d, \leq)$ is an ordered metric space. Define $f, g : X \to X$ by $fx = \frac{x^2}{4}$* and $gx = \frac{x^2}{2}$ *$\forall\, x \in X$. Define $\varphi : [0, \infty) \to [0, \infty)$ by $\varphi(t) = \alpha t\ \forall\, t \in [0, \infty)$, where $\alpha \in [\frac{1}{2}, 1)$. Observe that, $(X, d, \leq)$ is $\overline{\mathrm{O}}$-complete and rest of the conditions of Theorem 2.1 are satisfied ensuring the existence of the coincidence point (namely $x = 0$).*

The following result is relatively more natural than Theorem 2.1.

Theorem 2.2. *Let $(X, d, \preceq)$ be an ordered metric space and (f, g) a pair of self-mappings on X. Suppose that the following conditions hold:*
(a) $f(X) \subseteq g(X)$,
(b) f is g-increasing,
(c) there exists $x_0 \in X$ such that $gx_0 \preceq fx_0$,
(d) there exists a Matkowski-type function φ such that

$$d(fx, fy) \leq \varphi(d(gx, gy)),\ \ \forall\, x, y \in X \text{ with } gx \prec\succ gy,$$

and either
(e) (e1) (X, d) is complete,
(e2) (f, g) is O-compatible pair,
(e3) g is continuous,
(e4) either f is continuous or $(X, d, \preceq)$ enjoys g-ICC property,
or alternately
(è) (è1) either $(g(X), d)$ or $(f(X), d)$ is complete,
(è2) either f and g are continuous or $(g(X), d, \preceq)$ enjoys ICC property.
Then the pair (f, g) has a coincidence point.

Proof. In view of Remarks 1.1, 1.2, 1.3 and 1.4, the result follows via Theorem 2.1. □

Remark 2.1. *In the context of Example 2.1, the Cauchy sequence $x_n = -1 + \frac{1}{n}$ does not converge in X. Also, neither $f(X)$ nor $g(X)$ are complete metric spaces so that Theorem 2.2 can not be used in the context of the example. This demonstrates the advantage of Theorem 2.1 over Theorem 2.2.*

Owing to Remark 1.1 , we have the following.

Corollary 2.1. *Theorems 2.1–2.2 remain true if we replace condition (e2) by any of the following two conditions (besides retaining the rest of the hypotheses):*
(e2)′ *(f, g) is weakly commuting pair.*
(e2)″ *(f, g) is compatible pair.*

Corollary 2.2. *Theorems 2.1-2.2 remain true if we replace condition* (d) *by the following (besides retaining the rest of the hypotheses):*

$(d)'$ *there exits* $\alpha \in [0,1)$ *such that*

$$d(fx, fy) \leq \alpha d(gx, gy), \ \ \forall\ x, y \in X \text{ with } gx \prec\succ gy.$$

Proof. The result follows by setting $\varphi(t) = \alpha t$ with $\alpha \in [0,1)$. □

On setting $g = I$ in Theorem 2.2 (same can be done for Theorems 2.1), we deduce the following:

Corollary 2.3. *Let* $(X, d, \preceq)$ *be an ordered metric space and* f *a self-mapping on* X*. Suppose that the following conditions hold:*

(a) *either* $(f(X), d)$ *or* (X, d) *is complete,*
(b) f *is increasing,*
(c) *either* f *is continuous or* $(X, d, \preceq)$ *enjoys ICU property,*
(d) *there exists* $x_0 \in X$ *such that* $x_0 \preceq fx_0$,
(e) *there exists a Matkowski-type function* φ *such that*

$$d(fx, fy) \leq \varphi(d(x, y)), \ \ \forall\ x, y \in X \text{ with } x \prec\succ y.$$

Then f *has a fixed point.*

Remark 2.2. *Note that, above corollary covers Theorem 1.1, that is, the completeness of* $f(X)$ *is enough to get the result where* X *may or may not be complete.*

On setting $\varphi(t) = \alpha t$ (with $\alpha \in [0,1)$), and $g = I$ in Theorem 2.2, we deduce the following.

Corollary 2.4. *Corollary 2.3 remains true if the condition* (e) *is replaced by the following condition (besides retaining the rest of the hypotheses):*

$(e)'$ *there exists* $\alpha \in [0,1)$ *such that*

$$d(fx, fy) \leq \alpha d(x, y), \ \ \forall\ x, y \in X \text{ with } x \prec\succ y.$$

Remark 2.3. *Corollary 2.4 (and a fortiori to Corollary 2.3 and Theorem 2.2) covers Theorems 2.1, 2.2 and 2.4 of Nieto and Rodríguez-López [22] and Theorem 2.1 of Ran and Reurings [25]. Observe that, in mentioned theorems the completeness of* X *can be alternately replaced by the completeness of* $f(X)$.

As an application of Theorem 2.1 (same can be done for Theorem 2.2), we have the following coincidence point result for mappings satisfying Matkowski-integral type contraction in ordered metric space (see [14]).

Let Π be the set of functions $\omega : [0, \infty) \to [0, \infty)$ satisfying the following:

(a) ω is Lebesgue-integrable mapping on every compact subset of $[0, \infty)$;
(b) $\int_0^{\epsilon} \omega(t)dt > 0$ for all $\epsilon > 0$.
(c) $\int_0^{\int_0^{\vdots^{\int_0^x \omega(t)dt}} \omega(t)dt} \omega(t)dt$ tends to zero when the number of integral iterates tends to ∞.

Corollary 2.5. *Let* (f, g) *be a pair pf self-mappings on an ordered metric space* $(X, d, \preceq)$*. Suppose that for every* $x, y \in X$ *with* $gx \preceq gy$*, we have*

$$d(fx, fy) \leq \int_0^{\varphi(d(gx, gy))} \omega(t)dt, \tag{2.10}$$

where φ is a Matkowski-type function and $\omega \in \Pi$. If assumptions (a), (b) and (c) along with either (e) or (e′) of Theorem 2.1 are satisfied, then the pair (f, g) has a coincidence point.

Proof. Define $\Gamma : [0, \infty) \to [0, \infty)$ by $\Gamma(x) = \int_0^x \omega(t)dt$, then Γ is continuous as well as increasing function. Now, (2.10) can be written as

$$d(fx, fy) \leq (\Gamma \circ \varphi)(d(gx, gy)).$$

Owing to Theorem 2.1, it is enough to show that $\Gamma\circ\varphi$ is a Matkowski-type function. Obviously $\Gamma \circ \varphi$ is continuous and increasing. Set $I_n := (\Gamma \circ \varphi)^n(t)$, we show that $\lim_{n\to\infty} I_n = 0$.
$I_1 = (\Gamma \circ \varphi)(t) = \int_0^{\varphi(t)} \omega(s)ds \leq \int_0^t \omega(s)ds$ (due to part (a) of Lemma 1.1).
$I_2 = (\Gamma \circ \varphi)^2(t) = (\Gamma \circ \varphi)(I_1) = \int_0^{\varphi(I_1)} \omega(s)ds \leq \int_0^{I_1} \omega(s)ds \leq \int_0^{\int_0^t \omega(s)ds} \omega(s)ds.$
By induction, we have

$$I_n \leq \int_0^{\int_0^{\cdot^{\cdot^{\int_0^t \omega(s)ds}}} \omega(s)ds} \omega(s)ds,$$

where the integral is taken n times. On letting $n \to \infty$ and using the definition of Π, we get $\lim_{n\to\infty} I_n = 0$. □

3. Uniqueness Results

Theorem 3.1. *In addition to the hypotheses $(a) - (d)$ along with (e') in any one of the Theorems 2.1-2.2, if one of the following conditions holds:*
$(u_1) : C(fx, fy, \prec\succ, g(X))$ *is nonempty, for every* $x, y \in X$,
$(u_2) : (X, \prec)$ *is* $(f, g) -$ *directed,*
$(u_3) : (f(X), \prec)$ *is totally ordered,*
then the pair (f, g) has a unique point of coincidence.

Proof. We opt to prove this result corresponding to Theorem 2.1 when the condition (u_1) holds. The proof of results corresponding to other Theorem is similar, hence it is omitted once we show that each of the other conditions implies (u_1).

Assume that (u_1) holds *i.e.* there exist $x, y \in X$ such that

$$gx = fx = \overline{x} \text{ and } gy = fy = \overline{y}, \tag{3.1}$$

for some $\overline{x}, \overline{y} \in X$. We show that $\overline{x} = \overline{y}$.

Now, the proof of our theorem is similar to that of Theorem 4 of [3] except proving the claim (15) that is:

$$\lim_{n\to\infty} t_n^i = 0 \text{ for every } 1 \leq i \leq k-1.$$

On fixing i, two cases arise: If $t_m^i = d(gz_m^i, gz_m^{i+1}) = 0$ for some $m \in \mathbb{N}_0$, then the proof can be completed on the lines of Theorem 4 of [3]. Secondly, if $t_m^i >$

$0,\ for\ all\ n \in \mathbb{N}_0$. Then we have

$$\begin{aligned} t^i_{n+1} &= d(gz^i_{n+1}, gz^{i+1}_{n+1}) \\ &= d(fz^i_n, fz^{i+1}_n) \\ &\leq \varphi(d(gz^i_n, gz^{i+1}_n)) \\ &= \varphi(t^i_n) \\ &\leq \varphi^2(t^i_{n-1}) \\ &\leq \vdots \\ &\leq \varphi^n(t^i_1), \end{aligned}$$

which on making $n \to \infty$ on both the sides gives rise $\lim_{n\to\infty} t^i_n = 0$. Thus, in all, our claim is established.

In order to show that$(u_2) \Rightarrow (u_1)$, suppose that (u_2) holds. Then for every pair $x, y \in X$, $\exists\ z \in X$ such that

$$fx \prec\succ gz \prec\succ fy,$$

which together with condition (a) imply that $\{fx, gz, fy\}$ is a $\prec\succ$-chain in between fx and fy. It follows that $\mathrm{C}(fx, fy, \prec\succ, g(X))$ is nonempty for each $x, y \in X$, *i.e.*, (u_1) holds which amounts to say that (u_2) implies (u_1).

Similarly we can show that $(u_3) \Rightarrow (u_1)$. □

Remark 3.1. *Notice that, letting $g = I$ in Theorem 3.1 leads to a generalization of Theorem 4 in [23].*

Theorem 3.2. *In addition to the hypotheses of Theorem 3.1 if the following condition holds:*
$(u_4):$ *one of f and g is one − one mapping.*
then the pair (f, g) has a unique coincidence point.

Proof. Let f be one-one mapping (same argument holds if g is so). Assume there exist two coincidence points $x, y \in X$. Owing to Theorem 3.1, there exist a unique point of coincidence which in turn implies that $gx = fx = gy = fy$. As f is one-one, we have $x = y$. □

Theorem 3.3. *In addition to the hypotheses of Theorem 3.1 if the following condition holds:*
$(u_5) : (f, g)$ *is a weakly compatible pair ,*
then the pair (f, g) has a unique common fixed point.

Proof. Let $x \in X$ be a coincidence point of the pair (f, g), then there is $\overline{x} \in X$ such that $gx = fx = \overline{x}$. By Lemma 1.3, $\overline{x}$ itself is a coincidence point, that is $f\overline{x} = g\overline{x}$. Now, Theorem 3.1 ensures that $\overline{x} = x$ which concludes the proof. □

Theorem 3.4. *In addition to the conditions $(a) - (e)$ of the hypothesis of Theorem 2.1 (resp. Theorem 2.2), if any of the conditions (u_1), (u_2) or (u_3) of theorem 3.1 holds, then the pair (f, g) has a unique common fixed point.*

Proof. By remark 1.1, every $\overline{\mathrm{O}}$-compatibly pair (resp. $\mathtt{O}$-compatibly pair) is a weakly compatible pair so that (u_5) holds trivially. Now, proceeding on the lines similar to the proof of Theorem 3.1 (resp. Theorem 3.3) our result follows. □

The following example demonstrates the utility of Theorem 2.2 and the corresponding uniqueness theorems (*i.e.*, Theorems 3.1 and 3.2).

Example 3.1. *Consider $X = [-8, \infty)$ equipped with usual metric and usual order. Then, $(X, d, \leq)$ is an ordered metric space. Define $f, g : X \to X$ by $fx =$ 2 and $gx = x^2 - 7$, $\forall\ x \in X$. Define $\varphi : [0, \infty) \to [0, \infty)$ by $\varphi(t) = \alpha t$, where $\alpha \in [0, 1)$. Then, by a routine calculation one can verify that all the conditions of Theorem 2.2 are satisfied except condition (e) (wherein (e_2) does not hold). Hence, the pair (f, g) has a coincidence point in X. Moreover, (u_1) also holds and henceforth, in view of Theorem 3.1, the pair (f, g) has a unique point of coincidence namely: $\overline{x} = 2$. Finally, notice that, (u_4) does not hold (i.e. neither f nor g is one-one) so that Theorem 3.2 ensuring the uniqueness of the coincidence point can not be used in the present context. Observe that the involved maps have two coincidence points (namely: $x = 3$ and $x = -3$).*

Our next example demonstrates the utility of Theorem 2.2 and the corresponding uniqueness theorem (*i.e.* Theorems 3.4).

Example 3.2. *Consider $X = \mathbb{R}$ equipped with usual metric. Then, $(X, d, \leq)$ is an ordered metric space wherein the partial order is defined by: $x \leq y \Leftrightarrow |x| \leq |y|$ and $xy \geq 0$. Define $f, g : X \to X$ by $fx = \frac{x^2}{14}$ and $gx = \frac{x^2}{2}$ $\forall\ x, y \in X$. Clearly f is g-increasing. Define $\varphi : [0, \infty) \to [0, \infty)$ by*

$$\varphi(x) = \begin{cases} \frac{x}{5} & \text{if } x < 2, \\ \frac{x}{6} & \text{if } x \geq 2. \end{cases}$$

Notice that, φ is a Matkowski-type function. Now, for all $x, y \in X$ with $gx \leq gy$, we have

$$d(fx, fy) = \left|\frac{x^2}{14} - \frac{y^2}{14}\right| = \frac{2}{14}\left|\frac{x^2}{2} - \frac{y^2}{2}\right| = \frac{1}{7}d(gx, gy) < \varphi(d(gx, gy)).$$

Observe that, f, g and φ satisfy assumptions $(a) - (e)$ of Theorem 2.2 so that the pair (f, g) has a coincidence point in X. Furthermore, condition (u_2) holds, therefore, in view of Theorem 3.4, the pair (f, g) has a unique common fixed point namely: $x = 0$.

Remark 3.2. *One can obtain dual type results corresponding all forgoing results by replacing "$\overline{\text{O}}$-analogues" with "$\underline{\text{O}}$-analogues" and " ICC -property" with "DCC-property" provided the existence of $x_0 \in X$ such that $gx_0 \preceq fx_0$ is replaced by the existence of $x_0 \in X$ such that $gx_0 \succeq fx_0$.*

Remark 3.3. *One can obtain companied type results corresponding to all forgoing results by replacing "$\overline{\text{O}}$-analogues" with "O-analogues" and " ICC -property" with "MCC-property" provided the existence of $x_0 \in X$ such that $gx_0 \preceq fx_0$ is replaced by the existence of $x_0 \in X$ such that $gx_0 \prec\succ fx_0$.*

Acknowledgments. The authors would like to thank the anonymous referee for his/her comments that helped us improve this article.

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Rqeeb Gubran
Department of Mathematics, Aligarh Muslim University, Aligarh - 202002 UP, India
E-mail address: rqeeeb@gmail.com

Idrees A. Khan
Department of Mathematics Integral University, Lucknow - 226026 UP, India
E-mail address: idrees_maths@yahoo.com

Mohammad Imdad
Department of Mathematics, Aligarh Muslim University, Aligarh - 202002 U.P.India
E-mail address: mhimdad@gmail.com