Journal of Inequalities and Special Functions
ISSN: 2217-4303, URL: http://ilirias.com/jiasf
Volume 9 Issue 3(2018), Pages 19-27.

ORTHOGONAL POLYNOMIALS IN TWO VARIABLES: FIVE-TERM RECURRENCE RELATION AND RELATED RESULTS

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Abstract. We prove that a family of orthogonal polynomials $P_{n,m}(x,y)$ satisfy a five-term recurrence relation, which is a "natural" generalization of the three-term recurrence relation satisfied by the classical orthogonal polynomials. Moreover, we show that these polynomials satisfy a theorem similar to Favard's theorem. We also give a Darboux-Christoffel type formula for the polynomials $P_{n,m}(x,y)$.

1. Introduction

The study of orthogonal polynomials (OP) in two variables is an old subject and goes back as least as far as Hermite. In [10], a historical review as well as the results concerning OP in two variables until 1988, were reported in detail by Suetin. Recently in [3], Dunckl and Xu presented developments of the current research on multivariable OP, using vector matrices notation. Both of the aforementioned books contain an exhaustive bibliography.

It is well-known [1, 11], that one of the most important properties of a family of OP, $\{\phi_n(x)\}_{n\geq 0}$, in one variable is the three-term recurrence relation:

$$\alpha_n\phi_{n+1}(x) + \beta_n\phi_n(x) + \alpha_{n-1}\phi_{n-1}(x) = x\phi_n(x), \quad n \geq 0$$

$$\phi_{-1}(x) = 0, \quad \phi_0(x) = 1$$

where $\alpha_n > 0$. It is also known that $\{\phi_n(x)\}_{n\geq 0}$ satisfy Favard's theorem. A survey on the "Favard theorem" and its extensions, for OP in one variable, can be found in [9]. Recurrence relations satisfied by OP in two variables were studied by Krall and Sheffer in [8] and by Suetin in [10]. In addition, the three-term recurrence relation satisfied by OP in several variables, in connection with Favard's theorem were studied in [2, 3, 6, 7, 9, 12, 13] using vector notation.

Recently in [5], a family of polynomials $P_{n,m}(x,y)$ was studied with respect to their zeroes. The polynomials $P_{n,m}(x,y)$ were assumed to satisfy the five-term

2000 *Mathematics Subject Classification.* 42C05, 33C47.
Key words and phrases. orthogonal polynomials in two variables, recurrence relation, Darboux-Christoffel formula.

Submitted December 20, 2017. Published Maj 6, 2018.
Communicated by Yilmaz Simsek.

recurrence relation

$$\begin{aligned}\alpha_{n,m}P_{n+1,m}(x,y)+\alpha_{n-1,m}P_{n-1,m}(x,y)+\beta_{n,m}P_{n,m}(x,y)+\\ \delta_{n,m}P_{n,m+1}(x,y)+\delta_{n,m-1}P_{n,m-1}(x,y)+\gamma_{n,m}P_{n,m}(x,y)\\ =(ax+by)P_{n,m}(x,y)\end{aligned} \tag{1.1}$$

with

$$\begin{aligned}P_{-1,m}(x,y)\equiv 0,\quad P_{n,-1}(x,y)\equiv 0,\quad P_{0,0}(x,y)\equiv 1\\ P_{0,m}(x,y)\equiv R_m(y),\quad P_{n,0}(x,y)\equiv Q_n(x)\end{aligned} \tag{1.2}$$

where $\alpha_{n,m}, \delta_{n,m}, \beta_{n,m}, \gamma_{n,m}$ real sequences with $\alpha_{n,m}, \delta_{n,m} > 0$ and a, b positive numbers.

We note that although the terms $\beta_{n,m}P_{n,m}(x,y)$ and $\gamma_{n,m}P_{n,m}(x,y)$ can be combined in one, it is better to keep them apart, since under the convetions

(C1): $\alpha_{i,j} \equiv \alpha_i$, $\beta_{i,j} \equiv \beta_i$, $\gamma_{i,j} \equiv 0$, $\delta_{i,j} \equiv 0$, $a = 1$, $b = 0$, $P_{i,j}(x,y) = P_i(x)$, $P_{-1}(x) \equiv 0$, $P_0(x) \equiv 1$ or

(C2): $\alpha_{i,j} \equiv 0$, $\beta_{i,j} \equiv 0$, $\gamma_{i,j} \equiv \gamma_j$, $\delta_{i,j} \equiv \delta_j$, $a = 0$, $b = 1$, $P_{i,j}(x,y) = P_j(y)$, $P_{-1}(y) \equiv 0$, $P_0(y) \equiv 1$,

(1.1)-(1.2) is reduced to the following three-term recurrence relation for the classical orthogonal polynomials (COP):

$$\begin{aligned}\alpha_i P_{i+1}(x)+\alpha_{i-1}P_{i-1}(x)+\beta_i P_i(x)=xP_i(x)\\ P_{-1}(x)\equiv 0,\quad P_0(x)\equiv 1.\end{aligned} \tag{1.3}$$

or

$$\begin{aligned}\delta_j P_{j+1}(y)+\delta_{j-1}P_{j-1}(y)+\gamma_j P_j(y)=yP_j(y)\\ P_{-1}(y)\equiv 0,\quad P_0(y)\equiv 1.\end{aligned} \tag{1.4}$$

The main result of [5] is that the zeroes of $P_{n,m}(x,y)$ satisfying (1.1)-(1.2) are connected with the eigenvalues of a specific tridiagonal operator. The method used to prove this, is a generalization of a functional analytic method introduced in [4] for classical OP. It also stated in [5], that it worths investigating whether the polynomials $P_{n,m}(x,y)$ satisfying (1.1)-(1.2) are orthogonal. The answer to this question is the main aim of the present paper. More precisely, we prove that a sequence of orthogonal polynomials (OPS) $\{P_{n,m}(x,y)\}_{n,m\geq 0}$ satisfy (1.1)-(1.2) (see theorem 3.1). Moreover, we also prove a theorem similar to Favard's theorem (see theorem 3.2). These results are given in §3. As a consequence of these results, a Darboux-Christoffel type formula is proved in §4 (see proposition 4.1). Similar formulae using vector notation were also found in [2, 3, 12, 13]. In §2 we use the definition of orthogonality for an OPS given in [2]. This is crusial for the proofs of the theorems in §3 and §4. As the reader will found out, the approach used in the present paper is the same with the one presented in [1], although the specific details are more complicated, as one would expect. We do believe that (1.1)-(1.2) is quite useful, since it is not only a natural generalization of (1.3), but also simple to handle.

2. Preliminaries

Let $\mathcal{P}$ be the linear space of real polynomials in two variables and $\mathcal{P}_{n,m}$ the subspace of the polynomials

$$P_{n,m}(x,y)=\sum_{\kappa=0}^{n}\sum_{\lambda=0}^{m}c_{\kappa,\lambda}^{n,m}x^{\kappa}y^{\lambda},\quad n\geq 0,\quad m\geq 0$$

of degree (n,m), which means, n is the degree of x, m is the degree of y and $n+m$ is the total degree of $P_{n,m}(x,y)$.

Definition 2.1 Let $\{\mu_{n,m}\}_{n\geq 0,m\geq 0}^{\infty}$ be a sequence of complex numbers and L be a complex valued function defined on the vector space of all polynomials in two variables by:

$$L[x^n y^m]=\mu_{n,m},\quad n,m\geq 0 \tag{2.1}$$

$$L[\alpha\pi_1(x,y)+\beta\pi_2(x,y)]=\alpha L[\pi_1(x,y)]+\beta L[\pi_2(x,y)], \tag{2.2}$$

for all complex numbers α,β and all polynomials $\pi_i(x,y), i=1,2$. The function L is called moment functional determined by the moment sequence $\{\mu_{n,m}\}_{n\geq 0,m\geq 0}$.

It is obvious that for every polynomial

$$\pi(x,y)=\sum_{\kappa=0}^{n}\sum_{\lambda=0}^{m}c_{\kappa,\lambda}x^{\kappa}y^{\lambda},$$

it is

$$L[\pi(x,y)]=\sum_{\kappa=0}^{n}\sum_{\lambda=0}^{m}c_{\kappa,\lambda}\mu_{\kappa,\lambda}. \tag{2.3}$$

Definition 2.2 A polynomial sequence $\{P_{n,m}(x,y)\}_{n\geq 0,m\geq 0}$ of degree (n,m) is an orthogonal polynomial sequence (OPS) with respect to L if:

$$L[P_{\kappa,\lambda}(x,y)P_{n,m}(x,y)]=0 \tag{2.4}$$

for every polynomial $P_{\kappa,\lambda}(x,y)$ of degree (κ,λ), with $0\leq\kappa\leq n, 0\leq\lambda\leq m$ and $\kappa+\lambda<n+m$. Furthermore, if $(\kappa,\lambda)=(n,m)$ then

$$L[P_{n,m}^2(x,y)]=k_{n,m}\neq 0. \tag{2.5}$$

If $k_{n,m}$ are all positive constants then the OPS is said to be positive-definite.

Definition 2.3 A moment functional L is positive-definite, if there exists a positive-definite OPS with respect to L.

The following theorem can be proved easily and is equivalent to the definition of orthogonality.

Theorem 2.4 Let L be a moment functional defined by (2.1), (2.2) and let $\{P_{n,m}(x,y)\}$ be a sequence of polynomials of degree (n,m). Then the following are equivalent:
(α) $\{P_{n,m}(x,y)\}$ is an OPS with respect to L
(β) $L[\pi(x,y)P_{n,m}(x,y)]=0$ for every polynomial $\pi(x,y)$ of degree (i,j) with

$0 \le i \le n, 0 \le j \le m$ and $i+j < n+m$, while $L[\pi(x,y)P_{n,m}(x,y)] \neq 0$ if $(i,j) = (n,m)$
(γ) $L[x^i y^j P_{n,m}(x,y)] = k_{n,m}\delta_{nm,ij}$ where $k_{n,m} \neq 0$, $n,m \ge 0$ and

$$\delta_{nm,ij} = \begin{cases} 0, & 0 \le i \le n, 0 \le j \le m \quad \text{and} \quad i+j < n+m \\ 1, & (i,j) = (n,m) \end{cases}.$$

3. Recurrence relation and Favard's theorem

In this section, we' ll prove that an OPS of two variables satisfy (1.1)-(1.2). More precisely,

Theorem 3.1 Let L be a positive-definite moment functional and $\{P_{n,m}(x,y)\}_{n,m\ge 0}$ be an OPS of degree (n,m) with respect to L. Then there exist real sequences $\beta_{n,m}, \gamma_{n,m}$ and $\alpha_{n,m}, \delta_{n,m}$ such that the recurrence relation (1.1) is satisfied.

Proof. Since $xP_{n,m}(x,y)$ is a polynomial of degree $n+1$ with respect to x and m with respect to y, of total degree $n+m+1$, it can be written as

$$xP_{n,m}(x,y) = \sum_{\kappa=0}^{n+1}\sum_{\lambda=0}^{m} c_{\kappa,\lambda}^{n,m} P_{\kappa,\lambda}(x,y) \tag{3.1}$$

where

$$c_{\kappa,\lambda}^{n,m} = \frac{L[xP_{n,m}(x,y)P_{\kappa,\lambda}(x,y)]}{L[P_{\kappa,\lambda}^2(x,y)]} = \frac{L[xP_{\kappa,\lambda}(x,y)P_{n,m}(x,y)]}{L[P_{\kappa,\lambda}^2(x,y)]} \tag{3.2}$$

The polynomial $xP_{\kappa,\lambda}(x,y)$ is of degree $(\kappa+1,\lambda)$ and $P_{n,m}(x,y)$ is of degree (n,m). Thus, due to the definition of orthogonality

$$c_{\kappa,\lambda}^{n,m} = 0, \quad (\kappa+1)+\lambda < n+m.$$

We need to consider the following two cases:
(I) $0 \le \kappa+1 < n$ and $0 \le \lambda \le m$ or (II) $0 \le \kappa+1 \le n$ and $0 \le \lambda < m$.
In both cases, if $0 \le \kappa+1 < n$ and $0 \le \lambda < m$ then $\kappa+1+\lambda < n+m$ so $c_{\kappa,\lambda}^{n,m} = 0$.
In the first case if $\lambda = m$ then for $\kappa < n-1$, $c_{n-2,m}^{n,m} = 0$ so (3.1) becomes:

$$xP_{n,m}(x,y) = c_{n-1,m}^{n,m}P_{n-1,m}(x,y) + c_{n,m}^{n,m}P_{n,m}(x,y) + c_{n+1,m}^{n,m}P_{n+1,m}(x,y) \tag{3.3}$$

In the second case if $\kappa+1 = n$ then $\kappa+1+\lambda < n+m$ so $c_{\kappa,\lambda}^{n,m} = 0$.

Similarly, $yP_{n,m}(x,y)$ is a polynomial of degree n with respect to x and $m+1$ with respect to y, of total degree $n+m+1$ and it can be written as

$$yP_{n,m}(x,y) = \sum_{\kappa=0}^{n}\sum_{\lambda=0}^{m+1} \tilde{c}_{\kappa,\lambda}^{n,m} P_{\kappa,\lambda}(x,y) \tag{3.4}$$

where

$$\tilde{c}_{\kappa,\lambda}^{n,m} = \frac{L[yP_{n,m}(x,y)P_{\kappa,\lambda}(x,y)]}{L[P_{\kappa,\lambda}^2(x,y)]} = \frac{L[yP_{\kappa,\lambda}(x,y)P_{n,m}(x,y)]}{L[P_{\kappa,\lambda}^2(x,y)]} \tag{3.5}$$

As before it is easily proved that (3.4) becomes

$$yP_{n,m}(x,y) = \tilde{c}_{n,m-1}^{n,m}P_{n,m-1}(x,y) + \tilde{c}_{n,m}^{n,m}P_{n,m}(x,y) + \tilde{c}_{n,m+1}^{n,m}P_{n,m+1}(x,y) \tag{3.6}$$

From (3.3) and (3.6) we obtain for $a, b > 0$:

$$\begin{aligned} axP_{n,m}(x,y) + byP_{n,m}(x,y) = \\ ac^{n,m}_{n-1,m}P_{n-1,m}(x,y) + ac^{n,m}_{n,m}P_{n,m}(x,y) + ac^{n,m}_{n+1,m}P_{n+1,m}(x,y)+ \\ b\tilde{c}^{n,m}_{n,m-1}P_{n,m-1}(x,y) + b\tilde{c}^{n,m}_{n,m}P_{n,m}(x,y) + b\tilde{c}^{n,m}_{n,m+1}P_{n,m+1}(x,y) \end{aligned} \tag{3.7}$$

which is of the form (1.1). □

The following theorem is analogous to Favard's theorem.

Theorem 3.2. Let $\alpha_{n,m}, \delta_{n,m}, \beta_{n,m}, \gamma_{n,m}$ real sequences with $\alpha_{n,m}, \delta_{n,m} > 0$ and a, b positive numbers. Consider also the polynomials $P_{n,m}(x,y)$ of degree (n,m) defined by (1.1)-(1.2). Then, there exists a unique positive-definite moment functional L such that:

$$L[1] = \mu_{0,0} > 0, \quad L[P^2_{n,m}(x,y)] = 1$$

and

$$L[P_{\kappa,\lambda}(x,y)P_{n,m}(x,y)] = 0,$$

for $0 \le \kappa \le n, 0 \le \lambda \le m$ with $\kappa + \lambda < n + m$.

Proof. We define the moment functional L inductively by:

$$L[1] = \mu_{0,0} > 0, \quad L[P_{n,m}(x,y)] = 0, \quad n \ge 0, \quad m \ge 0, \quad n + m > 0 \tag{3.8}$$

and we consider the following steps. In the first step, since for $m = 0$, $P_{n,0}(x,y) = Q_n(x)$ and for $n = 0$, $P_{0,m}(x,y) = R_m(y)$, using relation (1.1) and Favard's theorem for classical OP we obtain the moments $\mu_{n,0}$ and $\mu_{0,m}$ as well as the functionals $L[x^\kappa P_{n,0}(x,y)]$, $0 \le \kappa \le n$ and $L[y^\lambda P_{0,m}(x,y)]$, $0 \le \lambda \le m$. We also prove that $L[xP_{0,m}(x,y)] = 0$, $L[yP_{n,0}(x,y)] = 0$ and define the moments $\mu_{\kappa,\lambda}$ for $0 \le \kappa \le n$ and $0 \le \lambda \le m$. In each of the following steps, second, third until the $(n+1)$-th step, we follow the same procedure, that is: at the s-th step $2 \le s \le n+1$, we multiply both sides of (1.1) by the monomials of the form $x^\kappa y^\lambda$ with $\kappa + \lambda = s - 2$, then we apply L at the derived expression and using the results of the previous steps we prove $L[x^\kappa y^\lambda P_{n,m}(x,y)] = 0$ for $\kappa + \lambda < n + m$.
Hence:

Step 1: The recurrence relation (1.1) for $m = 0, n > 0$ takes the form:

$$\begin{aligned} \alpha_{n,0}P_{n+1,0}(x,y) + \alpha_{n-1,0}P_{n-1,0}(x,y) + \beta_{n,0}P_{n,0}(x,y) + \delta_{n,0}P_{n,1}(x,y)+ \\ \delta_{n,-1}P_{n,-1}(x,y) + \gamma_{n,0}P_{n,0}(x,y) = (ax + by)P_{n,0}(x,y) \end{aligned} \tag{3.9}$$

or

$$\begin{aligned} \alpha_{n,0}P_{n+1,0}(x,y) + \alpha_{n-1,0}P_{n-1,0}(x,y) + B_{n,0}P_{n,0}(x,y) - axP_{n,0}(x,y) \\ = byP_{n,0}(x,y) - \delta_{n,0}P_{n,1}(x,y) \end{aligned} \tag{3.10}$$

where $B_{n,0} = \beta_{n,0} + \gamma_{n,0}$.
Since the left-hand side of (3.10) is a polynomial of x and the right-hand side is a polynomial of x and y we obtain:

$$\alpha_{n,0}P_{n+1,0}(x,y) + \alpha_{n-1,0}P_{n-1,0}(x,y) + B_{n,0}P_{n,0}(x,y) = axP_{n,0}(x,y) \tag{3.11}$$

and

$$byP_{n,0}(x,y) = \delta_{n,0}P_{n,1}(x,y). \tag{3.12}$$

Since $a > 0$, equation (3.11) is a three term recurrence relation for the polynomials $\{P_{n,0}(x,y)\}$. Then due to Favard's theorem for one variable polynomials, there exists a positive moment functional L such that the following hold:

$$L[x^n] = \mu_{n,0}, \quad n \geq 0 \tag{3.13}$$

$$L[x^k P_{n,0}(x,y)] = 0, \quad 0 \leq k \leq n-1 \tag{3.14}$$

$$L[x^n P_{n,0}(x,y)] \neq 0 \tag{3.15}$$

Applying L in both sides of (3.12) and using (3.8) we have:

$$L[yP_{n,0}(x,y)] = 0, \quad n > 0 \tag{3.16}$$

Similarly, the recurrence relation (1.1) for $n = 0, m > 0$ takes the form:

$$\begin{gathered}\alpha_{0,m}P_{1,m}(x,y) + \alpha_{-1,m}P_{-1,m}(x,y) + \beta_{0,m}P_{0,m}(x,y) + \\ \delta_{0,m}P_{0,m+1}(x,y) + \delta_{0,m-1}P_{0,m-1}(x,y) + \gamma_{0,m}P_{0,m}(x,y) \\ = (ax+by)P_{0,m}(x,y)\end{gathered} \tag{3.17}$$

and the equations

$$\Gamma_{0,m}P_{0,m}(x,y) + \delta_{0,m}P_{0,m+1}(x,y) + \delta_{0,m-1}P_{0,m-1}(x,y) = byP_{0,m}(x,y) \tag{3.18}$$

where $\Gamma_{0,m} = \gamma_{0,m} + \beta_{0,m}$ and

$$axP_{0,m}(x,y) = \alpha_{0,m}P_{1,m}(x,y) \tag{3.19}$$

are obtained.
From equation (3.18) it follows, since $b > 0$, that:

$$L[y^m] = \mu_{0,m}, \quad m \geq 0 \tag{3.20}$$

$$L[y^\lambda P_{0,m}(x,y)] = 0, \quad 0 \leq \lambda \leq m-1 \tag{3.21}$$

$$L[y^m P_{0,m}(x,y)] \neq 0 \tag{3.22}$$

and from (3.19) we obtain

$$L[xP_{0,m}(y)] = 0, \quad m > 0 \tag{3.23}$$

Also, reccurence relation (1.1) for $n = 1$ and $m = 0$ becomes:

$$\begin{gathered}\alpha_{1,0}P_{2,0}(x,y) + \alpha_{0,0}P_{0,0}(x,y) + \beta_{1,0}P_{1,0}(x,y) + \delta_{1,0}P_{1,1}(x,y) + \\ \delta_{1,-1}P_{1,-1}(x,y) + \gamma_{1,0}P_{1,0}(x,y) = (ax+by)P_{1,0}(x,y)\end{gathered} \tag{3.24}$$

or

$$\delta_{1,0}P_{1,1}(x,y) = (ax+by-\beta_{1,0}-\gamma_{1,0})P_{1,0}(x,y) - \alpha_{1,0}P_{2,0}(x,y) - \alpha_{0,0} \tag{3.25}$$

and the polynomials $P_{1,0}(x,y), P_{2,0}(x,y)$ are already known since they are polynomials of one variable. Thus, the polynomial $P_{1,1}(x,y)$ takes the form:

$$P_{1,1}(x,y) = Axy + Bx + \Gamma y + \Delta, \quad A, B, \Gamma, \Delta : \text{constants}$$

Since $L[P_{1,1}(x,y)] = 0$, we have:

$$AL[xy] + BL[x] + \Gamma L[y] + \Delta L[1] = 0$$

or

$$A\mu_{1,1} + B\mu_{1,0} + \Gamma\mu_{0,1} + \Delta\mu_{0,0} = 0$$

and the moment $L[xy] = \mu_{1,1}$ can be defined.
Continuing in the same way the moments, $L[x^n y] = \mu_{n,1}$ and $L[xy^m] = \mu_{1,m}$ can be defined since $L[P_{n,1}(x,y)] = 0$ and $L[P_{1,m}(x,y)] = 0$ by using reccurence relation (1.1). Then $L[x^n y^2] = \mu_{n,2}$ and $L[x^2 y^m] = \mu_{2,m}$ can be defined since $L[P_{n,2}(x,y)] = 0$ and $L[P_{2,m}(x,y)] = 0$, etc. Thus, all $L[x^\kappa y^\lambda] = \mu_{k,\lambda}$ can be defined for $0 \leq k \leq n$ and $0 \leq \lambda \leq m$.

Step 2: We apply the moment functional L in both sides of (1.1) and using (3.8) we obtain for $n+m > 1$, with $n \geq 0, m > 1$ or $n > 0, m \geq 1$

$$L[xP_{n,m}(x,y)] = 0 \quad \text{and} \quad L[yP_{n,m}(x,y)] = 0 \tag{3.26}$$

or

$$L[x^\kappa y^\lambda P_{n,m}(x,y)] = 0, \quad 0 \leq \kappa \leq 1, \quad 0 \leq \lambda \leq 1, \quad \kappa + \lambda = 1 < n+m \tag{3.27}$$

Step 3: We multiply relation (1.1) by x or by y, apply L and using equation (3.27) , we obtain for $n+m > 2$, with $n \geq 2, m \geq 0$ or $n \geq 1, m \geq 1$ or $n \geq 0, m \geq 2$

$$L[x^2 P_{n,m}(x,y)] = 0, \quad L[xyP_{n,m}(x,y)] = 0, \quad L[y^2 P_{n,m}(x,y)] = 0, \tag{3.28}$$

or

$$L[x^\kappa y^\lambda P_{n,m}(x,y)] = 0, \quad 0 \leq \kappa \leq 2, \quad 0 \leq \lambda \leq 2, \quad \kappa + \lambda = 2 < n+m \tag{3.29}$$

Step 4: By multiplying relation (1.1) with x^2, y^2 and then applying L we prove that:

$$L[x^\kappa y^\lambda P_{n,m}(x,y)] = 0, \quad 0 \leq \kappa \leq 3, \quad 0 \leq \lambda \leq 3, \quad \kappa + \lambda = 3 < n+m \tag{3.30}$$

So continuing this procedure at the $(n+1)$-th step we prove:

$$L[x^\kappa y^\lambda P_{n,m}(x,y)] = 0, \quad 0 \leq \kappa \leq n, \quad 0 \leq \lambda \leq m, \quad \kappa + \lambda < n+m \tag{3.31}$$

which is the equivalent definition of orthogonality (see theorem 2.4). □

4. Darboux-Christoffel formula

The following proposition gives a Darboux-Christoffel formula for $P_{n,m}(x,y)$.

Proposition 4.1 Let $\{P_{n,m}(x,y)\}_{n\geq 0, m\geq 0}$ be an OPS of degree (n,m) satisfying (1.1)-(1.2). Then the following holds

$$[a(x-s)+b(y-t)]\sum_{i=0}^{n}\sum_{j=0}^{m}P_{i,j}(x,y)P_{i,j}(s,t)=$$
$$\sum_{j=0}^{m}\alpha_{n,j}[P_{n+1,j}(x,y)P_{n,j}(s,t)-P_{n+1,j}(s,t)P_{n,j}(x,y)]+ \tag{4.1}$$
$$\sum_{i=0}^{n}\delta_{i,m}[P_{i,m+1}(x,y)P_{i,m}(s,t)-P_{i,m+1}(s,t)P_{i,m}(x,y)]$$

for $(x,y)\neq(s,t)$.

Corollary 4.2 If we make the conventions (C_1) or (C_2), (4.1) becomes the well known Christoffel-Darboux formula [1, 13] for polynomials of one variable.

Proof of Proposition 4.1. Consider the recurrence relation (1.1), satisfied by the OPS $P_{i,j}(x,y)$

$$\begin{gathered}\alpha_{i,j}P_{i+1,j}(x,y)+\alpha_{i-1,j}P_{i-1,j}(x,y)+\beta_{i,j}P_{i,j}(x,y)+\delta_{i,j}P_{i,j+1}(x,y)+\\ \delta_{i,j-1}P_{i,j-1}(x,y)+\gamma_{i,j}P_{i,j}(x,y)=(ax+by)P_{i,j}(x,y),\end{gathered}\tag{4.2}$$

and $P_{i,j}(s,t)$

$$\begin{gathered}\alpha_{i,j}P_{i+1,j}(s,t)+\alpha_{i-1,j}P_{i-1,j}(s,t)+\beta_{i,j}P_{i,j}(s,t)+\delta_{i,j}P_{i,j+1}(s,t)+\\ \delta_{i,j-1}P_{i,j-1}(s,t)+\gamma_{i,j}P_{i,j}(s,t)=(as+bt)P_{i,j}(x,y),\end{gathered}\tag{4.3}$$

Multiplying (4.2) by $P_{i,j}(s,t)$, and (4.3) by $P_{i,j}(x,y)$ and substracting the derived equations we obtain:

$$\begin{gathered}[a(x-s)+b(y-t)]P_{i,j}(x,y)P_{i,j}(s,t)=\\ [K_{i,j}(x,y,s,t)-K_{i-1,j}(x,y,s,t)]-[\Lambda_{i,j}(x,y,s,t)-\Lambda_{i-1,j}(x,y,s,t)]\\ +[M_{i,j}(x,y,s,t)-M_{i,j-1}(x,y,s,t)]-[N_{i,j}(x,y,s,t)-N_{i,j-1}(x,y,s,t)]\end{gathered}\tag{4.4}$$

where

$$\begin{aligned}K_{i,j}(x,y,s,t)&=\alpha_{i,j}P_{i+1,j}(x,y)P_{i,j}(s,t),\\ \Lambda_{i,j}(x,y,s,t)&=\alpha_{i,j}P_{i+1,j}(s,t)P_{i,j}(x,y),\\ M_{i,j}(x,y,s,t)&=\delta_{i,j}P_{i,j+1}(x,y)P_{i,j}(s,t),\\ N_{i,j}(x,y,s,t)&=\delta_{i,j}P_{i,j+1}(s,t)P_{i,j}(x,y).\end{aligned}$$

By taking the sums of both parts of (4.4) with respect to i and j, it follows

$$[a(x-s)+b(y-t)]\sum_{i=0}^{n}\sum_{j=0}^{m}P_{i,j}(x,y)P_{i,j}(s,t)=$$
$$\sum_{i=0}^{n}\sum_{j=0}^{m}\Big\{[K_{i,j}(x,y,s,t)-K_{i-1,j}(x,y,s,t)]-[\Lambda_{i,j}(x,y,s,t)-\Lambda_{i-1,j}(x,y,s,t)]\Big\}+$$
$$\sum_{i=0}^{n}\sum_{j=0}^{m}\Big\{[M_{i,j}(x,y,s,t)-M_{i,j-1}(x,y,s,t)]-[N_{i,j}(x,y,s,t)-N_{i,j-1}(x,y,s,t)]\Big\}.$$

However, due to (1.2) it is $P_{i,-1}(x,y)=0$ or $P_{-1,j}(x,y)=0$ and thus

$$[a(x-s)+b(y-t)]\sum_{i=0}^{n}\sum_{j=0}^{m}P_{i,j}(x,y)P_{i,j}(s,t)=$$
$$\sum_{j=0}^{m}[K_{n,j}(x,y,s,t)-\Lambda_{n,j}(x,y,s,t)]+\sum_{i=0}^{n}[M_{i,m}(x,y,s,t)-N_{i,m}(x,y,s,t)]$$

which is (4.1).

□

Acknowledgments. The authors would like to thank the anonymous referee for his/her comments that helped us improve this article.

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