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SHARP INEQUALITY INVOLVING HYPERBOLIC AND INVERSE HYPERBOLIC FUNCTIONS

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Abstract. We prove that the inequality

$$\cosh\left(\operatorname{arcosh}(2\cosh u)\cdot\tanh u\right) < \exp\left(u\cdot\tanh u\right)$$

holds for all $u > 0$. We check with the computation program *Mathematica* that the ratio between the left-hand and the right-hand side is greater than 0,97 for all $u \geq 0$, so this is a quite sharp inequality. It is also equivalent to any of the two inequalities:

$$\cosh\left(\sqrt{1-\frac{1}{t^2}}\cdot\operatorname{arcosh}2t\right) < \exp\left(\sqrt{1-\frac{1}{t^2}}\cdot\operatorname{arcosh}t\right)$$

for all $t > 1$, and

$$\cosh\left(c\cdot\operatorname{arcosh}\frac{2}{\sqrt{1-c^2}}\right) < \exp\left(c\cdot\operatorname{arcosh}\frac{1}{\sqrt{1-c^2}}\right)$$

for all $c \in (0,1)$.

In several attemps to compute the numerical index of two-dimensional normed space equipped with an l^p-norm (see [1, Problems 2 and 3] or [2, Problem 5.1]) we find a quite sharp inequality that can be added to the existing list of inequalities involving the hyperbolic functions (see e.g. [3],[4], and [5]). We begin with two lemmas.

Lemma 1. *Let x and y be positive real numbers. Then*

$$\tanh x\cdot\tanh y < \tanh(x\cdot\tanh y). \tag{1}$$

Proof. We will make use of the fact that the Taylor series expansion of the function $\operatorname{artanh} t$ has nonnegative coefficients:

$$\operatorname{artanh} t = \sum_{k=0}^{\infty}\frac{t^{2k+1}}{2k+1}\ , \quad |t| < 1.$$

Since $0 < \tanh y < 1$, we have

$$\operatorname{artanh}(\tanh x\cdot\tanh y) = \sum_{k=0}^{\infty}\frac{(\tanh x\cdot\tanh y)^{2k+1}}{2k+1} <$$

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$$< \tanh y \cdot \sum_{k=0}^{\infty} \frac{(\tanh x)^{2k+1}}{2k+1} = \tanh y \cdot \operatorname{artanh}(\tanh x) = x \cdot \tanh y,$$

and so (1) follows. ☐

Lemma 2. *For $0 < K < 1$, the function $\phi : [1, \infty) \to \mathbb{R}$ defined by*

$$\phi(x) = \cosh(K \operatorname{arcosh} x)$$

is strictly increasing and concave.

Proof. The first derivative

$$\phi'(x) = \sinh(K \operatorname{arcosh} x) K \frac{1}{\sqrt{x^2 - 1}}$$

is clearly positive for all $x > 1$, and so ϕ is a strictly increasing function. To show that ϕ is concave, we must prove that the second derivative

$$\phi''(x) = \cosh(K \operatorname{arcosh} x)\, K^2 \frac{1}{x^2 - 1} - \sinh(K \operatorname{arcosh} x)\, K \frac{x}{(x^2 - 1)^{3/2}}$$

is negative for all $x > 1$, that is

$$\cosh(K \operatorname{arcosh} x)\, K \sqrt{x^2 - 1} < \sinh(K \operatorname{arcosh} x)\, x.$$

Setting $u = \operatorname{arcosh} x$ and $v = \operatorname{artanh} K$, this inequality rewrites to the inequality

$$\tanh u \cdot \tanh v < \tanh(u \cdot \tanh v)$$

that holds by (1). This completes the proof. ☐

We now prove the main result of this paper.

Theorem 3. *We have*

$$\cosh\left(\sqrt{1 - \frac{1}{t^2}} \cdot \operatorname{arcosh} 2t\right) < \exp\left(\sqrt{1 - \frac{1}{t^2}} \cdot \operatorname{arcosh} t\right) \tag{2}$$

for all $t > 1$, or equivalently

$$\cosh\left(c \cdot \operatorname{arcosh} \frac{2}{\sqrt{1 - c^2}}\right) < \exp\left(c \cdot \operatorname{arcosh} \frac{1}{\sqrt{1 - c^2}}\right) = \exp\left(c \cdot \operatorname{artanh} c\right) \tag{3}$$

for all $c \in (0, 1)$, or equivalently

$$\cosh\left(\operatorname{arcosh}(2 \cosh u) \cdot \tanh u\right) < \exp\left(u \cdot \tanh u\right) \tag{4}$$

for all $u > 0$.

Proof. Fix $t > 1$. By Lemma 2, the function $\phi : [1, \infty) \to \mathbb{R}$ defined by

$$\phi(x) = \cosh\left(\sqrt{1 - \frac{1}{t^2}} \operatorname{arcosh} x\right)$$

is strictly increasing and concave. Therefore, its derivative

$$\phi'(x) = \sinh\left(\sqrt{1 - \frac{1}{t^2}} \operatorname{arcosh} x\right) \sqrt{1 - \frac{1}{t^2}} \frac{1}{\sqrt{x^2 - 1}},$$

is decreasing, and so

$$\max_{t \le x \le 2t} \phi'(x) = \phi'(t) = \sinh\left(\sqrt{1 - \frac{1}{t^2}} \operatorname{arcosh} t\right) \frac{1}{t}.$$

Now, the inequality

$$\phi(2t)-\phi(t)<(2t-t)\max_{t\le x\le 2t}\phi'(x)$$

yields that

$$\cosh\left(\sqrt{1-\frac{1}{t^2}}\cdot\operatorname{arcosh}2t\right)-\cosh\left(\sqrt{1-\frac{1}{t^2}}\cdot\operatorname{arcosh}t\right)<\sinh\left(\sqrt{1-\frac{1}{t^2}}\cdot\operatorname{arcosh}t\right),$$

implying the inequality (2).

The substitution $t=\cosh u$ in (2) gives the inequality (4), while the substitution $c=\tanh u$ in (4) yields the inequality (3). This completes the proof. □

Let us further explore the inequality (4). Since

$$\operatorname{arcosh}(2\cosh u)=\ln(2\cosh u+\sqrt{4\cosh^2 u-1})>$$
$$>\ln(2\cosh u+2\sinh u)=\ln(2e^u)=\ln 2+u,$$

the left-hand side inequality of (4) is greater than

$$\cosh\left(\tanh u\cdot(\ln 2+u)\right)>\frac{1}{2}\exp\left(\tanh u\cdot(\ln 2+u)\right)=2^{\tanh u-1}\exp\left(u\cdot\tanh u\right),$$

and so we also have the inequality

$$2^{\tanh u-1}\exp\left(u\cdot\tanh u\right)<\cosh\left(\operatorname{arcosh}(2\cosh u)\cdot\tanh u\right)<\exp\left(u\cdot\tanh u\right)\quad(5)$$

for all $u>0$. Define the function $f:[0,\infty)\to\mathbb{R}$ by

$$f(u)=\frac{\cosh\left(\operatorname{arcosh}(2\cosh u)\cdot\tanh u\right)}{\exp\left(u\cdot\tanh u\right)}.$$

By (4), we have $f(u)<1$ for all $u>0$, while $f(0)=1$. Since $\lim_{u\to\infty}\tanh u=1$, the inequality (5) implies that

$$\lim_{u\to\infty}f(u)=1.$$

Furthermore, using the computation program *Mathematica* one can reveal a remarkable property that $f(u)>0,972$ for all $u\ge 0$. Thus, the inequalities in Theorem 3 are surprisingly sharp.

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