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On the Hadamard's type Inequalities for Convex Functions via Conformable Fractional Integrals

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Abstract. The purpose of this paper is to establish some new Hermite-Hadamard type inequalities which includes a new fractional derivative called the conformable in the literature.

1. Introduction

The Hermite-Hadamard Inequality: Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a function and $a, b \in I$ with $a < b$. The following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

is known in the literature as Hermite-Hadamard Inequality for convex functions. If f is concave, both inequalities hold in the reversed direction.

For several recent results concerning inequality (1), see ([4],[14],[15]) where further references are listed.

We recall here some concepts of convexity that are well known in the literature.

Definition 1. *A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex on I if inequality*

$$f(\tau a + (1-\tau)b) \leq \tau f(a) + (1-\tau)f(b), \tag{2}$$

holds for all $a, b \in I$ and $\tau \in [0, 1]$.

The following definitions and theorems with respect to conformable fractional derivative and integral were referred in ([1]-[3], [8]-[13], [15]). Firstly, we give a brief information about derivative of non-integer order.

The idea of derivative of non-integer order was motivated by the question, "What does it mean by $\frac{d^n f}{dx^n}$, if $n = \frac{1}{2}$?", asked by L'Hospital in 1695 in his letters to Leibnitz ([5]-[7]). Afterwards, many mathematicians tried to answer this question for centuries in several points of view. Various types of fractional derivatives were introduced by many authors, most of them are defined via fractional integrals, but

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many of those fractional derivatives have some non-local behaviors. Among the inconsistencies of the existing fractional derivatives are:

(1) Most of the fractional derivatives do not satisfy $D_{\alpha}^{a}(1) = 0$, if α is not a natural number.

(2) All fractional derivatives do not obey the familiar Product Rule and Quotient Rule for two functions.

(3) All fractional derivatives do not obey the Chain Rule.

(4) Fractional derivatives do not have a corresponding Rolle's Theorem and Mean Value Theorem.

(5) All fractional derivatives do not obey: $D_{\alpha}^{a} D_{\beta}^{a} f = D_{\alpha+\beta}^{a} f$, in general.

To solve some of these and other difficulties, Khalil et al. [11], introduced the following.

Definition 2. *(**Conformable fractional derivative**) Given a function $f : [0, \infty) \to \mathbb{R}$. Then the "conformable fractional derivative" of f of order α is defined by*

$$D_{\alpha}(f)(\tau) = \lim_{\epsilon \to 0} \frac{f\left(\tau + \epsilon \tau^{1-\alpha}\right) - f(\tau)}{\epsilon} \tag{3}$$

for all $\tau > 0,\ \alpha \in (0,1)$. If f is $\alpha-$differentiable in some $(0, a)$, $\alpha > 0$, $\lim_{\tau \to 0^+} f^{(\alpha)}(\tau)$ exist, then define

$$f^{(\alpha)}(0) = \lim_{\tau \to 0^+} f^{(\alpha)}(\tau). \tag{4}$$

We can write $f^{(\alpha)}(\tau)$ for $D_{\alpha}(f)(\tau)$ to denote the conformable fractional derivatives of f of order α. In addition, if the conformable fractional derivative of f of order α exists, then we simply say f is $\alpha-$differentiable.

Theorem 1. [10] *Let $\alpha \in (0,1]$ and f, g be $\alpha-$differentiable at a point $\tau > 0$. Then*

i. $D_{\alpha}(af + bg) = aD_{\alpha}(f) + bD_{\alpha}(g)$, for all $a, b \in \mathbb{R}$ (linearity).

ii. $D_{\alpha}(\lambda) = 0$, for all constant functions $f(\tau) = \lambda$,

iii. $D_{\alpha}(fg) = fD_{\alpha}(g) + gD_{\alpha}(f)$ (product rule).

iv. $D_{\alpha}\left(\frac{f}{g}\right) = \frac{fD_{\alpha}(g) - gD_{\alpha}(f)}{g^2}$ (Quotient Rule).

v. $D_{\alpha}(f \circ g)(\tau) = f'(g(\tau)) D_{\alpha}(g)(\tau)$ (chain rule).

vi. If, in addition, f is differentiable, then

$$D_{\alpha}(f)(\tau) = \tau^{1-\alpha} \frac{df}{d\tau}(\tau). \tag{5}$$

Definition 3. [10]*(**Conformable fractional integral**) Let $\alpha \in (0,1]$ and $0 \leq a < b$. A function $f : (a, b] \to \mathbb{R}$ is α-fractional integrable on $[a, b]$ if the integral*

$$\int_a^b f(x) d_{\alpha} x := \int_a^b f(x) x^{\alpha-1} dx. \tag{6}$$

Note that, *the usual Riemann improper integral is defined as follow*

$$I_{\alpha}^{a}(f)(\tau) = I_{1}^{a}(\tau^{\alpha-1} f) = \int_a^{\tau} \frac{f(x)}{x^{1-\alpha}} dx,$$

where $\alpha \in (0,1]$.

Theorem 2. [1]*Let $f : (a, b) \to \mathbb{R}$ be differentiable and $0 < \alpha \leq 1$. Then, for all $\tau > a$ we have*

$$I_{\alpha}^{a} D_{\alpha}^{a} f(\tau) = f(\tau) - f(a). \tag{7}$$

Theorem 3. [1]*(**Integration by parts**) Let $f,g:[a,b]\to\mathbb{R}$ be two functions such that fg is differentiable. Then*

$$\int_a^b f(x)\, D_\alpha^a(g)(x)\, d_\alpha(x,a) = fg|_a^b - \int_a^b g(x)\, D_\alpha^a(f)(x)\, d_\alpha(x,a). \tag{8}$$

Theorem 4. [10]*(**Inverse property**) Assume that $a\geq 0$, and $\alpha\in(0,1)$, and also let f be a continuous function such that $I_a^\alpha f$ exists. Then, for all $\tau>a$ we have*

$$D_\alpha^a I_\alpha^a f(\tau) = f(\tau).$$

In [15], Sarikaya et al. proved the following results connected with (1) and the right part of (1).

Theorem 5. [15] *Let $\alpha\in(0,1]$ and $f:[a^\alpha,b^\alpha]\to\mathbb{R}$ be an α-fractional differentiable mapping on (a^α,b^α) with $0\leq a<b$. Then, the following inequality for conformable fractional integral holds:*

$$f\left(\frac{a^\alpha+b^\alpha}{2}\right) \leq \frac{\alpha}{b^\alpha-a^\alpha}\int_a^b f(x^\alpha)\, d_\alpha x \leq \frac{f(a^\alpha)+f(b^\alpha)}{2}. \tag{9}$$

Theorem 6. [15]*Let $\alpha\in(0,1]$ and $f:[a^\alpha,b^\alpha]\to\mathbb{R}$ be an α-fractional differentiable function on (a^α,b^α) and $D_\alpha(f)$ be an α-fractional integrable function on $[a^\alpha,b^\alpha]$ with $0\leq a<b$. If $|D_\alpha(f)|$ be a convex function on $[a,b]$,then the following inequality for conformable fractional integral holds:*

$$\begin{aligned}&\left|\frac{f(a^\alpha)+f(b^\alpha)}{2} - \frac{\alpha}{b^\alpha-a^\alpha}\int_a^b f(x^\alpha)\, d_\alpha x\right| \\ &\quad\leq \frac{\alpha(b^\alpha-a^\alpha)}{2}\left(\tfrac{2^{3\alpha^2}+6\times 2^{\alpha^2}-8}{3\alpha\times 2^{3\alpha^2}}\right)\left[\tfrac{a^{\alpha(\alpha-1)}|D_\alpha(f)(a^\alpha)|+b^{\alpha(\alpha-1)}|D_\alpha(f)(b^\alpha)|}{2}\right].\end{aligned} \tag{10}$$

In this paper, we have established some conformable fractional integral inequalities. The results presented here would provide generalizations of those given in earlier works. In the case of $\alpha=1$, the fractional integrals reduces to the classical integral.

2. Some inequalities of Hermite-Hadamard type

In this section, we give some generalized inequalities in association with Hermite-Hadamard type for differentiable functions whose derivatives in the absolute value are convex using fractional integrals.

During this section, we assume will accept $I_f(f;\alpha;\beta;u)$ as the following identity

$$\begin{aligned}&I_f(f;\alpha;\beta;u) = \frac{-a^{\alpha-1}f(a^\alpha)+b^{\alpha-1}f(b^\alpha)}{2(b^\alpha-a^\alpha)^\alpha} \\ &-\tfrac{1}{2(b^\alpha-a^\alpha)^{\beta+1}}\int_a^b (b^\alpha-u^\alpha)^\beta u^{-\alpha}\left[\alpha-1-(\beta-\alpha+1)u^{2\alpha-1}(b^\alpha-u^\alpha)^{-\alpha}\right] f(u^\alpha)\, d_\alpha u \\ &-\tfrac{1}{2(b^\alpha-a^\alpha)^{\beta+1}}\int_a^b (u^\alpha-a^\alpha)^{\beta-\alpha+1} u^{-\alpha}\left[\alpha-1+(\beta-\alpha+1)u^{2\alpha-1}(u^\alpha-a^\alpha)^{-\alpha}\right] f(u^\alpha)\, d_\alpha u.\end{aligned}$$

where $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on I°, the interior of I and $a,b\in I$ with $a<b$, $x\in[a,b]$, $\lambda\in[0,1]$, $\alpha,\beta\in(0,1]$.

Now we turn our attention to obtain the new integral inequalities of Hermite-Hadamard type inequality for convex functions, for this we need the below lemma:

Lemma 7. *Let $\alpha,\beta\in(0,1]$ and $f:[a^{\alpha},b^{\alpha}]\to\mathbb{R}$ be an α-fractional differentiable function on (a^{α},b^{α}) and $D_{\alpha}(f)$ be an α-fractional integrable function on $[a^{\alpha},b^{\alpha}]$ with $0\le a<b$. If for $q>1$, $|D_{\alpha}(f)|^{q}$ be a convex function on $[a^{\alpha},b^{\alpha}]$, then the following identity holds:*

$$I_f(f;\alpha;\beta;u)\equiv\frac{-a^{\alpha-1}f(a^{\alpha})+b^{\alpha-1}f(b^{\alpha})}{2(b^{\alpha}-a^{\alpha})^{\alpha}}$$

$$-\tfrac{1}{2(b^{\alpha}-a^{\alpha})^{\beta+1}}\int_a^b(b^{\alpha}-u^{\alpha})^{\beta}u^{-\alpha}\left[\alpha-1-(\beta-\alpha+1)u^{2\alpha-1}(b^{\alpha}-u^{\alpha})^{-\alpha}\right]f(u^{\alpha})\,d_{\alpha}u$$

$$-\tfrac{1}{2(b^{\alpha}-a^{\alpha})^{\beta+1}}\int_a^b(u^{\alpha}-a^{\alpha})^{\beta-\alpha+1}u^{-\alpha}\left[\alpha-1+(\beta-\alpha+1)u^{2\alpha-1}(u^{\alpha}-a^{\alpha})^{-\alpha}\right]f(u^{\alpha})\,d_{\alpha}u$$

$$=\tfrac{1}{2}\int_0^1\left(\tau^{\alpha\beta}-(1-\tau^{\alpha})^{\beta}\right)D_{\alpha}(f)(a^{\alpha}\tau^{\alpha}+b^{\alpha}(1-\tau^{\alpha}))\,d_{\alpha}\tau. \tag{11}$$

Where $D_{\alpha}(f)(a^{\alpha}\tau^{\alpha}+b^{\alpha}(1-\tau^{\alpha}))=(a^{\alpha}\tau^{\alpha}+b^{\alpha}(1-\tau^{\alpha}))^{1-\alpha}f'(a^{\alpha}\tau^{\alpha}+b^{\alpha}(1-\tau^{\alpha}))$.

Proof. Integrating by parts and changing variable of definite integral and changing variables with $u^{\alpha}=a^{\alpha}\tau^{\alpha}+b^{\alpha}(1-\tau^{\alpha})$, we have:

$$I_1(f;\alpha;\beta;u)$$

$$=\int_0^1\tau^{\alpha\beta}D_{\alpha}(f)(a^{\alpha}\tau^{\alpha}+b^{\alpha}(1-\tau^{\alpha}))\,d_{\alpha}\tau$$

$$=\tfrac{1}{(b^{\alpha}-a^{\alpha})^{\beta+1}}\int_a^b(b^{\alpha}-u^{\alpha})^{\beta+\alpha-1-\alpha+1}D_{\alpha}(f)(u^{\alpha})u^{\alpha-1}du$$

$$=\tfrac{1}{(b^{\alpha}-a^{\alpha})^{\beta+1}}\int_a^b(b^{\alpha}-u^{\alpha})^{\beta-\alpha+1}D_{\alpha}(f)(u^{\alpha})u^{\alpha-1}d_{\alpha}(b^{\alpha},u^{\alpha})$$

$$=\tfrac{1}{(b^{\alpha}-a^{\alpha})^{\beta+1}}\int_a^b(b^{\alpha}-u^{\alpha})^{\beta-\alpha+1}D_{\alpha}(f)(u^{\alpha})u^{\alpha-1}d_{\alpha}(b^{\alpha},u^{\alpha})$$

$$=\tfrac{1}{(b^{\alpha}-a^{\alpha})^{\beta+1}}\left\{\left[(b^{\alpha}-u^{\alpha})^{\beta-\alpha+1}u^{\alpha-1}\right]f(u^{\alpha})\Big|_a^b\right.$$

$$\left.-\int_a^bD_{\alpha}\left((b^{\alpha}-u^{\alpha})^{\beta-\alpha+1}u^{\alpha-1}\right)f(u^{\alpha})\,d_{\alpha}(b^{\alpha},u^{\alpha})\right\}$$

$$=-\tfrac{1}{(b^{\alpha}-a^{\alpha})^{\alpha}}a^{\alpha-1}f(a^{\alpha})$$

$$-\tfrac{1}{(b^{\alpha}-a^{\alpha})^{\beta+1}}\int_a^b\frac{(b^{\alpha}-u^{\alpha})^{\beta-\alpha+1}}{u}\left[\alpha-1-(\beta-\alpha+1)u^{2\alpha-1}(b^{\alpha}-u^{\alpha})^{-\alpha}\right]f(u^{\alpha})\,d_{\alpha}(b^{\alpha},u^{\alpha}). \tag{12}$$

Similarly, we have

$$-I_2(f;\alpha;\beta;u)$$

$$=-\int_0^1(1-\tau^{\alpha})^{\beta}D_{\alpha}(f)(a^{\alpha}\tau^{\alpha}+b^{\alpha}(1-\tau^{\alpha}))\,d_{\alpha}\tau$$

$$= -\frac{1}{(b^\alpha - a^\alpha)^{\beta+1}} \int_b^a (u^\alpha - a^\alpha)^{\beta+\alpha-1-\alpha+1} D_\alpha(f)(u^\alpha) u^{\alpha-1} du$$

$$= \frac{1}{(b^\alpha - a^\alpha)^{\beta+1}} \int_a^b (u^\alpha - a^\alpha)^{\beta-\alpha+1} D_\alpha(f)(u^\alpha) u^{\alpha-1} d_\alpha(u^\alpha, a^\alpha)$$

$$= \frac{1}{(b^\alpha - a^\alpha)^{\beta+1}} \int_a^b (u^\alpha - a^\alpha)^{\beta-\alpha+1} D_\alpha(f)(u^\alpha) u^{\alpha-1} d_\alpha(u^\alpha, a^\alpha)$$

$$= \frac{1}{(b^\alpha - a^\alpha)^{\beta+1}} \left\{ \left[(u^\alpha - a^\alpha)^{\beta-\alpha+1} u^{\alpha-1} \right] f(u^\alpha) \Big|_a^b \right.$$

$$\left. - \int_a^b D_\alpha\left((u^\alpha - a^\alpha)^{\beta-\alpha+1} u^{\alpha-1} \right) f(u^\alpha) d_\alpha(u^\alpha, a^\alpha) \right\}$$

$$= \frac{1}{(b^\alpha - a^\alpha)^{\alpha}} b^{\alpha-1} f(b^\alpha)$$

$$-\frac{1}{(b^\alpha - a^\alpha)^{\beta+1}} \int_a^b \frac{(u^\alpha - a^\alpha)^{\beta-\alpha+1}}{u} \left[\alpha - 1 + (\beta - \alpha + 1) u^{2\alpha-1} (u^\alpha - a^\alpha)^{-\alpha} \right] f(u^\alpha) d_\alpha(u^\alpha, a^\alpha) \tag{13}$$

From these two equalities the lemma is complated. ∎

Now we turn our attention to establish new integral inequalities of Hermite-Hadamard for convex functions via conformable fractional integrals.

Theorem 8. *Let $\alpha, \beta \in (0,1]$ and $f : [a^\alpha, b^\alpha] \to \mathbb{R}$ be an α-fractional differentiable function on (a^α, b^α) and $D_\alpha(f)$ be an α-fractional integrable function on $[a^\alpha, b^\alpha]$ with $0 \le a < b$. If $|D_\alpha(f)|$ be a convex function on $[a^\alpha, b^\alpha]$, then the following inequality holds:*

$$|I_f(f;\alpha;\beta;u)| \le \frac{1}{\alpha(\beta+1)} \left(\frac{2^\beta - 1}{2^{\beta+1}} \right) \{ |D_\alpha(f)(a^\alpha)| + |D_\alpha(f)(b^\alpha)| \}.$$

Proof. Using Lemma 7 and the convexity of $|D_\alpha(f)|$ on $[a^\alpha, b^\alpha]$, we obtain the following inequality

$$|I_f(f;\alpha;\beta;u)| = \frac{1}{2} \left[\int_0^{2^{-\frac{1}{\alpha}}} \left((1-\tau^\alpha)^\beta - \tau^{\alpha\beta} \right) |D_\alpha(f)(a^\alpha \tau^\alpha + b^\alpha(1-\tau^\alpha))| d_\alpha \tau \right.$$

$$\left. + \int_{2^{-\frac{1}{\alpha}}}^1 \left(\tau^{\alpha\beta} - (1-\tau^\alpha)^\beta \right) |D_\alpha(f)(a^\alpha \tau^\alpha + b^\alpha(1-\tau^\alpha))| d_\alpha \tau \right]$$

$$\le \frac{1}{2} \int_0^{2^{-\frac{1}{\alpha}}} \left((1-\tau^\alpha)^\beta - \tau^{\alpha\beta} \right) \{ \tau^\alpha |D_\alpha(f)(a^\alpha)| + (1-\tau^\alpha) |D_\alpha(f)(b^\alpha)| \} d_\alpha \tau$$

$$+ \frac{1}{2} \int_{2^{-\frac{1}{\alpha}}}^1 \left(\tau^{\alpha\beta} - (1-\tau^\alpha)^\beta \right) \{ \tau^\alpha |D_\alpha(f)(a^\alpha)| + (1-\tau^\alpha) |D_\alpha(f)(b^\alpha)| \} d_\alpha \tau$$

$$= \frac{|D_\alpha(f)(a^\alpha)|}{2}\left[\int_0^{2^{-\frac{1}{\alpha}}} \tau^\alpha \left((1-\tau^\alpha)^\beta - \tau^{\alpha\beta}\right) d_\alpha\tau + \int_{2^{-\frac{1}{\alpha}}}^1 \left(\tau^{\alpha\beta} - (1-\tau^\alpha)^\beta\right)\tau^\alpha d_\alpha\tau\right]$$

$$+\frac{|D_\alpha(f)(b^\alpha)|}{2}\int_0^{2^{-\frac{1}{\alpha}}} \left((1-\tau^\alpha)^\beta - \tau^{\alpha\beta}\right)(1-\tau^\alpha) d_\alpha\tau + \int_{2^{-\frac{1}{\alpha}}}^1 \left(\tau^{\alpha\beta} - (1-\tau^\alpha)^\beta\right)(1-\tau^\alpha) d_\alpha\tau$$

$$= \frac{\{|D_\alpha(f)(a^\alpha)| + |D_\alpha(f)(b^\alpha)|\}}{2\alpha(\beta+1)(\beta+2)}\left[\beta+2-\frac{\beta+2}{2^\beta}\right]$$

$$= \frac{1}{\alpha(\beta+1)}\left(\frac{2^\beta-1}{2^{\beta+1}}\right)\{|D_\alpha(f)(a^\alpha)| + |D_\alpha(f)(b^\alpha)|\}.$$

∎

Theorem 9. *Let $\alpha,\beta \in (0,1]$ and $f:[a^\alpha,b^\alpha]\to\mathbb{R}$ be an α-fractional differentiable function on (a^α,b^α) and $D_\alpha(f)$ be an α-fractional integrable function on $[a^\alpha,b^\alpha]$ with $0\le a<b$. If for $q\ge 1$ with $\frac{1}{p}+\frac{1}{q}=1$, $|D_\alpha(f)|^q$ be a convex function on $[a^\alpha,b^\alpha]$, then the following inequality holds:*

$$|I_f(f;\alpha;\beta;u)| \le \frac{1}{2}\left(\frac{2-2^{1-\beta}}{\alpha(\beta+1)}\right)^{\frac{1}{p}}\left(\frac{(|D_\alpha(f)(a^\alpha)|^q + |D_\alpha(f)(b^\alpha)|^q)}{2\alpha}\right)^{\frac{1}{q}}. \tag{14}$$

Proof. From Lemma 7, the convexity of $|D_\alpha(f)|^q$ on $[a^\alpha,b^\alpha]$, and for $q>1$, the power-mean inequality, we yields

$$|I_f(f;\alpha;\beta;u)| \le \frac{1}{2}\int_0^1 \left|\tau^{\alpha\beta}-(1-\tau^\alpha)^\beta\right| |D_\alpha(f)(a^\alpha\tau^\alpha + b^\alpha(1-\tau^\alpha))| d_\alpha\tau$$

$$\le \frac{1}{2}\int_0^1 \left|\tau^{\alpha\beta}-(1-\tau^\alpha)^\beta\right| |D_\alpha(f)(a^\alpha\tau^\alpha + b^\alpha(1-\tau^\alpha))| d_\alpha\tau$$

$$+\frac{1}{2}\left(\int_0^1 \left|\tau^{\alpha\beta}-(1-\tau^\alpha)^\beta\right| d_\alpha\tau\right)^{\frac{1}{p}}\left(\int_0^1 |D_\alpha(f)(a^\alpha\tau^\alpha + b^\alpha(1-\tau^\alpha))|^q d_\alpha\tau\right)^{\frac{1}{q}}$$

$$\le \frac{1}{2}\left(\int_0^{2^{-\frac{1}{\alpha}}} \left((1-\tau^\alpha)^\beta - \tau^{\alpha\beta}\right) d_\alpha\tau + \int_{2^{-\frac{1}{\alpha}}}^1 \left(\tau^{\alpha\beta}-(1-\tau^\alpha)^\beta\right) d_\alpha\tau\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 (|D_\alpha(f)(a^\alpha\tau^\alpha + b^\alpha(1-\tau^\alpha))|)^q d_\alpha\tau\right)^{\frac{1}{q}}. \tag{15}$$

Using the convexity of $|D_\alpha(f)|^q$ on $[a^\alpha,b^\alpha]$

$$\int_0^1 |D_\alpha(f)(a^\alpha\tau^\alpha + b^\alpha(1-\tau^\alpha))|^q d_\alpha\tau$$

$$\le \int_0^1 \{\tau^\alpha |D_\alpha(f)(a^\alpha)|^q + (1-\tau^\alpha)|D_\alpha(f)(b^\alpha)|^q\} d_\alpha\tau$$

$$= |D_\alpha(f)(a^\alpha)|^q \int_0^1 \tau^\alpha d_\alpha\tau + |D_\alpha(f)(b^\alpha)|^q \int_0^1 \{1-\tau^\alpha\} d_\alpha\tau \tag{16}$$

$$= \frac{(|D_\alpha(f)(a^\alpha)|^q + |D_\alpha(f)(b^\alpha)|^q)}{2\alpha}.$$

Integrating by parts and changing variable of definite integral, we have:

$$\int_0^{2^{-\frac{1}{\alpha}}} \left((1-\tau^\alpha)^\beta - \tau^{\alpha\beta}\right) d_\alpha\tau + \int_{2^{-\frac{1}{\alpha}}}^1 \left(\tau^{\alpha\beta} - (1-\tau^\alpha)^\beta\right) d_\alpha\tau = \frac{2-2^{1-\beta}}{\alpha(\beta+1)}. \quad (17)$$

By substituting (16), (17) into (15), we get the desired result for $q>1$.
If we choose $q=1$, then we get the following inequality;

$$|I_f(f;\alpha;\beta;u)| \le \frac{(|D_\alpha(f)(a^\alpha)| + |D_\alpha(f)(b^\alpha)|)}{4\alpha}.$$

∎

Theorem 10. *Let $\alpha,\beta \in (0,1]$ and $f:[a^\alpha,b^\alpha] \to \mathbb{R}$ be an α-fractional differentiable function on (a^α,b^α) and $D_\alpha(f)$ be an α-fractional integrable function on $[a^\alpha,b^\alpha]$ with $0 \le a < b$. If for $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, $|D_\alpha(f)|^q$ be a convex function on $[a^\alpha,b^\alpha]$, then the following inequality holds:*

$$|I_f(f;\alpha;\beta;u)| \le \frac{1}{2}\left(\frac{\left(2^{\beta p}-1\right)}{\alpha(\beta p+1)2^{\beta p-1}}\right)^{\frac{1}{p}} \left\{\frac{\left[|D_\alpha(f)(a^\alpha)|^q + |D_\alpha(b^\alpha)|^q\right]}{2\alpha}\right\}^{q^{-1}}. \quad (18)$$

Proof. Using Lemma 7, the convexity of $|D_\alpha(f)|^q$ on $[a^\alpha,b^\alpha]$, and Hölder inequality, for $q>1$, we obtain the below inequality

$$\begin{aligned}
&|I_f(f;\alpha;\beta;u)| \\
&\le \tfrac{1}{2}\int_0^1 \left|\tau^{\alpha\beta} - (1-\tau^\alpha)^\beta\right| |D_\alpha(f)(a^\alpha\tau^\alpha + b^\alpha(1-\tau^\alpha))|\, d_\alpha\tau \\
&\le \tfrac{1}{2}\left(\int_0^1 \left|\tau^{\alpha\beta} - (1-\tau^\alpha)^\beta\right|^p d_\alpha\tau\right)^{\frac{1}{p}} \left(\int_0^1 |D_\alpha(f)(a^\alpha\tau^\alpha + b^\alpha(1-\tau^\alpha))|^q d_\alpha\tau\right)^{\frac{1}{q}} \\
&\le \tfrac{1}{2}\left\{\int_0^{2^{-\frac{1}{\alpha}}} \left[(1-\tau^\alpha)^\beta - \tau^{\alpha\beta}\right]^p d_\alpha\tau + \int_{2^{-\frac{1}{\alpha}}}^1 \left[\tau^{\alpha\beta} - (1-\tau^\alpha)^\beta\right]^p d_\alpha\tau\right\}^{\frac{1}{p}} \\
&\times \left(\int_0^1 |D_\alpha(f)(a^\alpha\tau^\alpha + b^\alpha(1-\tau^\alpha))|^q d_\alpha\tau\right)^{\frac{1}{q}} \\
&\le \tfrac{1}{2}\left\{\int_0^{2^{-\frac{1}{\alpha}}} \left[(1-\tau^\alpha)^{\beta p} - \tau^{\alpha\beta p}\right] d_\alpha\tau + \int_{2^{-\frac{1}{\alpha}}}^1 \left[\tau^{\alpha\beta p} - (1-\tau^\alpha)^{\beta p}\right] d_\alpha\tau\right\}^{\frac{1}{p}} \\
&\times \left(\int_0^1 |D_\alpha(f)(a^\alpha\tau^\alpha + b^\alpha(1-\tau^\alpha))|^q d_\alpha\tau\right)^{\frac{1}{q}} \\
&= \tfrac{1}{2}\left(\frac{2}{\alpha(\beta p+1)}\right)^{\frac{1}{p}} \left(1-\frac{1}{2^{\beta p}}\right)^{\frac{1}{p}} \left(\int_0^1 |D_\alpha(f)(a^\alpha\tau^\alpha + b^\alpha(1-\tau^\alpha))|^q d_\alpha\tau\right)^{\frac{1}{q}}.
\end{aligned} \quad (19)$$

On the other hand, for the left side of inequality (19), using the convexity of $|D_\alpha(f)|^q$ on $[a^\alpha, b^\alpha]$ we get

$$\begin{aligned}
&\int_0^1 |D_\alpha(f)(a^\alpha\tau^\alpha + b^\alpha(1-\tau^\alpha))|^q d_\alpha\tau \\
&\leq \int_0^1 \{\tau^\alpha |D_\alpha(f)(a^\alpha)|^q + (1-\tau^\alpha)|D_\alpha(f)(b^\alpha)|^q\} d_\alpha\tau \\
&\leq |D_\alpha(f)(a^\alpha)|^q \int_0^1 \tau^\alpha d_\alpha\tau \\
&+ |D_\alpha(f)(b^\alpha)|^q \int_0^1 (1-\tau^\alpha) d_\alpha\tau \\
&= \frac{|D_\alpha(f)(a^\alpha)|^q + |D_\alpha(f)(b^\alpha)|^q}{2\alpha}.
\end{aligned} \tag{20}$$

By adding the inequality (20) into (19), we get the desired result. ∎

Theorem 11. *Let $\alpha, \beta \in (0,1]$ and $f : [a^\alpha, b^\alpha] \to \mathbb{R}$ be an α-fractional differentiable function on (a^α, b^α) and $D_\alpha(f)$ be an α-fractional integrable function on $[a^\alpha, b^\alpha]$ with $0 \leq a < b$. If for $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $|D_\alpha(f)|^q$ be a convex function on $[a^\alpha, b^\alpha]$, then $I_f(.)$ is bounded and the following inequality holds:*

$$|I_f(f;\alpha;\beta;u)| \leq \frac{1}{2(b^\alpha - a^\alpha)^{\frac{2}{q}}} \|D_\alpha(f)\|_{L_q(a,b)} \left(\tfrac{1}{\alpha(\beta p+1)}\left\{2 - 2^{-\beta p+1}\right\}\right)^{\frac{1}{p}}. \tag{21}$$

Proof. Suppose that $q > 1$. By Lemma 7, the convexity of $|D_\alpha(f)|^q$ on $[a^\alpha, b^\alpha]$, and Hölder's integral inequality, it follows that

$$\begin{aligned}
|I_f(f;\alpha;\beta;u)| &\leq \tfrac{1}{2}\int_0^1 \left|\tau^{\alpha\beta} - (1-\tau^\alpha)^\beta\right| |D_\alpha(f)(a^\alpha\tau^\alpha + b^\alpha(1-\tau^\alpha))| d_\alpha\tau \\
&\leq \tfrac{1}{2}\left(\int_0^1 |D_\alpha(f)(a^\alpha\tau^\alpha + b^\alpha(1-\tau^\alpha))|^q d_\alpha\tau\right)^{\frac{1}{q}} \left(\int_0^1 \left(\left|\tau^{\alpha\beta} - (1-\tau^\alpha)^\beta\right|\right)^p d_\alpha\tau\right)^{\frac{1}{p}} \\
&= \frac{1}{2(b^\alpha - a^\alpha)^{\frac{1}{q}}}\left(\tfrac{1}{b^\alpha - a^\alpha}\int_a^b |D_\alpha(f)(u^\alpha)|^q d_\alpha u\right)^{\frac{1}{q}} \left(\int_0^1 \left(\left|\tau^{\alpha\beta} - (1-\tau^\alpha)^\beta\right|\right)^p d_\alpha\tau\right)^{\frac{1}{p}} \\
&= \frac{1}{2(b^\alpha - a^\alpha)^{\frac{2}{q}}} \|D_\alpha(f)\|_{L_q(a,b)} \left(\int_0^1 \left(\left|\tau^{\alpha\beta} - (1-\tau^\alpha)^\beta\right|\right)^p d_\alpha\tau\right)^{\frac{1}{p}}.
\end{aligned} \tag{22}$$

On the other hand, for the right hand of inequality, we get the following inequality;

$$\begin{aligned}\int_0^1 \left(\left|\tau^{\alpha\beta} - (1-\tau^{\alpha})^{\beta}\right|\right)^p \tau^{\alpha-1} d\tau &= \int_0^{2^{-\frac{1}{\alpha}}} \left((1-\tau^{\alpha})^{\beta} - \tau^{\alpha\beta}\right)^p d_{\alpha}\tau \\ &+ \int_{2^{-\frac{1}{\alpha}}}^1 \left(\tau^{\alpha\beta} - (1-\tau^{\alpha})^{\beta}\right)^p d_{\alpha}\tau \\ &\le \frac{1}{\alpha}\int_0^{2^{-1}} \left((1-u)^{\beta p} - u^{\beta p}\right) du \\ &+ \frac{1}{\alpha}\int_{2^{-1}}^1 \left(u^{\beta p} - (1-u)^{\beta p}\right) du \\ &= \frac{1}{\alpha(\beta p+1)} \left\{2 - 2^{-\beta p+1}\right\}.\end{aligned}$$

Thus the proof is completed. ∎

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