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FIXED POINTS THEOREMS FOR MULTIVALUED MAPPINGS IN G-CONE METRIC SPACES OVER BANACH ALGEBRA

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Abstract. Motivated by the recent work of Liu and Xu, we establish a generalized fixed point theorem for multivalued mapping in the setting of G-cone metric spaces over Banach algebra without assuming the normality of the underlying cone. Our results generalize and extend some known results of G-metric and G-cone metric spaces. An example and certain consequences are presented to illustrate significance of proved results.

1. Introduction and Preliminaries

In the theory of fixed point, Banach contraction mapping principle is simple and powerful result with a wide range of applications, including iterative methods for solving linear, nonlinear, differential, integral, and difference equations. Due to wide spreading importance of Banach contraction, many authors generalized it in several directions[1, 2, 3, 5, 7, 15, 16, 17, 22, 27].

The study of fixed points for multivalued mappings using the Hausdorff metric was initiated by Markin. Later, an interesting and rich fixed point theory for such mappings has been developed. The theory of multivalued mappings has applications in control theory, convex optimization, differential inclusions and economics. Following the Banach contraction principle Nadler [29] introduced the concept of multivalued contractions and established that a multivalued contraction possesses a fixed point in a complete metric space. Subsequently many authors generalized Nadler's fixed point theorem in different way.

Huang and Zhang[14], introduced cone metric space with normal cone with constant K, which is generalization of metric space. After that Rezapour and Hamlbarani [30] generalized cone metric space with non-normal cone. Afterwords many researchers [31, 32] studied fixed point results in cone metric spaces. In 2006, Mustafa et al.[26] introduced the notion of G-metric space as a generalization of the metric space which recovered the flaws of D-metric space [12, 13]. Many researchers proved many fixed point results using G-metric space[20, 25]. Anchalee Kaewcharoen and Attapol Kaewkhao [20] and Nedal et al. [28] proved fixed point

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results for multivalued maps in G-metric spaces. In 2009, Beg et al. [10] introduced the notion of G-cone metric space and generalized some fixed point results of cone metric spaces and G-metric spaces.

In 2011, Cho et al.[11] gave the notion of generalized Hausdorff distance function in the setting of cone metric spaces. Later on, Azam et al. [9] utilized this generalized Hausdorff distance function in G-cone metric spaces and exploit it to obtain the fixed points of multivalued mappings. Recently, Liu et al. [23] used the cones over a Banach algebra and proved some fixed point theorems in cone metric spaces. They improved the contractive condition by replacing the contractive constant by a vector of cone. In this paper, we utilize the generalized Hausdorff distance function in G-cone metric spaces and establish a generalized fixed point theorem for multivalued mapping in the setting of G- cone metric spaces over Banach algebra.

Now we give the concept of G- cone metric spaces over Banach algebra. For this, first we discuss some basic properties of Banach algebra. Let $\mathbb{A}$ be a real Banach

algebra, i.e. $\mathbb{A}$ is a real Banach space in which an operation of multiplication is defined, subject to the following properties:

for all $x, y, z \in \mathbb{A}$, $a \in \mathbb{R}$

(i) $x(yz) = (xy)z$;

(ii) $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$;

(iii) $a(xy) = (ax)y = x(ay)$;

(iv)$||xy|| \leq ||x||||y||$.

We shall assume that the Banach algebra $\mathbb{A}$ has a unit, i.e., a multiplicative identity e such that $ex = xe = x$ for all $x \in \mathbb{A}$. An element $x \in \mathbb{A}$ is said to be invertible if there is an inverse element $y \in \mathbb{A}$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1}.

Consistent with [23] , let $\mathbb{A}$ be a real Banach algebra with a unit e and $x \in \mathbb{A}$. If the spectral radius $\rho(x)$ of x is less than one, that is

$$\rho(x) = \lim_{n\to\infty} ||x^n||^{\frac{1}{n}} = \inf_{n\geq 1} ||x^n||^{\frac{1}{n}} < 1$$

thene $-x$ is invertible. Actually,

$$(e - x)^{-1} = \sum_{i=0}^{\infty} x_{i.}$$

A subset P of $\mathbb{A}$ is called a cone if and only if

(i) $\{e, \theta\} \subset P$;

(ii) $P^2 = PP \subset P$, $P \cap (-P) = \{\theta\}$;

$(iii) a, b \in \mathbb{R}, a, b \geq 0, aP + bP \subset P$.

For a given cone $P \subset \mathbb{A}$, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$; $x \prec y$ will stand for $x \preceq y$ and $x \neq y$,while $x \ll y$ stand for $y - x \in intP$, where $intP$ denotes the interior of P. If $intP \neq \emptyset$, then P is called a solid cone. Write $|| \cdot ||$ as the norm of $\mathbb{A}$. A cone P is called normal if there is a number $M > 0$ such that for all $x, y \in \mathbb{A}$, we have

$$\theta \preceq x \preceq y \Longrightarrow ||x|| \leq M||y||.$$

The least positive number satisfying above is called the normal constant of P. Note that, for any normal cone P we have $M \geq 1$.

In the following we always suppose that $\mathbb{A}$ is a real Banach algebra with a unit e, P is a solid cone and $\preceq$ is the partial ordering with respect respect to P.

The following lemmas and remark will be useful in the sequel.

Lemma 1.1. [23] *If $\mathbb{A}$ is a real Banach algebra with a cone P and if $a \preceq \lambda a$ with $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.*

Lemma 1.2. [23] *If $\mathbb{A}$ is a real Banach algebra with a solid cone P and if $\theta \preceq u \ll c$ for each $\theta \ll c$, then $u = \theta$.*

Lemma 1.3. [23] *If $\mathbb{A}$ is a real Banach algebra with a solid cone P and if $||x_n|| \to 0$ as $n \to \infty$, then for any $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that, $x_n \ll c$ for all $n < n_0$.*

Remark 1.4. [23] *If $\rho(x) < 1$, then $||x_n|| \to 0$ as $n \to \infty$.*

Definition 1.5. [23] *Let X be a nonempty set and $\mathbb{A}$ be a Banach algebra. A function $d : X \times X \to \mathbb{A}$ is said to be a cone metric, if the following conditions hold:*
$(C1)$ $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
$(C2)$ $d(x, y) = d(y, x)$ for all $x, y \in X$;
$(C3)$ $d(x, z) \preceq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

The pair (X, d) is then called a cone metric space over Banach algebra $\mathbb{A}$.

Lemma 1.6. [23] *Let (X, d) be a cone metric space over Banach algebra $\mathbb{A}$, $x \in X$, let $\{x_n\}$ be a sequence in X. Then*
(i) $\{x_n\}$ converges to x whenever for every $c \in \mathbb{A}$ with $\theta \ll c$ there is a natural number n_0 such that $d(x_n, x) \ll c$, for all $n \geq n_0$. We denote this by $\lim_{n\to\infty} x_n = x$;
(ii) $\{x_n\}$ is a Cauchy sequence whenever for every $c \in \mathbb{A}$ with $\theta \ll c$ there is a natural number n_0 such that $d(x_n, x_m) \ll c$, for all $n, m \geq n_0$;
(iii) (X, d) is complete cone metric if every Cauchy sequence in X is convergent.

Now with some modification in the concept of G-cone metric spaces of [10] , we give the notion G-cone metric spaces over Banach algebra $\mathbb{A}$. In the following we shall always assume that the cone P is solid and non-normal.

Definition 1.7. *Let X be a nonempty set. Suppose a mapping $G : X \times X \times X \to \mathbb{A}$ satisfies:*

$(G1)G(x, y, z) = \theta$ if $x = y = z$,
$(G2)\theta \prec G(x, x, y)$, whenever $x \neq y$, for all $x, y \in X$,
$(G3)G(x, x, y) \preccurlyeq G(x, y, z)$, whenever $y \neq z$,
$(G4)G(x, y, z) = G(x, z, y) = G(y, x, z) = ...$(Symmetric in all three variables),
$(G5)G(x, y, z) \preccurlyeq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then G is called a generalized cone metric on X, and X is called a generalized cone metric space over Banach algebra $\mathbb{A}$ or more specifically a G-cone metric space over Banach algebra $\mathbb{A}$.

Note that the concept of a G-cone metric space over Banach algebra $\mathbb{A}$ is more general than that of a G-cone metric spaces.

Definition 1.8. *A G-cone metric space over Banach algebra $\mathbb{A}$ is symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.*

Definition 1.9. *Let (X, d) be a cone metric space over Banach algebra $\mathbb{A}$. Define $G : X \times X \times X \to \mathbb{A}$, by $G(x, y, z) = d(x, y) + d(y, z) + d(z, x)$. Then (X, G) is G-cone metric space over Banach algebra $\mathbb{A}$.*

Proposition 1.10. *Let X be a G-cone metric space over Banach algebra $\mathbb{A}$, define $d_G : X \times X \to \mathbb{A}$ by*

$$d_G(x, y) = G(x, y, y) + G(y, x, x).$$

Then (X, d_G) is a cone metric space over Banach algebra $\mathbb{A}$.
If X is a symmetric G-cone metric space over Banach algebra $\mathbb{A}$, then $d_G(x, y) = 2G(x, y, y)$, for all $x, y \in X$.

Definition 1.11. *Let X be a G-cone metric space over Banach algebra $\mathbb{A}$ and $\{x_n\}$ be a sequence in X. If $\{x_n\}$ is a sequence in X, then $\{x_n\}$ is:*

(a) Cauchy sequence if for every $c \in \mathbb{A}$ with $\theta \ll c$, there is N such that for all $n, m, l > N, G(x_n, x_m, x_l) \ll c$.

(b) Convergent sequence if for every c in $\mathbb{A}$ with $\theta \ll c$, there is N such that for all $m, n > N$, $G(x_m, x_n, x) \ll c$ for some fixed x in X. Here x is called the limit of a sequence $\{x_n\}$ and is denoted by $\lim\limits_{n\to\infty} x_n = x$ *or* $x_n \to x$ *as* $n \to \infty$.

A G-cone metric space over Banach algebra $\mathbb{A}$ is said to be complete if every Cauchy sequence in it, is convergent.

Proposition 1.12. *Let X be a G-cone metric space over Banach algebra $\mathbb{A}$, then the following are equivalent.*
(i) $\{x_n\}$ is converges to x.
(ii) $G(x_n, x_n, x) \to \theta$, as $n \to \infty$.
(iii) $G(x_n, x, x) \to \theta$, as $n \to \infty$.
(iv) $G(x_m, x_n, x) \to \theta$, as $m, n \to \infty$.

Lemma 1.13. *Let $\{x_n\}$ be a sequence in a G-cone metric space X over Banach algebra $\mathbb{A}$ and if $\{x_n\}$ converges to $x \in X$, then $G(x_m, x_n, x) \to \theta$ as $m, n \to \infty$.*

Lemma 1.14. *Let $\{x_n\}$ be a sequence in a G-cone metric space X over Banach algebra $\mathbb{A}$ and $x \in X$. If $\{x_n\}$ converges to $x \in X$, then $\{x_n\}$ is a Cauchy sequence.*

Lemma 1.15. *Let $\{x_n\}$ be a sequence in a G-cone metric space X over Banach algebra $\mathbb{A}$ and if $\{x_n\}$ is a Cauchy sequence in X, then $G(x_m, x_n, x_l) \to \theta$, as $m, n, l \to \infty$.*

Denote $N(X)$, $B(X)$ and $C(X)$ the set of nonempty, bounded and sequentially closed subset of a G-cone metric space over Banach algebra $\mathbb{A}$ respectively.

Let (X, G) be a G-cone metric space over Banach algebra $\mathbb{A}$, we denote

$$s(p) = \{q \in \mathbb{A} : p \preccurlyeq q\} \text{ for } q \in \mathbb{A}$$

and

$$s(a, B) = \bigcup_{b\in B} s(d_G(a, b)) = \bigcup_{b\in B} \{x \in \mathbb{A} : \ d_G(a, b) \preccurlyeq x\}$$

for $a \in X$ and $B \in N(X)$. For $A, B \in B(X)$ we denote

$$\hat{s}(A, B) = \bigcup_{a\in A, b\in B} s(d_G(a, b)),$$

$$\begin{aligned} s(a, B, C) &= s(a, B) + \hat{s}(B, C) + s(a, C) \\ &= \{u + v + w : u \in s(a, B), v \in \hat{s}(B, C), w \in s(a, C)\}, \end{aligned}$$

and

$$s(A, B, C) = \left(\bigcap_{a\in A} s(a, B, C)\right) \cap \left(\bigcap_{b\in B} s(b, A, C)\right) \cap \left(\bigcap_{c\in C} s(c, A, B)\right).$$

Lemma 1.16. *Let (X,G) be a G-cone metric space over Banach algebra $\mathbb{A}$, P be a solid cone. Then the following conditions hold:*

(i) Let $p,q \in \mathbb{A}$. if $p \preccurlyeq q$, then $s(q) \subset s(p)$.
(ii) Let $x \in X$ and $A \in N(X)$. If $0 \in s(x,A)$, then $x \in A$.
(iii) Let $q \in P$ and let $A,B,C \in B(X)$ and $a \in A$. If $q \in s(A,B,C)$, then $q \in s(a,B,C)$.
(iv) Let $q \in P$ and let $\lambda \geq 0$, then $\lambda s(q) \subseteq s(\lambda q)$.

Remark 1.17. *Kaewcharoen and Kaewkhao* [20]*(see also*[28]*) . gave the following concepts on G-metric spaces. Let X be a G-metric space and $CB(X)$ the family of all nonempty, closed and bounded subsets of X. Let $H_G(.,.,.)$ be the Hausdorff G-distance on $CB(X)$, i.e.,*

$$H_G(A,B,C) = \max\{\sup_{a\in A} G(a,B,C), \sup_{b\in B} G(b,A,C), \sup_{c\in C} G(c,A,B)\}$$

$$H_{d_G}(A,B) = \max\{\sup_{a\in A} d_G(a,B), \sup_{b\in B} d_G(b,A)\},$$

where

$$\begin{aligned} G(x,B,C) &= d_G(x,B) + d_G(B,C) + d_G(x,C), \\ d_G(x,B) &= \inf\{d_G(x,y), y \in B\}, \\ d_G(A,B) &= \inf\{d_G(a,b), a \in A, b \in B\}, \\ G(a,b,C) &= \inf\{G(a,b,c),\ c \in C\}. \end{aligned}$$

The above expressions shows a relation between H_G and H_{d_G}. Moreover, note that if (X,G) is a G-cone metric space over Banach algebra $\mathbb{A}$, $\mathbb{A} = \mathbb{R}$, and $P = [0,\infty)$, then (X,G) is a G-metric space. Also for $A,B,C \in CB(X)$,

$$H_G(A,B,C) = \inf s(A,B,C).$$

Remark 1.18. *Let (X,G) be a G-cone metric space over Banach algebra $\mathbb{A}$. Then*

$$\begin{aligned} &(a) \qquad \hat{s}(\{a\},\{b\}) = s(d_G(a,b)) \qquad \textit{for} \quad a,b \in X. \\ &(b) \qquad \textit{If } x \in s(a,B,B) \textit{ then } x \in 2s(d_G(a,b)). \end{aligned}$$

Proof. Proof of this remark is same as in [9]. □

Remark 1.19. *Let (X,G) be a G-cone metric space over Banach algebra $\mathbb{A}$ and $T: X \to 2^X$ be a mutivalued mapping. Then*

$$s(Tx,Ty,Ty) \neq s(Ty,Tx,Tx)$$

Proof. Proof of this remark is same as in [9]. □

In 2013, Alghamdi et al. [4] introduced the concepts of α-admissible mapping in G-metric space and established some fixed point theorems.

Definition 1.20. *Let T be a self-mapping on X and $\alpha : X \times X \times X \to [0,+\infty)$ be a function. Then T is said to be an α-admissible mapping if*

$$x,y,z \in X, \quad \alpha(x,y,z) \geq 1 \quad \Longrightarrow \quad \alpha(Tx,Ty,Tz) \geq 1.$$

In this paper, using the concept of α-admissibility we prove fixed point theorems for multivalued mappings in the context of complete G-cone metric space over a Banach algebra $\mathbb{A}$.

2. Main Result

Theorem 2.1. *Let (X, G) be a complete G-cone metric space over a Banach algebra $\mathbb{A}$ and P be the underlying solid cone. If there exist a function $\alpha : X \times X \times X \to [0, +\infty)$ and $k \in P$ with $\rho(\frac{k}{2}) < 1$ and a multivalued mapping $T : X \to C(X)$ such that*

$$kG(x, y, z) \in \alpha(x, y, z)\, s(Tx, Ty, Tz) \tag{2.1}$$

for all $x, y, z \in X$. If there exist $x_0 \in X$, $x_1 \in Sx_0$ such that $\alpha(x_0, x_1, x_1) \geq 1$. Assume that if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, x_{n+1}) \geq 1$ for all n and $x_n \to x^*$ as $n \to +\infty$ then $\alpha(x_n, x^*, x^*) \geq 1$ for all n. Then, T has a fixed point in X.

Proof. Let x_0 be an arbitrary point in X , then $Tx_0 \in C(X)$, so $Tx_0 \neq \varnothing$. Let $x_1 \in Tx_0$. We may suppose that $x_0 \neq x_1$. Then $\alpha(x_0, x_1, x_1) \geq 1$. From (2.1), we have

$$kG(x_0, x_1, x_1) \in \alpha(x_0, x_1, x_1)\, s(Tx_0, Tx_1, Tx_1).$$

Thus by Lemma 1.8 (iii), we get

$$kG(x_0, x_1, x_1) \in \alpha(x_0, x_1, x_1)\, s(x_1, Tx_1, Tx_1).$$

By Remark 1.3 (b), we can take $x_2 \in Tx_1$ such that

$$kG(x_0, x_1, x_1) \in 2\alpha(x_0, x_1, x_1)\, s(d_G(x_1, x_2)) = s(2\alpha(x_0, x_1, x_1)\, d_G(x_1, x_2)).$$

Thus,

$$2\alpha(x_0, x_1, x_1)\, d_G(x_1, x_2) \preccurlyeq kG(x_0, x_1, x_1). \tag{2.2}$$

Now if $x_1 = x_2$,then x_1 is the required fixed point. And we have nothing to prove. So we suppose that $x_1 \neq x_2$, then $x_2 \notin Tx_2$. Now from (2.1), we hav

$$kG(x_1, x_2, x_2) \in \alpha(x_1, x_2, x_2)\, s(Tx_1, Tx_2, Tx_2)$$

and by Lemma 1.8 (iii), we get

$$kG(x_1, x_2, x_2) \in \alpha(x_1, x_2, x_2)\, s(x_2, Tx_2, Tx_2).$$

By Remark 1.3 (b), we can take $x_3 \in Tx_2$ such that

$$kG(x_1, x_2, x_2) \in 2\alpha(x_1, x_2, x_2)\, s(d_G(x_2, x_3)) = s(2\alpha(x_1, x_2, x_2)\, d_G(x_2, x_3)).$$

Thus,

$$2\alpha(x_1, x_2, x_2)\, d_G(x_2, x_3) \preccurlyeq kG(x_1, x_2, x_2). \tag{2.3}$$

It implies that

$$\begin{aligned} 2\alpha(x_1, x_2, x_2)\, d_G(x_2, x_3) &\preccurlyeq kG(x_1, x_2, x_2), \\ &\preccurlyeq kG(x_1, x_2, x_2) + kG(x_2, x_1, x_1), \\ &\preccurlyeq k[G(x_1, x_2, x_2) + G(x_2, x_1, x_1)] \\ &= kd_G(x_1, x_2). \end{aligned}$$

This implies that

$$d_G(x_2, x_3) \preccurlyeq \frac{k}{2} d_G(x_1, x_2). \tag{2.4}$$

By continuing this process, we obtain a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n, x_{n+1} \neq x_n$, $\alpha(x_n, x_{n+1}, x_{n+1}) \geq 1$ and

$$d_G(x_n, x_{n+1}) \preceq l d_G(x_{n-1}, x_n) \preceq ... \preceq l^n d_G(x_0, x_1) \tag{2.5}$$

for all n, where $l = \frac{k}{2}$. Then, for $n < m$ we have

$$\begin{aligned} d_G(x_n, x_m) &\preceq d_G(x_n, x_{n+1}) + d_G(x_{n+1}, x_{n+2}) + d_G(x_{n+2}, x_{n+3}) + ... + d_G(x_{m-1}, x_m) \\ &\preceq l^n d_G(x_0, x_1) + l^{n+1} d_G(x_0, x_1) + l^{n+2} d_G(x_0, x_1) + ... + l^{m-1} d_G(x_0, x_1) \\ &\preceq l^n (e + l^1 + l^2 + ... + l^{m-n-1}) d_G(x_0, x_1) \\ &\preceq l^n (e - l)^{-1} d_G(x_0, x_1). \end{aligned}$$

Since $\rho(l) < 1$, we have $||l^n|| \to 0$ as $n \to \infty$. Therefore for every $c \in \mathbb{A}$ with $\theta \ll c$ there exists $n_0 \in \mathbb{N}$ such that

$$d_G(x_n, x_m) \preceq l^n (e - l)^{-1} d_G(x_0, x_1) \ll c$$

for all $n > n_0$. This is sufficient to conclude that $\{x_n\}$ is a Cauchy sequence. Then there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. As we have a sequence $\{x_n\}$ in X such that $\alpha(x_n, x_{n+1}, x_{n+1}) \geq 1$ for all n and $x_n \to x^*$ as $n \to +\infty$ then by asumption , we have $\alpha(x_n, x^*, x^*) \geq 1$ for all n. From (2.1), we have

$$kG(x_n, x^*, x^*) \in \alpha(x_n, x^*, x^*) s(Tx_n, Tx^*, Tx^*).$$

By Lemma 1.8 (iii), we have

$$kG(x_n, x^*, x^*) \in \alpha(x_n, x^*, x^*) s(x_{n+1}, Tx^*, Tx^*).$$

By Remark 1.3 (b), there exists $u_n \in Tx^*$, such that

$$kG(x_n, x^*, x^*) \in 2\alpha(x_n, x^*, x^*) s(d_G(x_{n+1}, u_n)) = s(2\alpha(x_n, x^*, x^*) d_G(x_{n+1}, u_n)).$$

It implies that

$$\begin{aligned} 2\alpha(x_n, x^*, x^*) d_G(x_{n+1}, u_n) &\preccurlyeq kG(x_n, x^*, x^*), \\ \alpha(x_n, x^*, x^*) d_G(x_{n+1}, u_n) &\preccurlyeq \frac{1}{2} kG(x_n, x^*, x^*), \\ &\preccurlyeq \frac{1}{2} k[G(x_n, x^*, x^*) + G(x_n, x_n, x^*)] = k d_G(x_n, x^*). \end{aligned}$$

Since $\alpha(x_n, x^*, x^*) \geq 1$ for all n, so we have

$$d_G(x_{n+1}, u_n) \preccurlyeq \alpha(x_n, x^*, x^*) d_G(x_{n+1}, u_n) \preccurlyeq k d_G(x_n, x^*).$$

As $x_n \to x^*$ as $n \to +\infty$, so for a given $c \in IntP$, there exists $k \in \mathbb{N}$ such that $d_G(x_n, x^*) \ll \frac{c}{2}$ and $d_G(x_{n+1}, x^*) \ll \frac{c}{2}$ for $n \geq k = k(c)$. Now from triangular property, we have

$$\begin{aligned} d_G(x^*, u_n) &\preccurlyeq d_G(x_{n+1}, x^*) + d_G(x_{n+1}, u_n), \\ &\preccurlyeq d_G(x_{n+1}, x^*) + k d_G(x_n, x^*) . \\ &\prec d_G(x_{n+1}, x^*) + d_G(x_n, x^*), \\ d_G(x^*, u_n) &\ll \frac{c}{2} + \frac{c}{2} = c. \end{aligned}$$

for $n \geq k = k(c)$. Therefore $\lim_{n \to \infty} u_n = x^*$. Since Tx^* is closed so $x^* \in Tx^*$. □

Corollary 2.2. *Let (X, G) be a complete G-cone metric space over a Banach algebra $\mathbb{A}$ and P be the underlying solid cone. If there exist $k \in P$ with $\rho(\frac{k}{2}) < 1$ and a multivalued mapping $T : X \to C(X)$ such that*

$$kG(x,y,z) \in s(Tx,Ty,Tz)$$

Then, T has a fixed point in X.

By Remark 1.2, we have the following result.

Corollary 2.3. *Let (X,G) be a complete G-cone metric space over a Banach algebra $\mathbb{A}$ and P be the underlying solid cone and let $T : X \longrightarrow CB(X)$ be multivalued mapping. There exist a function $\alpha : X \times X \times X \to [0,+\infty)$ and $k \in P$ with $\rho(\frac{k}{2}) < 1$ such that*

$$\alpha(x,y,z) H_G(Tx,Ty,Tz) \leq kG(x,y,z)$$

for all $x,y,z \in X$. Also suppose that the following assertions holds:

(a) There exist $x_0 \in X$, $x_1 \in Sx_0$ such that $\alpha(x_0,x_1,x_1) \geq 1$.

(b)If $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1},x_{n+1}) \geq 1$ for all n and $x_n \to x^*$ as $n \to +\infty$ then $\alpha(x_n,x^*,x^*) \geq 1$ for all n.

Then T has a fixed point in X.

Corollary 2.4. *Let (X,G) be a complete G-cone metric space over a Banach algebra $\mathbb{A}$ and P be the underlying solid cone and let $T : X \longrightarrow CB(X)$ be multivalued mapping. There exists $k \in P$ with $\rho(\frac{k}{2}) < 1$ such that*

$$H_G(Tx,Ty,Tz) \leq kG(x,y,z)$$

for all $x,y,z \in X$. Then T has a fixed point in X.

In the following we formulate an illustrative example regarding our main theorem.

Example 2.5. *Let $X = [0,1]$, $\mathbb{A} = C^1_{\mathbb{R}}[0,1] \times C^1_{\mathbb{R}}[0,1]$ with the norm*

$$||u|| = ||u||_\infty + ||u'||_\infty.$$

Define the multiplication on $\mathbb{A}$ as just pointwise multiplication. Then $\mathbb{A}$ is a real Banach algebra with unit $e = 1(e(t) = 1$ for all $t \in X)$.

Let $P = \{u(t) \in \mathbb{A} : u(t) \geq 0 : t \in [0,1]\}$. Then P is a non-normal solid cone. Define $G : X \times X \times X \to E$ by

$$G(x,y,z)(t) = \max\{|x-y|, |y-z|, |x-z|\}e^t.$$

Then (X,G) is a G-cone metric space over Banach algebra $\mathbb{A}$. Define $T : X \to 2^X$ by $Tx = [0, \frac{x}{20}]$ for all $x \in X$ and

$$\alpha(x,y,z) = \begin{cases} 1 \text{ if } x \neq y, x \neq z, y \neq z \\ 0 \text{ if } x = y = z. \end{cases}$$

Then clearly T is α-admissible. Take $k = \frac{1}{2} + \frac{1}{4}t$, where $t \in [0,1]$, then $\rho(k) = \frac{2}{3} < 1$. The contractive condition of main theorem is trivial for the case when $x = y = z = 0$. Suppose without any loss of generality that all x, y and z are nonzero and $x < y < z$.

Then

$$G(x,y,z) = |x-z| e^t,$$

and

$$d_G(x,y) = 2|x-y| e^t.$$

Now

$$s(x,Ty)=\left\{\begin{array}{c} 0 \text{ if } x\leq \frac{y}{20} \\ \left|x-\frac{y}{20}\right|e^{t} \text{ if } x>\frac{y}{20}\end{array}\right.$$

$$s(y,Tz)=\left\{\begin{array}{c} 0 \text{ if } y\leq \frac{z}{20} \\ \left|y-\frac{z}{20}\right|e^{t} \text{ if } y>\frac{z}{20}.\end{array}\right.$$

Now for $s(x,Ty)=0=s(y,Tz)$, we have

$$s\left(x,Ty,Tz\right)=\underset{x\in Tx}{\cap}s\left(x,Ty,Tz\right)=s(x,Ty)+\hat{s}(Ty,Tz)+s(x,Tz)=s(0),$$

$$s\left(y,Tx,Tz\right)=s(y,Tx)+\hat{s}(Tx,Tz)+s(y,Tz)=s(2\left|y-\frac{x}{20}\right|e^{t}),$$

$$\underset{y\in Ty}{\cap}s\left(y,Tx,Tz\right)=s(2\left|\frac{y}{20}-\frac{x}{20}\right|e^{t}),$$

$$s\left(z,Tx,Ty\right)=s(z,Tx)+\hat{s}(Tx,Ty)+s(z,Ty)=s(2\left|z-(\frac{x}{20}+\frac{y}{20})\right|e^{t}),$$

and

$$\underset{z\in Tz}{\cap}s\left(z,Tx,Ty\right)=s(2\left|\frac{z}{20}-(\frac{x}{20}+\frac{y}{20})\right|e^{t})=s(2\left|\frac{z}{20}-\frac{x}{20}-\frac{y}{20}\right|e^{t}).$$

So we have

$$\begin{aligned} s\left(Tx,Ty,Tz\right) &= \left(\underset{x\in Tx}{\cap}s\left(x,Ty,Tz\right)\right)\cap\left(\underset{y\in Ty}{\cap}s\left(y,Tx,Tz\right)\right)\cap\left(\underset{z\in Tz}{\cap}s\left(z,Tx,Ty\right)\right), \\ &= \left(s(0)\right)\cap\left(s(2\left|\frac{y}{20}-\frac{x}{20}\right|e^{t})\right)\cap\left(s(2\left|\frac{z}{20}-\frac{x}{20}-\frac{y}{20}\right|e^{t})\right). \end{aligned}$$

Now we discuss the following three cases.

(i). If $s\left(Tx,Ty,Tz\right)=s(2\left|\frac{z}{20}-\frac{x}{20}-\frac{y}{20}\right|e^{t})$, then for $t\in[0,1]$, we have

$$\begin{aligned} 2\left|\frac{z}{20}-\frac{x}{20}-\frac{y}{20}\right|e^{t} &\preceq 2\left|\frac{z}{20}-\frac{x}{20}\right|e^{t}, \\ &= \frac{1}{10}\left|z-x\right|e^{t} \\ &\prec (\frac{1}{2}+\frac{1}{4}t)\left|z-x\right|e^{t}=(\frac{1}{2}+\frac{1}{4}t)\max\{\left|x-y\right|,\left|y-z\right|,\left|x-z\right|\}e^{t} \\ &= kG(x,y,z). \end{aligned}$$

So by definition we have

$$kG(x,y,z)\in\alpha(x,y,z)s\left(Tx,Ty,Tz\right).$$

(ii). If $s\left(Tx,Ty,Tz\right)=s(2\left|\frac{y}{20}-\frac{x}{20}\right|e^{t})$,then for $t\in[0,1]$, we have

$$\begin{aligned} 2\left|\frac{y}{20}-\frac{x}{20}\right|e^{t} &\preceq 2\left|\frac{z}{20}-\frac{x}{20}\right|e^{t} \\ &= \frac{1}{10}\left|z-x\right|e^{t} \\ &\prec (\frac{1}{2}+\frac{1}{4}t)\left|z-x\right|e^{t}=(\frac{1}{2}+\frac{1}{4}t)\max\{\left|x-y\right|,\left|y-z\right|,\left|x-z\right|\}e^{t} \end{aligned}$$

Thus we have

$$kG(x,y,z)\in\alpha(x,y,z)s\left(Tx,Ty,Tz\right).$$

(iii). If $s(Tx, Ty, Tz) = s(0)$, then

$$\begin{aligned} 0 &\preceq 2\left|\frac{z}{20} - \frac{x}{20}\right| e^t, \\ &= \frac{1}{10}|z-x|\,e^t \\ &\prec (\frac{1}{2} + \frac{1}{4}t)\,|z-x|\,e^t = (\frac{1}{2} + \frac{1}{4}t)\max\{|x-y|\,, |y-z|\,, |x-z|\}e^t \end{aligned}$$

Thus we have

$$kG(x, y, z) \in \alpha(x, y, z)s(Tx, Ty, Tz).$$

Similarly one can prove for other possible values of $s(x, Ty)$ and $s(y, Tz)$.

Hence all the conditions of our main theorem are satisfied and 0 is a fixed point of mapping T.

Competing interests

The authors declare that they have no competing interests.

Authors'contribution

Both authors read and approved the final manuscript.

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References

[1] A. Ahmad, A. Al-Rawashdeh, A. Azam, *Fixed point results for $\{\alpha, \xi\}$-expansive locally contractive mappings*, Journal of Inequalities and Applications 2014, 2014:364

[2] J. Ahmad, A. E. Al-Mazrooei, Y. J. Cho, Y. O. Yang, *Fixed point results for generalized Θ-contractions*, J. Nonlinear Sci. Appl., **10** (2017), 2350–2358

[3] J. Ahmad, A. Al-Rawashdeh and A. Azam, *New Fixed Point Theorems for Generalized F-Contractions in Complete Metric Spaces*, Fixed Point Theory and Applications (2015)

[4] M. A. Alghamdi and E. Karapınar, *$G-\beta-\psi$-contractive type mappings in G-metric spaces*, Fixed Point Theory and Applications 2013, 2013:123

[5] A. Al-Rawashdeh and J. Ahmad, *Common Fixed Point Theorems for JS- Contractions*, Bull. Math. Anal. Lapp.,Volume **8** Issue 4(2016), Pages 12-22.

[6] M. Arshad and J. Ahmad, *On multivalued contractions in cone metric spaces without normality*, The Scientific World Journal, 2013, Article ID 481601, 3 pages.

[7] Z. Aslam, J. Ahmad, N. Sultana, *New common fixed point theorems for cyclic compatible contractions*, J. Math. Anal., **8** (2017), 1–12.

[8] A. Azam, N. Mehmood, J. Ahmad and S. Radenović, *Multivalued fixed point theorems in cone b-metric spaces*, Journal of Inequalities and Applications20132013:582

[9] A. Azam, N. Mehmood, *Fixed point theorems for multivalued mappings in G-cone metric spaces*, Journal of Inequalities and Applications, (2013) 2013: 354.

[10] I. Beg, M. Abbas., T. Nazir, *Generalized cone metric spaces*, J. Nonlinear Sci. Appl., 2010, 3(1): 21 - 31.

[11] SH. Cho, JS. Bae, *Fixed point theorems for multi-valued maps in cone metric spaces,* Fixed Point Theory Appl 2011, 87 (2011). doi: 10.1186/1687-1812-2011-87.

[12] BC. Dhage, *Generalized metric space and mapping with fixed point*, Bulletin of the Calcutta Mathematical Society, 84 (1992), 329-336.

[13] BC. Dhage, *Generalized metric space and topological structure I*, Analele Stiintifice ale Universitatii "Al. I. Cuza" din Iasi. Serie Noua. Mathematica, 46 (1) (2000), 3-24.

[14] LG. Huang, X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J Math Anal Appl. 332(2), 1468–1476 (2007). doi:10.1016/j.jmaa.2005.03.087.

[15] N. Hussain, E. Karapinar, P. Salimi, P. Vetro, *Fixed point results for G^m-Meir-Keeler contractive and $G-(\alpha,\psi)$-Meir-Keeler contractive mappings*, Fixed Point Theory and Applications 2013, 2013:34.

[16] N. Hussain, A. E. Al-Mazrooei, J. Ahmad, *Fixed point results for generalized (α-η)-Θ contractions with applications*, J. Nonlinear Sci. Appl., **10** (2017), 4197–4208

[17] N. Hussain, J. Ahmad and A. Azam, *On Suzuki-Wardowski type fixed point theorems*, J. Nonlinear Sci. Appl. **8** (2015), 1095-1111

[18] S. Janković, Z. Kadelburg, S. Radenović, *On cone metric spaces: A survey,* Nonlinear Anal. 74(2011)2591-2601.

[19] M. Jlelli, B. Samet, *Remarks on G-metric spaces and fixed point theorems,* Fixed Point Theory Appl 2012: 210.

[20] A. Kaewcharoen, A. Kaewkhao, *Common fixed points for single-valued and multi-valued mappings in G-metric spaces,* Int. J. Math. Anal. Vol. 5, 2011, no. 36, 1775 - 1790.

[21] M.A. Kutbi, J. Ahmad and A. Azam, *On fixed points of $\alpha-\psi$- contractive multi- valued mappings in cone metric spaces*, Abstract and Applied Analysis, Volume 2013, Article ID 313782.

[22] A. Latif, N. Hussain, J. Ahmad, *Coincidence points for hybrid contractions in cone metric spaces*, Journal of Nonlinear and Convex Analysis, 17(2016), 899-906.

[23] H. Liu and S.-Y. Xu, *Cone metric spaces with Banach algebras and fixed point theorems of generalized Lipschitz mappings*, Fixed Point Theory Appl., 2013, 2013:320.

[24] H. Liu and S.-Y. Xu, *Fixed point theorems of quasi-contractions on cone metric spaces with Banach algebras*, Abstarct and Applied Analysis, Volume 2013, Article ID 187348, 5 pages.

[25] Z. Mustafa, B. Sims, *Some Remarks concerning D- metric spaces*, Proc. Int. Conf. on Fixed Point Theory and Appl., Valencia (Spain), July 2003,189 - 198.

[26] Z. Mustafa. B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear and Convex Analysis., 2006,7(2): 289 - 297.

[27] W. Onsod , T. Saleewong, J. Ahmad, A. E. Al-Mazrooei and P. Kumam, *Fixed points of a Θ-contraction on metric spaces with a graph*, Commun. Nonlinear Anal. **2** (2016), 139–14

[28] N. Tahat, H. Aydi, E. Karapinar, W. Shatanawi, *Common fixed points for single-valued and multi-valued maps satisfying a generalized contraction in G-metric spaces,* Fixed Point Theory and Appl.,2012, 2012:48.

[29] SB. Nadler, *Multi-valued contraction mappings*, Pacific J Math. 30, 475–478 (1969).

[30] Sh. Rezapour, R. Hamlbarani, *Some notes on the paper "Cone metric spaces and fixed point theorems of contractive mappings"*. J Math Anal Appl. 345, 719–724 (2008).

[31] W. Shatanawi, *Some coincidence point results in cone metric spaces.* Math Comput Model (2011).

[32] W. Shatanawi, *On w-compatible mappings and common coupled coincidence point in cone metric spaces*, Appl Math, Lett (2011).

[33] S. Radenović, B.E. Rhoades, *Fixed point theorem for two non-self mappings in cone metric spaces*, Comput. Math. Appl., 57, 1701-1707 (2009).

[34] B. Samet, C. Vetro, P. Vetro, *Fixed point theorem for $\alpha-\psi$ contractive type mappings*, Nonlinear Anal. 75 (2012) 2154–2165.

[35] S. Xu, S. Radenović, *Fixed point theorems of generalized Lipschitz mappings on cone metric spaces over Banach algebras without assumption of normality*, Fixed Point Theory Appl., 2014, 2014:102.

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