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EQUIVALENT ASYMPTOTIC FORMULAS FOR R-STIRLING NUMBERS OF THE FIRST KIND

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Abstract. Two asymptotic formulas for the r-Stirling numbers of the first kind obtained using different methods will be shown to be asymptotically equivalent valid within certain range of a parameter.

1. Introduction

The r-Stirling numbers of the first kind count the number of permutations of the set $\{1, 2, \ldots, n\}$ with m cycles such that the first r elements are in distinct cycles. These numbers were first introduced by Andrei Broder [2]. The r-Stirling numbers of the second kind were also studied in [2] but focus here will be on the first kind. This study is motivated by the work of Chelluri, Richmond and Temme [5].

Andrei Broder denoted the r-Stirling numbers of the first kind by $\begin{bmatrix} n \\ m \end{bmatrix}_r$. Since $\begin{bmatrix} n \\ m \end{bmatrix}_r = 0$ for $m < r$, this study considers the r-Stirling numbers of the first kind $\begin{bmatrix} n+r \\ m+r \end{bmatrix}_r$, where n, m, r are positive integers. These numbers satisfy the relation

$$z(z+1)(z+2)\ldots(z+n-1) = \sum_{m=0}^{n} \begin{bmatrix} n+r \\ m+r \end{bmatrix}_r (z-r)^m. \tag{1.1}$$

The generalized Stirling numbers of the first kind as generalized by Hsu and Shuie [7] denoted by $S_{n,m}^{\alpha,\gamma}$ satisfy the relation

$$z(z-\alpha)(z-2\alpha)\ldots(z-(n-1)\alpha) = \sum_{m=0}^{n} S_{n,m}^{\alpha,\gamma}(z-\gamma)^m, \tag{1.2}$$

where α, γ are complex numbers. Taking $\alpha = -1$ and $\gamma = r$, (1.2) becomes

$$z(z+1)(z+2)\ldots(z+(n-1)) = \sum_{m=0}^{n} S_{n,m}^{-1,\gamma}(z-r)^m,$$

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which is exactly (1.1). Thus,

$$\begin{bmatrix} n+r \\ m+r \end{bmatrix}_r = S_{n,m}^{-1,r}. \tag{1.3}$$

In this paper two asymptotic formulas for the r-Stirling numbers of the first kind obtained using different methods will be discussed and will be shown to be asymptotically equivalent within certain range of the parameter m.

2. Asymptotic Formulas for r-Stirling Numbers

Let C be any closed contour enclosing r. Applying the Cauchy-Integral Formula to (1) gives

$$\begin{bmatrix} n+r \\ m+r \end{bmatrix}_r = \frac{1}{2\pi i}\int_C \frac{z(z+1)(z+2)\dots(z+n-1)}{(z-r)^{m+1}}dz. \tag{2.1}$$

A modified saddle point method used in [12] was applied to the integral above to obtain the following asymptotic approximation:

Theorem 2.1. [3] *For positive integers m, n and r, the asymptotic formula holds,*

$$\begin{bmatrix} n+r \\ m+r \end{bmatrix}_r \sim e^B g(s_0)\frac{(n-1)_m r^{n-m-1}}{m!}, \tag{2.2}$$

as $n \to \infty$ valid uniformly in the range $0 < m < n$, where

$$s_0 = \frac{nr}{n-m}, \tag{2.3}$$

$$B = \phi(z_0) - n\log s_0 + m\log(s_0 - r), \tag{2.4}$$

and

$$g(s_0) = \frac{1}{(z_0 - r)}\sqrt{\frac{s_0(s_0-r)(n-m)}{\phi''(z_0)}}. \tag{2.5}$$

The number z_0 is the unique positive solution to the equation $\phi'(z) = 0$, the function $\phi(z)$ is

$$\phi(z) = \log[z(z+1)(z+2)\dots(z+n-1)] - m\log(z-r), \tag{2.6}$$

and $(n-1)_m = (n-1)(n-2)\dots(n-1-m+1)$.

Remark. The number z_0 may be computed using mathematica.

Using the method in [9], Vega and Corcino [13] obtained an asymptotic formula for the generalized Stirling numbers of the first kind which is given by

$$S_{n,m}^{\alpha,\gamma} \sim \frac{(-\alpha)^{n-m}\Gamma(R-\nu+n)}{(2\pi H)^{1/2}R^m\Gamma(R-\nu)} \quad \left\{1 + \frac{3C_4}{H^2} - \frac{15C_3^2}{2H^3}\right\}, \tag{2.7}$$

as $n \to \infty$ valid for m in the range $h(n) < m < n - O(n^\delta)$, where $h(n)$ is a function such that $\lim_{n\to\infty} h(n) = \infty$ and $0 < \delta < 1$, $\Gamma(x)$ is the gamma function, $\nu = \frac{\gamma}{\alpha} < 1$. In this paper, $h(n) = \log n$ and $\delta = 1/2$. The H that appears in (2.7) is

$$H = \sum_{h=1}^{n-1}\frac{(h-\nu)R}{(R+h-\nu)^2}, \tag{2.8}$$

and R is the unique positive solution to the equation

$$\sum_{h=1}^{n-1} \frac{R}{R+h-\nu} = m-1. \tag{2.9}$$

The constants C_3 and C_4 are given by

$$C_3 = \frac{1}{6}\left[3H - 2(m-1) + 2\sum_{h=1}^{n-1} \frac{R^3}{(R+h-\nu)^3}\right] \tag{2.10}$$

and

$$C_4 = \frac{1}{24}\left[36C_3 - 11H + 6(m-1) - 6\sum_{h=1}^{n-1} \frac{R^4}{(R+h-\nu)^4}\right]. \tag{2.11}$$

With a little modification in the computations in [13], the same formula as (2.7) is obtained when

$$H = \sum_{h=0}^{n-1} \frac{(h-\nu)R}{(R+h-\nu)^2}, \tag{2.12}$$

and R is the unique positive solution to the equation

$$\sum_{h=0}^{n-1} \frac{R}{R+h-\nu} = m. \tag{2.13}$$

Since $\left[\begin{array}{c} n+r \\ m+r \end{array}\right]_r = S_{n,m}^{-1,r}$ [see (1.2)], taking $\alpha = -1, \gamma = r$ in (2.7), the following asymptotic formula for the r-Stirling numbers of the first kind is obtained:

Theorem 2.2. *For positive integers m, n, r and as $n \to \infty$, the following asymptotic formula for the r-Stirling numbers of the first kind holds:*

$$\left[\begin{array}{c} n+r \\ m+r \end{array}\right]_r = \frac{\Gamma(R+r+n)}{(2\pi H)^{1/2} R^m \Gamma(R+r)} \left\{1 + \frac{3C_4}{H^2} - \frac{15C_3^2}{2H^3}\right\}, \tag{2.14}$$

valid for m in the range $\log n < m < n - O(n^{1/2})$, where R is the unique positive solution to the equation

$$\sum_{h=0}^{n-1} \frac{R}{R+h+r} = m. \tag{2.15}$$

and

$$H = \sum_{h=0}^{n-1} \frac{(h+r)R}{(R+h+r)^2}. \tag{2.16}$$

The corresponding constants C_3 and C_4 are as follows,

$$C_3 = \frac{1}{6}\sum_{h=0}^{n-1} \frac{R(h+r)(3R+h+r)}{(R+h+r)^3}, \tag{2.17}$$

$$C_4 = \frac{1}{24}\sum_{h=0}^{n-1} \frac{R(h+r)[-3R^2 + 4R(h+r) + (h+r)^2]}{(R+h+r)^4}. \tag{2.18}$$

3. Approximation for z_0

The goal in this section is to find the asymptotics of the unique positive solution z_0 of the equation $\phi'(z) = 0$, where $\phi(z)$ is given in (8).
By definition, z_0 is the solution of the algebraic equation

$$\frac{1}{z} + \frac{1}{z+1} + \cdots + \frac{1}{z+n-1} = \frac{m}{z-r}. \tag{3.1}$$

We are going to prove the following theorem.

Theorem 3.1. *If m is a fixed positive integer, then, as $n \to \infty$*

$$z_0 \sim n - m + \frac{1}{2} - \frac{4m^2 - 4m + 1}{8n} + O\left(\frac{1}{n^2}\right).$$

Proof. First note that (3.1) can be written by using the Digamma function $\psi = (\log \Gamma)'$ as

$$\psi(z+n) - \psi(z) = \frac{m}{z-r}.$$

Now making use of the fact that

$$\psi(z) \sim (\log z) - \frac{1}{2z} + O\left(\frac{1}{z^2}\right),$$

it can be seen that z must tend to infinity as n tends to infinity, at least when m is fixed. That what happens when m grows together with n will be discussed in the Remark after the proof.
We can leave the $O\left(\frac{1}{n^2}\right)$ term in the approximation of the Digamma function and get that, asymptotically, (3.1) is equivalent to

$$\log(z+n) - \frac{1}{2(z+n)} - \log z + \frac{1}{2z} = \frac{m}{z-r}.$$

This is equivalent to

$$\log\left(1 + \frac{n}{z}\right) + \frac{1}{2}\frac{n}{nz + z^2} = \frac{m}{z-r}.$$

As n tends to infinity $\log\left(1 + \frac{n}{z}\right) \sim \log n - \log z$, and $\frac{n}{nz+z^2} \sim \frac{1}{z}$. Also, as $z \to \infty$, $\frac{m}{z-r} \sim \frac{m}{z}$. In this step lose some weak r dependence, but get an equation exactly solvable. At this point the equation is asymptotically equivalent to

$$\log z + \frac{1}{z}\left(m - \frac{1}{2}\right) = \log n.$$

This equation can be solved in terms of the Lambert W function [6]:

$$z_0 \sim \frac{1-2m}{2W\left(\frac{1-2m}{2n}\right)}. \tag{3.2}$$

If m is fixed then the Lambert function asymptotics [6] around $x = 0$,

$$W(x) \sim x - x^2 + O(x^3) \tag{3.3}$$

yields that

$$z_0 \sim \frac{1-2m}{2W\left(\frac{1-2m}{2n}\right)} \sim n - m + \frac{1}{2} - \frac{4m^2 - am + 1}{8n} + O\left(\frac{1}{n^2}\right),$$

as stated in the theorem. $\square$

Remark 3.2. *It is interesting to see what happens when m is not fixed, but grows together with n. If we still want z_0 to tend to infinity,* (3.2) *can be used. The expression* $\frac{1-2m}{2W\left(\frac{1-2m}{2n}\right)}$ *is a positive real number for all $m > 0$ such that $m \leq \frac{n}{e} + \frac{1}{2}$ (this fact comes from the shape of the Lambert function). If $m = O(n^{\delta})$ with $\delta < 1$, then $z_0 \to \infty$. Indeed, with such an m the argument of the Lambert function tends to* 0 *and* (3.3) *can be applied. Hence, keeping only one term in* (3.3)*, we have that*

$$z_0 \sim \frac{1-2m}{2\left(\frac{1-2m}{2n}\right)} = n.$$

4. Equivalence of the formulas

First we compare the quantities z_0 and R. The following Lemma gives the connection formula between z_0 and R.

Lemma 4.1.

$$z_0 = R + r.$$

Proof. Note that

$$\phi'(z) = \frac{1}{z} + \frac{1}{z+1} + \frac{1}{z+2} + \ldots + \frac{1}{z+n-1} - \frac{m}{z-r},$$

and

$$\phi'(z_0) = 0.$$

Thus,

$$\sum_{h=0}^{n-1} \frac{z_0 - r}{z_0 + h} = m. \tag{4.1}$$

On the other hand, let

$$P(R, n, r) = \sum_{h=0}^{n-1} \frac{R}{R+h+r}. \tag{4.2}$$

By (15), $P(R, n) = m$. Let $w = R + r$, then

$$P(R, n, r) = \sum_{h=0}^{n-1} \frac{w-r}{w+h} = m. \tag{4.3}$$

Comparing (24) and (26) and using the fact that z_0 is unique, we conclude that

$$z_0 = w = R + r. \tag{4.4}$$

Lemma 4.2.

$$\frac{1}{H} = O\left(\frac{1}{m}\right).$$

Proof. From (17)

$$H = \sum_{h=o}^{n-1} \frac{(h+r)R}{(R+h+r)^2},$$

and

$$P(R,n,r) = \sum_{h=0}^{n-1} \frac{R}{R+h+r} = m.$$

By partial fractions, H can be written

$$\begin{aligned} H &= \sum_{h=0}^{n-1} \left(\frac{R}{R+h+r} - \frac{R^2}{(R+h+r)^2} \right), \\ &= \sum_{h=0}^{n-1} \frac{R}{R+h+r} - \sum_{h=0}^{n-1} \frac{R^2}{(R+h+r)^2} \\ &= m - \sum_{h=0}^{n-1} \frac{R^2}{(R+h+r)^2} \\ &= m - R^2 \sum_{h=0}^{n-1} \frac{1}{(R+h+r)^2}. \end{aligned}$$

Note that by Integral Test, $\sum_{h=0}^{n-1} \frac{1}{(R+h+r)^2}$ is convergent.
Let $\mu = \sum_{h=0}^{n-1} \frac{1}{(R+h+r)^2}$.
Then

$$\begin{aligned} H &= m - R^2\mu \\ \frac{1}{H} &= \frac{1}{m - R^2\mu} \end{aligned}$$

We will show that $\frac{1}{m-R^2\mu} = O\left(\frac{1}{m}\right)$.

$$\begin{aligned} \frac{\frac{1}{m-R^2\mu}}{\frac{1}{m}} &= \frac{\frac{1}{m\left(1-\frac{R^2\mu}{m}\right)}}{\frac{1}{m}} \\ &= \frac{1}{1-\frac{R^2\mu}{m}} \end{aligned}$$

From Theorem 5.1,

$$\begin{aligned} \frac{R}{m} &\sim \frac{-r+n-m+\frac{1}{2} - \frac{4m^2}{8n} + \frac{4m}{8n} - \frac{1}{8n} + O\left(\frac{1}{n^2}\right)}{m} \\ &\sim \frac{-r}{m} + \frac{n-m}{m} + \frac{1}{2m} - \frac{4m}{8n} + \frac{4}{8n} - \frac{1}{8nm} + O\left(\frac{1}{n^2 m}\right) \\ &\sim -\frac{4}{8} + O\left(\frac{1}{m}\right) \end{aligned}$$

So,

$$\begin{aligned}\frac{R^2\mu}{m} &= \frac{R}{m}\cdot R\mu\\ &\sim \left[-\frac{4}{8}+O\left(\frac{1}{m}\right)\right]R\mu\end{aligned}$$

Then

$$\begin{aligned}\frac{1}{1-\frac{R^2\mu}{m}} &\sim \frac{1}{1+\left[-\frac{4}{8}+O\left(\frac{1}{m}\right)\right]R\mu}\\ &= \frac{1}{1+\frac{1}{2}R\mu+R\mu O\left(\frac{1}{m}\right)} < 1\end{aligned}$$

Thus,

$$\frac{1}{H} = \frac{1}{m-R^2\mu} = O\left(\frac{1}{m}\right)$$

□

Theorem 4.3. *The formula in (22) can be written in the form*

$$\left[\begin{array}{c} n+r \\ m+r \end{array}\right]_r \sim \frac{\Gamma(R+r+n)}{(2\pi H)^{1/2}R^m\Gamma(R+r)} \qquad \left\{1+O\left(\frac{1}{m}\right)\right\}. \tag{4.5}$$

Proof. This follows from Lemma 4.2 and the fact that for each $k \geq 2, |c_k| \leq H$ (This is (3.6) in [13]). □

The following lemma gives the connection formula between $\phi''(z_0)$ and H.

Lemma 4.4.

$$\phi''(z_0) = \frac{H}{(z_0-r)^2}. \tag{4.6}$$

Proof. Recall

$$\phi''(z) = \frac{m}{(z-r)^2} - \frac{1}{z^2} - \frac{1}{(z+1)^2} - \frac{1}{(z+2)^2} - \dots - \frac{1}{(z+n-1)^2},$$

which can be written

$$(z-r)^2\phi''(z) = m - \sum_{h=0}^{n-1}\frac{(z-r)^2}{(z+h)^2}. \tag{4.7}$$

At $z = z_0$,

$$(z_0-r)^2\phi''(z_0) = m - \sum_{h=0}^{n-1}\frac{(z_0-r)^2}{(z_0+h)^2} = m - \sum_{h=0}^{n-1}\frac{R^2}{(R+r+h)^2}. \tag{4.8}$$

It remains to show that

$$m - \sum_{h=0}^{n-1}\frac{R^2}{(R+r+h)^2} = H. \tag{4.9}$$

Note that

$$m = P(R,n) = \sum_{h=0}^{n-1}\frac{R}{R+h+r}.$$

Thus,

$$\begin{aligned} m - \sum_{h=0}^{n-1} \frac{R^2}{(R+r+h)^2} &= \sum_{h=0}^{n-1} \frac{R}{R+h+r} - \sum_{h=0}^{n-1} \frac{R^2}{(R+r+h)^2} \\ &= \sum_{h=0}^{n-1} \frac{(r+h)R}{(R+r+h)^2} = H. \end{aligned}$$

□

Lemma 4.5.

$$e^B g(s_0) \frac{(n-1)_m r^{n-m-1}}{m!} = \frac{\Gamma(R+r+n)}{\Gamma(R+r)(R)^m \sqrt{H}} D, \tag{4.10}$$

where

$$D = \frac{m^m}{n^n} (n-m)^{n-m} \frac{(n-1)_m}{m!} \sqrt{\frac{nm}{n-m}}. \tag{4.11}$$

Proof. It follows from (4.8) and Lemma 4.4 that

$$\phi''(z_0) = \frac{H}{(z_0 - r)^2}. \tag{4.12}$$

Thus, from (2.5) we have,

$$\begin{aligned} g(s_0) &= \frac{1}{z_0 - r} \sqrt{\frac{s_0(s_0-r)(n-m)}{\phi''(z_0)}} \\ &= \frac{1}{z_0 - r} \sqrt{\frac{s_0(s_0-r)(n-m)}{H/(z_0-r)^2}} \\ &= \frac{1}{\sqrt{H}} \sqrt{s_0(s_0-r)(n-m)} \\ &= \frac{r}{\sqrt{H}} \sqrt{\frac{nm}{n-m}}. \end{aligned}$$

Note that $z_0 = R + r > r$.
We turn to the factor e^B, where

$$B = \phi(z_0) - n \log s_0 + m \log(s_0 - r).$$

Then,

$$e^B = e^{\phi(z_0)} \frac{(s_0 - r)^m}{s_0^n}.$$

With $\phi(z)$ given in (2.6) we have

$$e^{\phi(z_0)} = \frac{\Gamma(z_0 + n)}{\Gamma(z_0)(z_0 - r)^m} \tag{4.13}$$

and

$$e^B = \frac{\Gamma(z_0 + n)}{\Gamma(z_0)(z_0 - r)^m} \frac{m^m}{n^n} \left(\frac{n-m}{r} \right)^{n-m}. \tag{4.14}$$

Thus,

$$\begin{aligned}
e^B g(s_0)\frac{(n-1)_m r^{n-m-1}}{m!} &= \frac{\Gamma(z_0+n)}{\Gamma(z_0)(z_0-r)^m}\frac{m^m}{n^n}\left(\frac{n-m}{r}\right)^{n-m} \\
&\quad \times \frac{r}{\sqrt{H}}\sqrt{\frac{nm}{n-m}}\frac{(n-1)_m r^{n-m-1}}{m!} \\
&= \frac{\Gamma(R+r+n)}{\Gamma(R+r)R^m}\frac{m^m}{n^n}\frac{(n-m)^{n-m}}{r^{n-m}} \\
&\quad \times \frac{r}{\sqrt{H}}\sqrt{\frac{nm}{n-m}}(n-1)_m\frac{r^{n-m}r^{-1}}{m!} \\
&= \frac{\Gamma(R+r+n)}{\Gamma(R+r)R^m\sqrt{H}}\frac{m^m}{n^n}(n-m)^{n-m}\frac{(n-1)_m}{m!}\sqrt{\frac{nm}{n-m}} \\
&= \frac{\Gamma(R+r+n)}{\Gamma(R+r)R^m\sqrt{H}}D
\end{aligned}$$

□

Lemma 4.6. *Let*

$$D = \frac{m^m}{n^n}(n-m)^{n-m}\frac{(nm)^{1/2}}{(n-m)^{1/2}}\frac{(n-1)_m}{m!}. \tag{4.15}$$

Then

$$D = \frac{1}{\sqrt{2\pi}}\left[1+O(1/m)\right], \tag{4.16}$$

as $n\to\infty$ such that $n-m=O(n^{1/2})$.

Proof.

$$\begin{aligned}
D &= \frac{m^m}{n^n}(n-m)^{n-m}\frac{(nm)^{1/2}}{(n-m)^{1/2}}\frac{(n-1)_m}{m!} \\
&= \frac{m^m}{n^n}(n-m)^{n-m}\frac{(nm)^{1/2}}{(n-m)^{1/2}}\frac{(n-1)!}{(n-m-1)!m!} \\
&= \frac{(n-m)^{n-m}}{n^n}\frac{1}{\sqrt{2\pi}e^{-m+\theta_1/12m}}\frac{n^{1/2}}{(n-m)^{1/2}}\frac{(n-1)!}{(n-m-1)!} \\
&= \frac{1}{(1-\frac{1}{n-m})^{n-m-1/2}}\frac{1}{\sqrt{2\pi}}exp\left[\frac{-\theta_1}{12m}+\frac{\theta_2}{12(n-1)}-\frac{\theta_3}{n-m-1}\right] \\
&\quad \times\left(1-\frac{1}{n}\right)^{n-1/2},
\end{aligned}$$

where $0<\theta_i<1$, $i=1,2,3$. The last equality in the array above follows from Stirling's formula for $n!$ (see [1]). Thus,

$$
\begin{aligned}
D &= \frac{\left(1-\frac{1}{n}\right)^n}{\left(1-\frac{1}{n-m}\right)^{n-m}}\left(\frac{1-\frac{1}{n}}{1-\frac{1}{n-m}}\right)^{1/2} \\
&\qquad \times \frac{1}{\sqrt{2\pi}} exp\left[\frac{-\theta_1}{12m}+\frac{\theta_2}{12(n-1)}-\frac{\theta_3}{n-m-1}\right] \\
&= \frac{1}{\sqrt{2\pi}}\left[1+O(1/m)\right].
\end{aligned}
$$

□

The following theorem follows from Lemma 4.1, Lemma 4.5 and Lemma 4.6.

Theorem 4.7. *Let r, m, and n be positive integers. Then*

$$
\begin{bmatrix} n+r \\ m+r \end{bmatrix}_r = e^B g(s_0)\frac{(n-1)_m r^{n-m-1}}{m!} = \frac{\Gamma(R+r+n)}{\Gamma(R+r)R^m\sqrt{2\pi H}}\left[1+O(1/m)\right], \quad (4.17)
$$

as $n \to \infty$ such that $m = n - O(n^{1/2})$, where s_0 is defined in (2.3) *and R is the unique solution to* (2.15)*.*

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