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HERMITE-HADAMARD TYPE INTEGRAL INEQUALITIES FOR GENERALIZED CONVEX FUNCTIONS

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Abstract. This paper deals with some results about (η_1, η_2)-convex functions as a generalization of convex functions. Hermite-Hadamard, trapezoid and mid-point type inequalities are obtained. Also the classes of $(\eta_1, \eta_2)_b$-convex and $(\eta_1, \eta_2)_E$-convex functions are introduced and their relations with (η_1, η_2)-convex functions are investigated.

1. Introduction and Preliminaries

Preinvex function as a generalization of a convex function was considered by Ben-Israel and Mond in [5] and Hanson and Mond in [11], but so named by Jeyakumar [12]. Later the concept of invex sets (see [14]) was considered as the domain of a preinvex function.

Definition 1. [5, 11] A set $I \subseteq \mathbb{R}$ is *invex with respect to a real bifunction* $\eta : I \times I \to \mathbb{R}$, if

$$x, y \in I, \lambda \in [0,1] \Rightarrow y + \lambda\eta(x,y) \in I.$$

If I is an invex set with respect to η, then a function $f : I \to \mathbb{R}$ is said to be *preinvex* if $x, y \in I$ and $\lambda \in [0,1]$ implies

$$f\big(y + \lambda\eta(x,y)\big) \le \lambda f(x) + (1-\lambda)f(y).$$

In fact in an invex set for any $x, y \in I$, there is a path starting from y to $y+\eta(x,y)$ which is contained in I. The point x is not necessarily the end point of the path. If for every $x, y \in I$ we need that x should be an end point of the path, then I reduces to a convex set.

Also the concept of η-convex functions (at the beginning was named by φ-convex functions), considered in [8, 9, 16, 17], has been introduced as the following.

Definition 2. *Consider a convex set $I \subseteq \mathbb{R}$ and a bifunction $\eta : f(I) \times f(I) \to \mathbb{R}$. A function $f : I \to \mathbb{R}$ is called convex with respect to η (briefly η-convex), if*

$$f\big(\lambda x + (1-\lambda)y\big) \le f(y) + \lambda\eta\big(f(x), f(y)\big),$$

for all $x, y \in I$ and $\lambda \in [0,1]$.

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Geometrically it says that if a function is η-convex on I, then for any $x, y \in I$, its graph is on or under the path starting from $\big(y, f(y)\big)$ and ending at $\Big(x, f(y) + \eta(f(x), f(y))\Big)$. If $f(x)$ should be the end point of the path for every $x, y \in I$, then we have $\eta(x, y) = x - y$ and the function reduces to a convex one.

Motivated by above works and references therein we use the concept of (η_1, η_2)-convex functions as a generalization of preinvex and η-convex functions to obtain some basic results and Hermite-Hadamard type inequalities. We give trapezoid and mid-point type inequalities when the absolute value of the derivatives of a function is (η_1, η_2)-convex. Also we introduce the classes of $(\eta_1, \eta_2)_b$-convex functions and $(\eta_1, \eta_2)_E$-convex functions as a generalization of B-vex functions [4] and E-convex functions [21] respectively and investigate about relations between (η_1, η_2)-convex $\big((\eta_1, \eta_2)$-quasiconvex$\big)$ functions and these classes of functions.

2. Generalized Convex Functions

In this section the concept of (η_1, η_2)-convex functions is used to obtain some basic results.

Definition 3. [15] Let $I \subseteq \mathbb{R}$ be an invex set with respect to $\eta_1 : I \times I \to \mathbb{R}$. Consider $f : I \to \mathbb{R}$ and $\eta_2 : f(I) \times f(I) \to \mathbb{R}$. The function f is said to be (η_1, η_2)*-convex* if

$$f\big(x + \lambda\eta_1(y, x)\big) \le f(x) + \lambda\eta_2\big(f(y), f(x)\big),$$

for all $x, y \in I$ and $\lambda \in [0, 1]$. Also f is said to be (η_1, η_2)*-quasiconvex* if

$$f\big(x + \lambda\eta_1(y, x)\big) \le \max\big\{f(x), f(x) + \eta_2\big(f(y), f(x)\big)\big\},$$

for all $x, y \in I$ and $\lambda \in [0, 1]$.

Remark. *An* (η_1, η_2)*-convex function reduces to*

(i) an η-convex function if we consider $\eta_1(x, y) = x - y$ for all $x, y \in I$.

(ii) a preinvex function if we consider $\eta_2(x, y) = x - y$ for all $x, y \in f(I)$.

(iii) a convex function if satisfies (i) and (ii).

We can find an (η_1, η_2)-convex function which is not convex.

Example 2.1. [15] Consider the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$f(x) = \begin{cases} x, & 0 \le x \le 1; \\ 1, & x > 1. \end{cases}$$

Define two bifunction $\eta_1 : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ and $\eta_2 : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ by

$$\eta_1(x, y) = \begin{cases} -y, & 0 \le y \le 1; \\ x + y, & y > 1, \end{cases}$$

and

$$\eta_2(x, y) = \begin{cases} x + y, & x \le y; \\ 2(x + y), & x > y. \end{cases}$$

Then f is an (η_1, η_2)-convex function. But f is not preinvex with respect to η_1 and it is not convex (consider $x = 0$, $y = 2$ and $\lambda > 0$).

The composition of an (η_1,η_2)-convex function with an (η_2,η_3)-convex function, under the certain conditions, is an (η_1,η_3)-convex function.

Theorem 2.2. *Consider $I, J, K \subseteq \mathbb{R}$ such that I is an invex set with respect to η_1 and J is an invex set with respect to η_2. Suppose that $f : I \to J$ is an (η_1,η_2)-convex function and $g : J \to K$ is a nondecreasing (η_2,η_3)-convex function. Then $g \circ f : I \to K$ is an (η_1,η_3)-convex function.*

Proof. Consider $x, y \in I$ and $\lambda \in [0,1]$. Since g is a nondecreasing (η_2,η_3)-convex function on J, so from inequality

$$f\big(y+\lambda\eta_1(x,y)\big) \le f(y) + \lambda\eta_2\big(f(x), f(y)\big)$$

we have

$$\begin{aligned} g\Big(f\big(y+\lambda\eta_1(x,y)\big)\Big) &\le g\Big(f\big(y\big)+\lambda\eta_2(f(x),f(y))\Big) \\ &\le g\big(f(y)\big) + \lambda\eta_3\big(g(f(x)), g(f(y))\big), \end{aligned}$$

which means that gof is an (η_1,η_3)-convex function. □

Definition 4. *The bifunction η_2 in Definition 3 is said to be*

(i) *nonnegatively homogeneous* if $\eta_2(\lambda x, \lambda y) = \lambda\eta_2(x,y)$ for all $x, y \in \mathbb{R}$ and all $\lambda \ge 0$.

(ii) *additive* if $\eta_2(x_1,y_1)+\eta_2(x_2,y_2) = \eta_2(x_1+x_2, y_1+y_2)$ for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

(iii) *nonnegatively linear* if satisfies conditions (i) and (ii).

Proposition 2.3. *From above definition if $I \subseteq \mathbb{R}$ is an invex set with respect to η_1 and if*

(1) $f : I \to \mathbb{R}$ is an (η_1,η_2)-convex function such that η_2 is nonnegatively homogeneous, then for any $\lambda \ge 0$ the function $\lambda f : I \to \mathbb{R}$ is (η_1,η_2)-convex.

(2) $f, g : I \to \mathbb{R}$ are two (η_1,η_2)-convex functions such that η_2 is additive, then $f+g : I \to \mathbb{R}$ is (η_1,η_2)-convex.

(3) $f_i : I \to \mathbb{R}$ $(i = 1, ...n)$ are (η_1,η_2)-convex functions such that η_2 is nonnegatively linear, then for $\lambda_i \ge 0$ the function $f = \sum_{i=1}^{n} \lambda_i f_i : I \to \mathbb{R}$ is (η_1,η_2)-convex.

The class of (η_1,η_2)-convex functions is closed under "Sup" operation under special condition. Also this fact is true for the class of (η_1,η_2)-quasiconvex functions.

Theorem 2.4. *Suppose that $I \subseteq \mathbb{R}$ and $\{f_j : I \to \mathbb{R}, j \in J\}$ is a nonempty collection of (η_1,η_2)-convex* $\big((\eta_1,\eta_2)$-quasiconvex$\big)$ functions such that

(a) there exist $\alpha \in [0,\infty]$ and $\beta \in [-1,\infty]$ such that $\eta_2(x,y) = \alpha x + \beta y$ for all $x, y \in \mathbb{R}$,

(b) for each $x \in I$, $\sup_{j\in J} f_j(x)$ exists in $\mathbb{R}$.

Then the function $f : I \to \mathbb{R}$ defined by

$$f(x) = \sup_{j\in J} f_j(x) \;\; for\ each\ x \in I,$$

is an (η_1,η_2)-convex $\big((\eta_1,\eta_2)$-quasiconvex$\big)$ function.

Proof. For any $x, y \in I$ and $\lambda \in [0,1]$, we can derive following relations

$$\begin{aligned}
&f\big(y+\lambda\eta_1(x,y)\big) = \sup_{j\in J} f_j\big(y+\lambda\eta_1(x,y)\big) \le \sup_{j\in J}\big\{f_j(y)+\lambda\eta_2\big(f_j(x),f_j(y)\big)\big\}\\
&= \sup_{j\in J}\big\{f_j(y)+\lambda\big(\alpha f_j(x)+\beta f_j(y)\big)\big\} = \sup_{j\in J}\big\{f_j(y)(1+\beta\lambda)+\alpha\lambda(f_j(x))\big\}\\
&\le \sup_{j\in J}(1+\beta\lambda)f_j(y)+\alpha\lambda\sup_{j\in J} f_j(x) = (1+\beta\lambda)f(y)+\alpha\lambda f(x)\\
&= f(y)+\lambda\big(\alpha f(x)+\beta f(y)\big) = f(y)+\lambda\eta_2\big(f(x),f(y)\big).
\end{aligned}$$

In the case that f is $\big((\eta_1,\eta_2)$-quasiconvex$\big)$ function, the proof is similar. □

If we set $\alpha = 1$ and $\beta = -1$ in Theorem 2.4, we obtain the following.

Corollary 2.5. *Suppose that $I \subseteq \mathbb{R}$ and $\{f_j : I \to \mathbb{R}, j \in J\}$ is a nonempty collection of preinvex functions such that for each $x \in I$, $\sup_{j\in J} f_j(x)$ exists in $\mathbb{R}$. Then the function $f : I \to \mathbb{R}$, defined by*

$$f(x) = \sup_{j\in J} f_j(x) \ \text{ for each } x \in I,$$

is a preinvex function.

3. Hermite-Hadamard Inequality

The Hermite-Hadamard inequality for preinvex functions established in [1] and for η-convex functions obtained in [16]. In this section we prove Hermite-Hadamard inequality related to (η_1,η_2)-convex functions.

Theorem 3.1. *Let $I \subseteq \mathbb{R}$ be an invex set with respect to η_1 such that*

$$\eta_1\Big(x_2+\lambda_2\eta_1(x_1,x_2), x_2+\lambda_1\eta_1(x_1,x_2)\Big) = \big(\lambda_2-\lambda_1\big)\eta_1(x_1,x_2), \tag{3.1}$$

for all $x_1, x_2 \in I$ and $\lambda_1, \lambda_2 \in [0,1]$. Also suppose that $f : I \to \mathbb{R}$ is an (η_1,η_2)-convex function. For any $a, b \in I$ with $\eta_1(b,a) > 0$, if
$f \in L^1[a, a+\eta_1(b,a)]$ and η_2 is integrable on $f(I)\times f(I)$ then

$$\begin{aligned}
&f\left(\frac{2a+\eta_1(b,a)}{2}\right) - \frac{1}{2\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} \eta_2\Big(f(x), f(2a+\eta_1(b,a)-x)\Big)dx \qquad (3.2)\\
&\le \frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f(x)dx \le f(a)+\frac{\eta_2\big(f(b),f(a)\big)}{2}.
\end{aligned}$$

Proof. For the right side of inequality (3.2), consider arbitrary $x = a+\lambda\eta_1(b,a)$ with $\lambda \in [0,1]$. So we have $f(x) \le f(a)+\lambda\eta_2\big(f(b),f(a)\big)$ with $\lambda = \frac{x-a}{\eta_1(b,a)}$. Then

$$\begin{aligned}
&\frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f(x)dx\\
&\le \frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} \Big[f(a)+\frac{x-a}{\eta_1(b,a)}\eta_2\big(f(b),f(a)\big)\Big]dx\\
&= \frac{1}{\eta_1(b,a)}\left[f(a)\eta_1(b,a)+\frac{\eta_2\big(f(b),f(a)\big)}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)}(x-a)dx\right]\\
&= f(a)+\frac{\eta_2\big(f(b),f(a)\big)}{\eta_1^2(b,a)}\frac{\eta_1^2(b,a)}{2} = f(a)+\frac{\eta_2\big(f(b),f(a)\big)}{2}.
\end{aligned}$$

For the left side of (3.2) we need some explanation. From relation (3.1) we obtain

$$
\begin{aligned}
f\left(\frac{2a+\eta_1(b,a)}{2}\right) &= f\left(\frac{2a+\eta_1(b,a)+\lambda\eta_1(b,a)}{2} - \frac{\lambda}{2}\eta_1(b,a)\right) \\
&= f\left(\frac{2a+\eta_1(b,a)+\lambda\eta_1(b,a)}{2}\right. \\
&\left. + \frac{1}{2}\eta_1\left(a+\frac{(1-\lambda)}{2}\eta_1(b,a), a+\frac{(1+\lambda)}{2}\eta_1(b,a)\right)\right) \\
&= f\left(\frac{2a+\eta_1(b,a)+\lambda\eta_1(b,a)}{2}\right. \\
&\left. + \frac{1}{2}\eta_1\left(\frac{2a+\eta_1(b,a)-\lambda\eta_1(b,a)}{2}, \frac{2a+\eta_1(b,a)+\lambda\eta_1(b,a)}{2}\right)\right) \\
&\leq f\left(\frac{2a+\eta_1(b,a)+\lambda\eta_1(b,a)}{2}\right) \\
&+ \frac{1}{2}\eta_2\left(f\left(\frac{2a+\eta_1(b,a)-\lambda\eta_1(b,a)}{2}\right), f\left(\frac{2a+\eta_1(b,a)+\lambda\eta_1(b,a)}{2}\right)\right),
\end{aligned} \tag{3.3}
$$

and with the same argument as above

$$
\begin{aligned}
f\left(\frac{2a+\eta_1(b,a)}{2}\right) &\leq f\left(\frac{2a+\eta_1(b,a)-\lambda\eta_1(b,a)}{2}\right) \\
&+ \frac{1}{2}\eta_2\left(f\left(\frac{2a+\eta_1(b,a)+\lambda\eta_1(b,a)}{2}\right), f\left(\frac{2a+\eta_1(b,a)-\lambda\eta_1(b,a)}{2}\right)\right).
\end{aligned} \tag{3.4}
$$

Using relations (3.3), (3.4) and the change of variable

$$x = \frac{2a+\eta_1(b,a)-\lambda\eta_1(b,a)}{2}, \tag{3.5}$$

and

$$x = \frac{2a+\eta_1(b,a)+\lambda\eta_1(b,a)}{2}, \tag{3.6}$$

we deduce the following inequality.

$$
\begin{aligned}
&\frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f(x)dx \\
&= \frac{1}{\eta_1(b,a)}\left[\int_a^{a+\frac{\eta_1(b,a)}{2}} f(x)dx + \int_{a+\frac{\eta_1(b,a)}{2}}^{a+\eta_1(b,a)} f(x)dx\right] \\
&= \frac{1}{2}\left[\int_0^1 f\left(\frac{2a+\eta_1(b,a)-\lambda\eta_1(b,a)}{2}\right)d\lambda\right. \\
&\left. + \int_0^1 f\left(\frac{2a+\eta_1(b,a)+\lambda\eta_1(b,a)}{2}\right)d\lambda\right]
\end{aligned} \tag{3.7}
$$

$$\geq \frac{1}{2}\int_0^1 \left[f\left(\frac{2a+\eta_1(b,a)}{2}\right) - \frac{1}{2}\eta_2\left(f\left(\frac{2a+\eta_1(b,a)+\lambda\eta_1(b,a)}{2}\right),\right.\right.$$
$$\left.\left. f\left(\frac{2a+\eta_1(b,a)-\lambda\eta_1(b,a)}{2}\right)\right)\right]d\lambda + \frac{1}{2}\int_0^1 \left[f\left(\frac{2a+\eta_1(b,a)}{2}\right)\right.$$
$$\left. - \frac{1}{2}\eta_2\left(f\left(\frac{2a+\eta_1(b,a)-\lambda\eta_1(b,a)}{2}\right), f\left(\frac{2a+\eta_1(b,a)+\lambda\eta_1(b,a)}{2}\right)\right)\right]d\lambda.$$

Again using the change of variable (3.5) and (3.6) in relations obtained in (3.7) respectively we have

$$\frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f(x)dx$$
$$\geq f\left(\frac{2a+\eta_1(b,a)}{2}\right) - \frac{1}{2\eta_1(b,a)}\int_{a+\frac{\eta_1(b,a)}{2}}^{a+\eta_1(b,a)} \eta_2\Big(f(x), f\big(2a+\eta_1(b,a)-x\big)\Big)dx$$
$$- \frac{1}{2\eta_1(b,a)}\int_a^{a+\frac{\eta_1(b,a)}{2}} \eta_2\Big(f(x), f\big(2a+\eta_1(b,a)-x\big)\Big)dx$$
$$= f\left(\frac{2a+\eta_1(b,a)}{2}\right) - \frac{1}{2\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} \eta_2\Big(f(x), f\big(2a+\eta_1(b,a)-x\big)\Big)dx.$$

Thus we arrive at the desired result. □

Corollary 3.2 (Theorem 5 in [16]). *If in the proof of Theorem 3.1 we consider $\eta_1(x,y) = x - y$ for all $x, y \in I$, then the following inequalities hold.*

$$f\left(\frac{a+b}{2}\right) - \frac{1}{2(b-a)}\int_a^b \eta_2\Big(f(x), f(a+b-x)\Big)dx$$
$$\leq \frac{1}{b-a}\int_a^b f(x)dx \leq f(a) + \frac{\eta_2\big(f(b), f(a)\big)}{2}.$$

Corollary 3.3 (Theorem 3.1 in [1]). *If in Theorem 3.1 we set $\eta_2(x,y) = x - y$ for all $x, y \in f(I)$, then we recapture Hermite-Hadamard inequality related to preinvex functions.*

$$f\left(\frac{2a+\eta_1(b,a)}{2}\right) \leq \frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f(x)dx \leq \frac{f(a) + f\big(a+\eta_1(b,a)\big)}{2}. \quad (3.8)$$

Corollary 3.4. *If in Theorem 3.1 we consider $\eta_1(x,y) = x - y$ for all $x, y \in I$ and $\eta_2(x,y) = x - y$ for all $x, y \in f(I)$, then we have the classic Hermite-Hadamard inequality.*

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \quad (3.9)$$

4. Differentiable (η_1, η_2)-Convex Function

An interesting problem in (3.8) is estimating the difference between left and middle terms or between right and middle terms. In this section we investigate about first and second part of this problem when the absolute value of the first and then second derivatives of a function is (η_1, η_2)-convex. Also at the end of this section two inequalities for differentiable (η_1, η_2)-convex functions are obtained.

In the first part we estimate the difference between left and middle terms and between right and middle terms in (3.8) respectively, when the absolute value of the first derivative of a function is (η_1, η_2)-convex.

The following result is Lemma 3.10 in [2].

Lemma 4.1. *Suppose that (interior of I) $I^\circ \subseteq \mathbb{R}$ is an invex set with respect to η_1 and $f : I^\circ \longrightarrow \mathbb{R}$ is a differentiable mapping on I°. For any $a, b \in I^\circ$ with $\eta_1(b,a) > 0$ If $f' \in L^1[a, a + \eta_1(b,a)]$, then*

$$\begin{aligned}&\frac{1}{\eta_1(b,a)} f\left(\frac{2a + \eta_1(b,a)}{2}\right) - \frac{1}{\eta_1^2(b,a)} \int_a^{a+\eta_1(b,a)} f(x)dx \\ &= \left[\int_0^{1/2} t f'\big(a + t\eta_1(b,a)\big)dt + \int_{1/2}^1 (t-1) f'\big(a + t\eta_1(b,a)\big)dt\right].\end{aligned}$$

Using Lemma 4.1, we can prove the following theorem to estimate the difference between the middle and left terms in (3.8).

Theorem 4.2. *Suppose that $I^\circ \subseteq \mathbb{R}$ is an invex set with respect to η_1 and $f : I^\circ \longrightarrow \mathbb{R}$ is a differentiable mapping on I°. For any $a, b \in I^\circ$ with $\eta_1(b,a) > 0$ If $|f'| \in L^1[a, a + \eta_1(b,a)]$ is an (η_1, η_2)-convex mapping, then*

$$\begin{aligned}&\left|\frac{1}{\eta_1(b,a)} \int_a^{a+\eta_1(b,a)} f(x)dx - f\left(\frac{2a + \eta_1(b,a)}{2}\right)\right| \\ &\leq \frac{1}{8}\eta_1(b,a)\left\{2|f'(a)| + \eta_2\big(|f'(b)|, |f'(a)|\big)\right\}.\end{aligned}$$

Proof. Lemma 4.1 with (η_1, η_2)-convexity of $|f'|$ imply that

$$\begin{aligned}&\left|\frac{1}{\eta_1(b,a)} \int_a^{a+\eta_1(b,a)} f(x)dx - f\left(\frac{2a + \eta_1(b,a)}{2}\right)\right| \\ &\leq \eta_1(b,a)\left\{\int_0^{1/2} t\big|f'\big|\big(a + t\eta_1(b,a)\big)dt\right. \\ &\left. + \int_{1/2}^1 \big(1-t\big)\big|f'\big|\big(a + t\eta_1(b,a)\big)dt\right\} \\ &\leq \eta_1(b,a)\left\{\int_0^{1/2} t\Big(\big|f'(a)\big| + t\eta_2\big(|f'(b)|, |f'(a)|\big)\Big)dt\right. \\ &\left. + \int_{1/2}^1 \big(1-t\big)\Big(\big|f'(a)\big| + t\eta_2\big(|f'(b)|, |f'(a)|\big)\Big)dt\right\} \\ &= \frac{1}{8}\eta_1(b,a)\Big[2|f'(a)| + \eta_2\big(|f'(b)|, |f'(a)|\big)\Big],\end{aligned}$$

which gives the respected result. □

Remark. *With the assumptions of Theorem 4.2 if we consider*

(i) $\eta_1(x,y) = x-y$ for all $x,y \in I$, then we obtain Theorem 3 in [17].

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\Big(\frac{a+b}{2}\Big)\right| \le \frac{1}{8}(b-a)\Big\{2|f'(a)| + \eta_2\big(|f'(b)|,|f'(a)|\big)\Big\}.$$

(ii) $\eta_2(x,y) = x-y$ for all $x,y \in f(I)$, then we obtain Theorem 5 in [18].

$$\left|\frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f(x)dx - f\left(\frac{2a+\eta_1(b,a)}{2}\right)\right| \le \frac{1}{8}\eta_1(b,a)\Big\{|f'(a)| + |f'(b)|\Big\}.$$

(iii) condition (i) and (ii) together, then we recapture Theorem 2.2 in [13].

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\Big(\frac{a+b}{2}\Big)\right| \le \frac{1}{8}(b-a)\Big\{|f'(a)| + |f'(b)|\Big\}.$$

The following is Lemma 3.6 in [3].

Lemma 4.3. *Suppose that $I^\circ \subseteq \mathbb{R}$ is an invex set with respect to η_1 and $f : I^\circ \longrightarrow \mathbb{R}$ is a differentiable mapping on I°. For any $a,b \in I^\circ$ with $\eta_1(b,a) > 0$, If $f' \in L^1[a, a+\eta_1(b,a)]$ then*

$$\begin{aligned}&\frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f(x)dx - \frac{f(a) + f\big(a+\eta_1(b,a)\big)}{2}\\&= \frac{\eta_1(b,a)}{2}\int_0^1 (1-2t)f'\big(a+t\eta_1(b,a)\big)dt.\end{aligned}$$

Using Lemma 4.5, we can prove the following theorem to estimate the difference between the middle and right terms in (3.8).

Theorem 4.4. *Suppose that $I^\circ \subseteq \mathbb{R}$ is an invex set with respect to η_1 and $f : I^\circ \longrightarrow \mathbb{R}$ is a differentiable mapping on I°. For any $a,b \in I^\circ$ with $\eta_1(b,a) > 0$, If $|f'| \in L^1[a, a+\eta_1(b,a)]$ is an (η_1,η_2)-convex mapping then*

$$\begin{aligned}&\left|\frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f(x)dx - \frac{f(a) + f\big(a+\eta_1(b,a)\big)}{2}\right|\\&\le \frac{1}{8}\eta_1(b,a)\Big\{2|f'(a)| + \eta_2\big(|f'(b)|,|f'(a)|\big)\Big\}.\end{aligned}$$

Proof. Lemma 4.5 along with (η_1,η_2)-convexity of $|f'|$ imply that

$$\begin{aligned}&\left|\frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f(x)dx - \frac{f(a) + f\big(a+\eta_1(b,a)\big)}{2}\right|\\&= \frac{\eta_1(b,a)}{2}\left|\int_0^1 (1-2t)f'\big(a+t\eta_1(b,a)\big)dt\right|\\&\le \frac{\eta_1(b,a)}{2}\int_0^1 \big|(1-2t)\big|\big|f'\big(a+t\eta_1(b,a)\big)\big|dt\\&\le \frac{\eta_1(b,a)}{2}\int_0^1 \big|1-2t\big|\Big[|f'(a)| + t\eta_2\big(|f'(b)|,|f'(a)|\big)\Big]dt\\&= \frac{\eta_1(b,a)}{8}\Big[2|f'(a)| + \eta_2\big(|f'(b)|,|f'(a)|\big)\Big].\end{aligned}$$

This completes the proof. □

Remark. *Similar to the results obtained in Remark 4, If in the proof of Theorem 4.4 we consider*
(i) $\eta_1(x,y) = x - y$ for all $x, y \in I$, then we obtain Theorem 6 in [17].
(ii) $\eta_2(x,y) = x - y$ for all $x, y \in f(I)$, then we obtain Theorem 2.1 in [3].
(iii) condition (i) and (ii) together, then we recapture Theorem 2.2 in [7].

In this part we estimate the difference between left and middle terms and between right and middle terms in (3.8) respectively, when the absolute value of the second derivative of a function is (η_1, η_2)-convex.

Lemma 4.5. *Suppose that $I^\circ \subseteq \mathbb{R}$ is an invex set with respect to η_1 and $f : I^\circ \longrightarrow \mathbb{R}$ is a twice differentiable mapping on I°. For any $a, b \in I^\circ$ with $\eta_1(b,a) > 0$ If $f'' \in L^1[a, a+\eta_1(b,a)]$, then*

$$\int_a^{a+\eta_1(b,a)} f(t)dt - \eta_1(b,a) f\Big(\frac{2a+\eta_1(b,a)}{2}\Big)$$
$$= \frac{1}{2}\Big[\int_a^{\frac{2a+\eta_1(b,a)}{2}} (t-a)^2 f''(t)dt + \int_{\frac{2a+\eta_1(b,a)}{2}}^{a+\eta_1(b,a)} (a+\eta_1(b,a)-t)^2 f''(t)dt\Big].$$

Proof. Using integration by parts we obtain the following inequalities

$$\int_a^{\frac{2a+\eta_1(b,a)}{2}} (t-a)^2 f''(t)dt + \int_{\frac{2a+\eta_1(b,a)}{2}}^{a+\eta_1(b,a)} \big(a+\eta_1(b,a)-t\big)^2 f''(t)dt$$
$$= (t-a)^2 f'(t)\Big|_a^{\frac{2a+\eta_1(b,a)}{2}} - 2\int_a^{\frac{2a+\eta_1(b,a)}{2}} (t-a)f'(t)dt$$
$$+ \big(a+\eta_1(b,a)-t\big)^2 f'(t)\Big|_{\frac{2a+\eta_1(b,a)}{2}}^{a+\eta_1(b,a)} + 2\int_{\frac{2a+\eta_1(b,a)}{2}}^{a+\eta_1(b,a)} \big(a+\eta_1(b,a)-t\big)f'(t)dt$$
$$= \Big(\frac{\eta_1(b,a)}{2}\Big)^2 f'\Big(\frac{2a+\eta_1(b,a)}{2}\Big) - 2\int_a^{\frac{2a+\eta_1(b,a)}{2}} (t-a)f'(t)dt$$
$$- \Big(\frac{\eta_1(b,a)}{2}\Big)^2 f'\Big(\frac{2a+\eta_1(b,a)}{2}\Big) + 2\int_{\frac{2a+\eta_1(b,a)}{2}}^{a+\eta_1(b,a)} \big(a+\eta_1(b,a)-t\big)f'(t)dt$$
$$= -2\int_a^{\frac{2a+\eta_1(b,a)}{2}} (t-a)f'(t)dt + 2\int_{\frac{2a+\eta_1(b,a)}{2}}^{a+\eta_1(b,a)} \big(a+\eta_1(b,a)-t\big)f'(t)dt$$
$$= -2(t-a)f(t)\Big|_a^{\frac{2a+\eta_1(b,a)}{2}} + 2\int_a^{\frac{2a+\eta_1(b,a)}{2}} f(t)dt$$
$$+ 2\big(a+\eta_1(b,a)-t\big)f(t)\Big|_{\frac{2a+\eta_1(b,a)}{2}}^{a+\eta_1(b,a)} + 2\int_{\frac{2a+\eta_1(b,a)}{2}}^{a+\eta_1(b,a)} f(t)dt$$

$$
\begin{aligned}
&= -2\Big(\frac{\eta_1(b,a)}{2}\Big) f\Big(\frac{2a+\eta_1(b,a)}{2}\Big) + 2\int_a^{\frac{2a+\eta_1(b,a)}{2}} f(t)dt \\
&- 2\Big(\frac{\eta_1(b,a)}{2}\Big) f\Big(\frac{2a+\eta_1(b,a)}{2}\Big) + 2\int_{\frac{2a+\eta_1(b,a)}{2}}^{a+\eta_1(b,a)} f(t)dt \\
&= -2\eta_1(b,a) f\Big(\frac{2a+\eta_1(b,a)}{2}\Big) + 2\int_a^{a+\eta_1(b,a)} f(t)dt.
\end{aligned}
$$

So

$$
\begin{aligned}
&\int_a^{a+\eta_1(b,a)} f(t)dt - \eta_1(b,a) f\Big(\frac{2a+\eta_1(b,a)}{2}\Big) \\
&= \frac{1}{2}\Big[\int_a^{\frac{2a+\eta_1(b,a)}{2}} (t-a)^2 f''(t)dt + \int_{\frac{2a+\eta_1(b,a)}{2}}^{a+\eta_1(b,a)} (a+\eta_1(b,a)-t)^2 f''(t)dt\Big].
\end{aligned}
$$

□

Corollary 4.6. [6] Suppose $f : I^\circ \longrightarrow \mathbb{R}$ is a twice differentiable mapping on I°. For any $a, b \in I^\circ$ if $f'' \in L^1[a,b]$ then

$$
\begin{aligned}
&\int_a^b f(t)dt - (b-a) f\Big(\frac{a+b}{2}\Big) \\
&= \frac{1}{2}\Big[\int_a^{\frac{a+b}{2}} (t-a)^2 f''(t)dt + \int_{\frac{a+b}{2}}^{b} (b-t)^2 f''(t)dt\Big].
\end{aligned}
$$

Theorem 4.7. *Suppose that $I^\circ \subseteq \mathbb{R}$ is an invex set with respect to η_1 and $f : I^\circ \longrightarrow \mathbb{R}$ is a twice differentiable mapping on I°. For any $a, b \in I^\circ$ with $\eta_1(b,a) > 0$ If $|f''| \in L^1[a, a+\eta_1(b,a)]$ is an (η_1,η_2)-convex mapping, then*

$$
\begin{aligned}
&\Big|\frac{1}{\eta_1(b,a)} \int_a^{a+\eta_1(b,a)} f(t)dt - f\Big(\frac{2a+\eta_1(b,a)}{2}\Big)\Big| \\
&\le \big(\eta_1(b,a)\big)^2 \Big[\frac{1}{24}|f''(a)| + \frac{1}{48}\eta_2(|f''(b)|, |f''(a)|)\Big].
\end{aligned}
$$

Proof. Using the change of variable $t = a + x\eta_1(b,a)$ and Lemma 4.5 we have

$$
\begin{aligned}
&\Big|\int_a^{\frac{2a+\eta_1(b,a)}{2}} (t-a)^2 f''(t)dt + \int_{\frac{2a+\eta_1(b,a)}{2}}^{a+\eta_1(b,a)} (a+\eta_1(b,a)-t)^2 f''(t)dt\Big| \\
&= \big(\eta_1(b,a)\big)^3 \Big|\int_0^{\frac{1}{2}} x^2 f''\big(a+x\eta_1(b,a)\big)dx + \int_{\frac{1}{2}}^{1} (1-x)^2 f''\big(a+x\eta_1(b,a)\big)dx\Big| \\
&\le \big(\eta_1(b,a)\big)^3 \Big[\int_0^{\frac{1}{2}} x^2 |f''|\big(a+x\eta_1(b,a)\big)dx \\
&+ \int_{\frac{1}{2}}^{1} (1-x)^2 |f''|\big(a+x\eta_1(b,a)\big)dx\Big]
\end{aligned}
$$

$$\le \big(\eta_1(b,a)\big)^3\bigg[\int_0^{\frac{1}{2}} x^2\Big(|f''(a)| + x\eta_2(|f''(b)|,|f''(a)|)\Big)dx$$
$$+\int_{\frac{1}{2}}^1 (1-x)^2\Big(|f''(a)| + x\eta_2(|f''(b)|,|f''(a)|)\Big)dx\bigg]$$
$$= \big(\eta_1(b,a)\big)^3\bigg[\frac{1}{24}|f''(a)| + \frac{1}{64}\eta_2(|f''(b)|,|f''(a)|)$$
$$+\frac{1}{24}|f''(a)| + \frac{5}{192}\eta_2(|f''(b)|,|f''(a)|)\bigg]$$
$$= \big(\eta_1(b,a)\big)^3\bigg[\frac{1}{12}|f''(a)| + \frac{1}{24}\eta_2(|f''(b)|,|f''(a)|)\bigg].$$

So

$$\left|\int_a^{a+\eta_1(b,a)} f(t)dt - \eta_1(b,a)f\Big(\frac{2a+\eta_1(b,a)}{2}\Big)\right|$$
$$\le \big(\eta_1(b,a)\big)^3\bigg[\frac{1}{24}|f''(a)| + \frac{1}{48}\eta_2\big(|f''(b)|,|f''(a)|\big)\bigg].$$

□

Corollary 4.8. *Suppose $f : I^\circ \longrightarrow \mathbb{R}$ is a twice differentiable mapping on I°. For any $a, b \in I^\circ$ if $f'' \in L^1[a, b]$ is a convex function then*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - f\Big(\frac{a+b}{2}\Big)\right| \le \frac{(b-a)^2}{48}\big[|f''(b)| + |f''(a)|\big].$$

For difference between the right and middle terms in (3.8) we need the following lemma (see [3]).

Lemma 4.9. *Suppose that $I^\circ \subseteq \mathbb{R}$ is an invex set with respect to η_1 and $f : I^\circ \longrightarrow \mathbb{R}$ is a twice differentiable mapping on I°. For any $a, b \in I^\circ$ with $\eta_1(b, a) > 0$ If $f'' \in L^1[a, a + \eta_1(b, a)]$, then*

$$\frac{f\big(a+\eta_1(b,a)\big)+f(a)}{2} - \frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f(t)dt$$
$$= \frac{\big(\eta_1(b,a)\big)^2}{2}\int_0^1 t(1-t)f''\big(a+t\eta_1(b,a)\big)dt.$$

Theorem 4.10. *Suppose that $I^\circ \subseteq \mathbb{R}$ is an invex set with respect to η_1 and $f : I^\circ \longrightarrow \mathbb{R}$ is a twice differentiable mapping on I°. For any $a, b \in I^\circ$ with $\eta_1(b, a) > 0$ If $|f''| \in L^1[a, a + \eta_1(b, a)]$ is an (η_1, η_2)-convex mapping, then*

$$\left|\frac{f\big(a+\eta_1(b,a)\big)+f(a)}{2} - \frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f(t)dt\right|$$
$$\le \big(\eta_1(b,a)\big)^2\bigg[\frac{1}{12}|f''(a)| + \frac{1}{24}\eta_2\big(|f''(b)|,|f''(a)|\big)\bigg].$$

Proof. Using Lemma 4.9 we have

$$\begin{aligned}
&\left|\frac{f\big(a+\eta_1(b,a)\big)+f(a)}{2}-\frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)}f(t)dt\right|\\
&\leq \frac{\big(\eta_1(b,a)\big)^2}{2}\int_0^1 |t(1-t)|\Big(|f''(a)|+t\eta_2(|f''(b)|,|f''(a)|)\Big)dt\\
&=\frac{\big(\eta_1(b,a)\big)^2}{2}\int_0^1\Big[t(1-t)|f''(a)|+t^2(1-t)\eta_2\big(|f''(b)|,|f''(a)|\big)\Big]dt\\
&=\frac{\big(\eta_1(b,a)\big)^2}{2}\Big[\frac{1}{6}|f''(a)|+\frac{1}{12}\eta_2(|f''(b)|,|f''(a)|)\Big]\\
&=\big(\eta_1(b,a)\big)^2\Big[\frac{1}{12}|f''(a)|+\frac{1}{24}\eta_2(|f''(b)|,|f''(a)|)\Big].
\end{aligned}$$

This completes the proof. ☐

Corollary 4.11. *Suppose that $I^\circ\subseteq\mathbb{R}$ is a convex set and $f:I^\circ\longrightarrow\mathbb{R}$ is a twice differentiable mapping on I°. If for any $a,b\in I^\circ$, $|f''|\in L^1[a,a+\eta_1(b,a)]$ is a convex mapping then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{\big(b-a\big)^2}{24}\big(|f''(a)|+|f''(b)|\big).$$

In the case that an (η_1,η_2)-convex function is differentiable, there exist two inequalities that we can obtain.

Theorem 4.12. *Suppose that $I^\circ\subseteq\mathbb{R}$ is an invex set with respect to η_1 and $f: I^\circ\longrightarrow\mathbb{R}$ is a differentiable mapping on I°. For any $a,b\in I^\circ$ with $\eta_1(b,a)>0$ If f is an (η_1,η_2)-convex function on $[a,a+\eta_1(b,a)]$ and η_2 is a measurable bifunction on $f\big([a,a+\eta_1(b,a)]\big)\times f\big([a,a+\eta_1(b,a)]\big)$, then*

$$f'(y)\int_a^{a+\eta_1(b,a)}\eta_1(x,y)dx\leq\int_a^{a+\eta_1(b,a)}\eta_2\big(f(x),f(y)\big)dx, \tag{4.1}$$

for every $y\in\big[a,a+\eta_1(b,a)\big]$.

Proof. From the definition of an (η_1,η_2)-convex function we have

$$\frac{f\big(y+t\eta_1(x,y)\big)-f(y)}{t}\leq\eta_2\big(f(x),f(y)\big),$$

for $t\in(0,1]$. Taking the limit over $t\to 0+$ we get

$$f'(y)\,\eta_1(x,y)\leq\eta_2\big(f(x),f(y)\big), \tag{4.2}$$

for any $x,y\in[a,a+\eta_1(b,a)]$.
Since η is measurable on $f\big([a,a+\eta_1(b,a)]\big)\times f\big([a,a+\eta_1(b,a)]\big)$, then the integral

$$\int_a^{a+\eta_1(b,a)}\eta_2\big(f(x),f(y)\big)dx,$$

exists for any $y\in\big[a,a+\eta_1(b,a)\big]$. Integrating (4.2) over x on $[a,a+\eta_1(b,a)]$ we deduce (4.3). ☐

Corollary 4.13. *In Theorem 4.12 if we consider $\eta_1(x,y)=x-y$, then we get*

$$f'(y)\left(\frac{a+b}{2}-y\right)\leq\frac{1}{b-a}\int_a^b\eta_2\big(f(x),f(y)\big)dx,\tag{4.3}$$

for every $y\in[a,b]$.

Theorem 4.14. *Suppose that $I^\circ\subseteq\mathbb{R}$ is an invex set with respect to η_1 such that η_1 is continuously differentiable on the second variable and $f:I^\circ\longrightarrow\mathbb{R}$ is a differentiable mapping on I°. For any $a,b\in I^\circ$ with $\eta_1(b,a)>0$ If f is an (η_1,η_2)-convex function on $[a,a+\eta_1(b,a)]$ and η_2 is a measurable bifunction on $f\big([a,a+\eta_1(b,a)]\big)\times f\big([a,a+\eta_1(b,a)]\big)$, then*

$$\begin{aligned}&-\int_a^{a+\eta_1(b,a)}f(y)\,\frac{\partial\eta_1(x,y)}{\partial y}dy\\&\leq\eta_1(x,a)f(a)-\eta_1\big(x,a+\eta_1(b,a)\big)f\big(a+\eta_1(b,a)\big)\\&+\int_a^{a+\eta_1(b,a)}\eta_2\big(f(x),f(y)\big)dy,\end{aligned}\tag{4.4}$$

for any $x\in[a,a+\eta_1(b,a)]$.

Proof. From inequality (4.2) obtained in Theorem 4.12, we have

$$f'(y)\,\eta_1(x,y)\leq\eta_2\big(f(x),f(y)\big),\tag{4.5}$$

for any $x,y\in[a,a+\eta_1(b,a)]$.

Since η_2 is measurable on $f\big([a,a+\eta_1(b,a)]\big)\times f\big([a,a+\eta_1(b,a)]\big)$, then the integral

$$\int_a^{a+\eta_1(b,a)}\eta\big(f(x),f(y)\big)dy$$

exists for any $x\in[a,a+\eta_1(b,a)]$.

Integrating in (4.2) over y on the interval $[a,a+\eta_1(b,a)]$ we get

$$\int_a^{a+\eta_1(b,a)}f'(y)\,\eta_1(x,y)dy\leq\int_a^{a+\eta_1(b,a)}\eta_2\big(f(x),f(y)\big)dy\tag{4.6}$$

for any $x\in[a,a+\eta_1(b,a)]$.

Integrating by parts, we also have

$$\begin{aligned}&\int_a^{a+\eta_1(b,a)}f'(y)\,\eta_1(x,y)dy\\&=\eta_1\big(x,a+\eta_1(b,a)\big)f\big(a+\eta_1(b,a)\big)-\eta_1(x,a)f(a)\\&-\int_a^{a+\eta_1(b,a)}f(y)\frac{\partial\eta_1(x,y)}{\partial y}dy,\end{aligned}$$

and by (4.6) we get the desired result (4.4). □

Corollary 4.15. *If in Theorem 4.14 we consider $\eta_1(x,y)=x-y$ then we get*

$$\int_a^b f(y)\,dy\leq(x-a)\,f(a)+(b-x)\,f(b)+\int_a^b\eta_2\big(f(x),f(y)\big)dy,\tag{4.7}$$

for every $x \in [a,b]$. Also by integrating over $x \in [a, a+\eta_1(b,a)]$ in (4.7) we obtain the double integral inequality

$$\frac{1}{b-a}\int_a^b f(y)dy \le \frac{f(a)+f(b)}{2} + \frac{1}{(b-a)^2}\int_a^b\int_a^b \eta_2\big(f(x),f(y)\big)dxdy.$$

5. $(\eta_1,\eta_2)_b$-Convex, $(\eta_1,\eta_2)_E$-Convex

In this section we introduce the class of $(\eta_1,\eta_2)_b$-convex function and $(\eta_1,\eta_2)_E$-convex function and we show that the class of $(\eta_1,\eta_2)_b$-convex functions for some b, is equivalent to the class of (η_1,η_2)-quasiconvex functions. Moreover under some special conditions we show that the restriction of an $(\eta_1,\eta_2)_E$-convex function is an (η_1,η_2)-convex function.

Definition 5. *Let $I \subseteq \mathbb{R}$ be an invex set with respect to η_1 and $b : I\times I\times[0,1] \to \mathbb{R}^+$ be a function such that $\lambda b(x,y,\lambda) \in [0,1]$ for all $x,y \in I$ and $\lambda \in [0,1]$. A function $f : I \to \mathbb{R}$ is called $(\eta_1,\eta_2)_b$-convex if for any $x,y \in I$ and $\lambda \in [0,1]$ we have*

$$f\big(y+\lambda\eta_1(x,y)\big) \le f(y) + \lambda b(x,y,\lambda)\eta_2\big(f(x),f(y)\big).$$

Remark. *Note that Definition 5 is equivalent to*
(i) Definition 2.7 in [19] if we consider $\eta_2(x,y) = x-y$, for all $x,y \in f(I)$.
(ii) Definition 4.1 in [9] if $\eta_1(x,y) = x-y$ for all $x,y \in I$.
(iii) If η_1 and η_2 satisfy conditions (i) and (ii) simultaneously, then an $(\eta_1,\eta_2)_b$-convex function reduces to a B-vex function [4].

Theorem 5.1. *Suppose that $I \subseteq \mathbb{R}$ is an invex set with respect to η_1 and $f : I \to \mathbb{R}$ is a function. The following assertions are equivalent:*
(a) f is $(\eta_1,\eta_2)_b$-convex, for some function b.
(b) f is (η_1,η_2)-quasiconvex.

Proof. (a)⇒(b)
For $x,y \in I$ and $\lambda \in [0,1]$, from the definition of an $(\eta_1,\eta_2)_b$-convex function we have

$$\begin{aligned} f\big(y+\lambda\eta_1(x,y)\big) &\le f(y) + \lambda b(x,y,\lambda)\eta_2\big(f(x),f(y)\big) \\ &\le \max\big\{f(y), f(y)+\eta_2(f(x),f(y))\big\}. \end{aligned}$$

(b)⇒(a) For $x,y \in I$ and $\lambda \in [0,1]$, define the function b as the following.

$$b(x,y,\lambda) = \begin{cases} 1/\lambda, & \text{if } \lambda \in (0,1] \textit{ and } f(y) \le f(y)+\eta_2(f(x),f(y)); \\ 0, & \lambda = 0 \textit{ or } f(y) > f(y)+\eta_2(f(x),f(y)). \end{cases}$$

Notice that $\lambda b(x,y,\lambda) \in [0,1]$. For a such function b we have

$$\begin{aligned} f\big(y+\lambda\eta_1(x,y)\big) &\le \max\big\{f(y), f(y)+\eta_2(f(x),f(y))\big\} \\ = \lambda b(x,y,\lambda)\big(f(y)&+\eta_2(f(x),f(y)\big) + \big(1-\lambda b(x,y,\lambda)\big)f(y) \\ &= f(y) + \lambda b(x,y,\lambda)\eta_2(f(x),f(y)). \end{aligned}$$

□

Consider a function $E : A \to B$ for appropriate $A,B \subseteq \mathbb{R}$. We can state the following results related to the function E.

Definition 6. [10] A set $A \subseteq \mathbb{R}$ is called to be *E-invex with respect to η* if

$$x, y \in A, \lambda \in [0,1] \Rightarrow E(y) + \lambda\eta\big(E(x), E(y)\big) \in A.$$

For a nonempty subset M of $\mathbb{R}$, define $E(M) = \{E(x) : x \in M\}$.

Lemma 5.2. [10] Suppose that $A \subseteq \mathbb{R}$ is E-invex with respect to some η. Then $E(A) \subseteq A$.

Definition 7. *Let $I \subseteq \mathbb{R}$ be a nonempty E-invex set with respect to η_1. A function $f : I \to \mathbb{R}$ is said to be $(\eta_1, \eta_2)_E$-convex if*

$$f\big(E(y) + \lambda\eta_1(E(x), E(y))\big) \leq f(E(y)) + \lambda\eta_2\big(f(E(x)), f(E(y))\big),$$

for all $x, y \in I$ and $\lambda \in [0,1]$.
Also f is said to be $(\eta_1, \eta_2)_E$-quasiconvex if

$$\begin{aligned} &f\big(E(y) + \lambda\eta_1(E(x), E(y))\big) \\ &\leq \max\big\{f(E(y)), f(E(y)) + \eta_2\big(f(E(x)), f(E(y))\big)\big\}, \end{aligned}$$

for all $x, y \in I$ and $\lambda \in [0,1]$.

Remark. *Note that if in Definition 7 we*
(i) set $\eta_1(x, y) = x - y$ for all $x, y \in E(I)$, then we achieve Definition 4.5 in [9].
(ii) consider $\eta_2(x, y) = x - y$ for all $x, y \in f(E(I))$, then we reach to Definitions 2.2 and 2.6 in [10].
(iii) If η_1 and η_2 satisfy conditions (i) and (ii) simultaneously, then an $(\eta_1, \eta_2)_E$-convex $\big((\eta_1, \eta_2)_E$-prequsiinvex$\big)$ function reduces to an E-convex function [21] $\big(E$-quasiconvex function [20]$\big)$.

The restriction of an $(\eta_1, \eta_2)_E$-convex function is an (η_1, η_2)-convex function. This fact is shown in Theorem 5.3.

Theorem 5.3. *Suppose that $I \subseteq \mathbb{R}$ is an E-invex set with respect to η_1 and $J \subseteq E(I)$ is a nonempty invex set with respect to η_1. If $f : I \to \mathbb{R}$ is an $(\eta_1, \eta_2)_E$-convex $\big((\eta_1, \eta_2)_E$-quasiconvex$\big)$, then it's restriction $\bar{f} : J \to \mathbb{R}$ defined as*

$$\bar{f}(x') = f(x') \ for\ all\ x' \in J,$$

is (η_1, η_2)-convex $\big((\eta_1, \eta_2)$-quasiconvex$\big)$.

Proof. For $x', y' \in J$, there are $x, y \in I$ such that $x' = E(x)$ and $y' = E(y)$. Since J is invex with respect to η_1, then $y' + \lambda\eta_1(x', y') \in J$. So

$$\begin{aligned} &\bar{f}\big(y' + \lambda\eta_1(x', y')\big) = f\big(E(y) + \lambda\eta_1(E(x), E(y))\big) \\ &\leq f(E(y)) + \lambda\eta_2\big(f(E(x)), f(E(y))\big) = \bar{f}(y') + \lambda\eta_2\big(\bar{f}(x'), \bar{f}(y')\big). \end{aligned}$$

□

In Theorem 5.4, we show that if an $(\eta_1, \eta_2)_E$-convex $\big((\eta_1, \eta_2)$-quasiconvex$\big)$ function is restricted on a suitable domain it is equivalent to an (η_1, η_2)-convex $\big((\eta_1, \eta_2)$-quasiconvex$\big)$ function.

Theorem 5.4. *Suppose that $I \subseteq \mathbb{R}$ is an E-invex set with respect to η_1 and $E(I)$ is an invex set with respect to η_1. The function $f : I \to \mathbb{R}$ is an $(\eta_1, \eta_2)_E$-convex $\big((\eta_1, \eta_2)_E$-quasiconvex$\big)$ if and only if it's restriction $\bar{f} : E(I) \to \mathbb{R}$ defined as*

$$\bar{f}(x') = f(x') \ for\ all\ x' \in E(I),$$

is (η_1, η_2)-convex $\big((\eta_1, \eta_2)$-quasiconvex$\big)$ on $E(I)$.

Proof. The straight direction is clear from Theorem 5.3. Conversely suppose that $\bar{f} : E(I) \to \mathbb{R}$ is an (η_1, η_2)-convex function on $E(I)$. For $x, y \in I$ we have $E(x), E(y) \in E(I)$ and since $E(I)$ is invex with respect to η_1, then $E(y)+\lambda\eta_1\big(E(x), E(y)\big) \in E(I)$ for all $\lambda \in [0, 1]$. Then

$$\begin{aligned} &f\big(E(y) + \lambda\eta_1(E(x), E(y))\big) = \bar{f}\big(E(y) + \lambda\eta_1(E(x), E(y))\big) \\ &\leq \bar{f}(E(y)) + \lambda\eta_2\big(\bar{f}(E(x)), \bar{f}(E(y))\big) \\ &= f(E(y)) + \lambda\eta_2\big(f(E(x)), f(E(y))\big). \end{aligned}$$

In the case that f is an $(\eta_1, \eta_2)_E$-quasiconvex function and $\bar{f}$ is an (η_1, η_2)-convex function, the proof is the same as above argument. □

Conclusion. The convexity of a function plays an important role in obtaining many results and inequalities in mathematics. Also in many new problems in nonlinear programming and optimization theory related to the concept of convexity, generalized notions about convex functions are required to obtain applicable results. One of this generalizations may be the notion of (η_1, η_2)-convex functions which can generalizes many results and inequalities (Hermite-Hadamard, Fejér,...) related to the convex functions.

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References

[1] M. Aslam Noor, *Hermite-Hadamard integral inequalities for log-preinvex functions*, Journal of Mathematical Analysis and Approximation Theory, **2** (2007) 126–131. **189** (1995) 901–908.

[2] M. Aslam Noor, K. Inayat Noor, M. Uzair Awan and J. Li, *On Hermite-Hadamard inequalities for h-preinvex functions*, Filomat, **28**, (2014) 1463–1474.1

[3] A. Barani, A. G. Ghazanfari and S. S. Dragomir, *Hermite-Hadamard inequality for functions whose derivatives absolute values are preinvex*, Journal of Inequalities and Applications, 2012, 2012:247.

[4] C.R. Bector and C. Singh, *B-vex functions*, Journal of Optimization Theory and Applications, **71** (1991) 237–254.

[5] A. Ben-Israel and B. Mond, *What is invexity?*, Jornal of Australian Mathematical Society, Ser. B. **28** (1986) 1–9.

[6] P. Cerone and S. S. Dragomir, *New bounds for the three-point rule involving the Riemann-Stieltjes integrals*, Advances in Statistics Combinatorics and Related Areas, C. Gulati, etal. (Eds.), World Science Publishing, (2002), 53–62.

[7] S.S. Dragomir and R.P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Applied Mathematics Letters, **11** (1998) 91–95.

[8] M. Eshaghi Gordji, S. S. Dragomir and M. Rostamian Delavar, *An inequality related to η-convex functions (II)*, International Journal of Nonlinear Analysis and Applications, **6**(2) (2016) 26–32.

[9] M. Eshaghi Gordji, M. Rostamian Delavar and M. De La Sen, *On φ-convex functions*, Journal of Mathematical Inequalities, **10**(1) (2016) 173–183.

[10] C. Fulga and V. Preda, *Nonlinear programming with E-preinvex and local E-preinvex functions*, European Journal of Operational Research, **192** (2009) 737–743.

[11] M.A. Hanson and B. Mond, *Convex transformable programming problems and invexity*, Journal of Information and Optimization Sciences, **8** (1987) 201–207.

[12] W. Jeyakumar, *Strong and weak invexity in mathematical programming*, European Journal of Operational Research, **55** (1985) 109–125.

[13] U.S. Kirmaci, *Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula*, Appl. Math. Comp. **147**(1) (2004) 137–146.

[14] S. Mititelu, *Invex sets*, Studii Cercetări de Matematică, **46**(5) (1994) 529–532.

[15] S. Mohammadi Aslani, M. Rostamian Delavar and S. M. Vaezpour, *Inequalities of Fejér type related to generalized convex functions with applications*, International Journal of Analysis and Applications, **16**(1) (2018) 38–49.
[16] M. Rostamian Delavar and S.S. Dragomir, *On η-convexity*, Mathematical Inequalities and Applications, **20** (2017) 203–216.
[17] M. Rostamian Delavar and M. De La Sen, *Some generalizations of Hermite-Hadamard type inequalities*, SpringerPlus, (2016) 5:1661.
[18] M. Z. Sarikaya, H. Bozkurt and N. Alp, *On Hermite-Hadamard type integral inequalities for preinvex and log-preinvex functions*, arXiv:1203.4759v1.
[19] S.K. Suneja, C. Singh and C.R. Bector, *Generalization of preinvex and B-vex functions*, Journal of Optimization Theory and Applications, **76** (1993) 577–587.
[20] Y.R. Syau and E.S. Lee, *Some properties of E-convex functions*, Applied Mathematics Letters, **18** (2005) 1074–1080.
[21] E.A. Youness, *E-convex sets, E-convex functions and E-convex programming*, Journal of Optimization Theory and Applications, **102** (1999) 439–450.

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