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IDENTITIES AND RELATION ON THE POLY-GENOCCHI POLYNOMIALS WITH A q-PARAMETER

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Abstract. Komatsu generalized the poly-Cauchy numbers with a q-parameter. Recently, Cenkci and Komatsu defined and investigated poly-Bernoulli numbers and polynomials with a q-parameter. They proved some relations between these polynomials and weighted Stirling numbers of the first and second kind.

In this work, we define poly-Genocchi numbers and polynomials with a q-parameter. We give a recurrence relation for the poly-Genocchi polynomials. Also, we prove some relations and closed formulae for the poly-Genocchi numbers with a q-parameter.

1. Introduction

The classical Bernoulli numbers and polynomials are defined the following generating functions respectively,

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t-1},\ |t|<2\pi, \tag{1}$$

$$\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \frac{2t}{e^t+1}e^{xt},\ |t|<2\pi. \tag{2}$$

Similarly, the classical Genocchi numbers and polynomials are defined the following generating functions respectively [1],

$$\sum_{n=0}^{\infty} G_n \frac{t^n}{n!} = \frac{2t}{e^t+1},\ |t|<\pi, \tag{3}$$

$$\sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} = \frac{2t}{e^t+1}e^{xt},\ |t|<\pi. \tag{4}$$

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Let $k\in\mathbb{Z}$, $k>1$. Then k-th polylogarithm functions is defined

$$L_{i_k}(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^k} \tag{5}$$

([1]-[22]). This function is convergences for $|z|<1$. If $k<0$, the polylogarithm function is a rational function. For some $k\in\mathbb{Z}^+$,

$$L_{i_0}(x)=\frac{x}{1-x},\ L_{i_{-1}}(x)=\frac{x}{(1-x)^2}$$

$$L_{i_{-2}}(x)=\frac{x^2+x}{(1-x)^3},\ L_{i_{-3}}(x)=\frac{x^3+4x^2+x}{(1-x)^4},\ \cdots.$$

For $k=1$,

$$L_{i_1}(z)=-\log(1-z).$$

The Stirling numbers of the second kind is defined as:

$$\sum_{n=0}^{\infty}\mathcal{S}_2(n,m)\frac{t^n}{n!}=\frac{(e^t-1)^m}{m!}. \tag{6}$$

The poly-Bernoulli numbers are defined by following generating functions by Arakawa-Kaneko in [1]:

$$\sum_{n=0}^{\infty}B_n^{(k)}\frac{t^n}{n!}=\frac{L_{i_k}(1-e^{-t})}{1-e^{-t}}. \tag{7}$$

Moreover, the poly-Bernoulli polynomials are defined by following generating functions by Hamahata-Bayad in ([4], [9])

$$\sum_{n=0}^{\infty}B_n^{(k)}(x)\frac{t^n}{n!}=\frac{L_{i_k}(1-e^{-t})}{1-e^{-t}}e^{xt} \tag{8}$$

and by Coppo and Candelperger in [8]

$$\sum_{n=0}^{\infty}B_n^{(k)}(x)\frac{t^n}{n!}=\frac{L_{i_k}(1-e^{-t})}{1-e^{-t}}e^{-xt}. \tag{9}$$

Kim *et al.* in [15] are defined the poly-Bernoulli polynomials and the poly-Genocchi polynomials respectively

$$\sum_{n=0}^{\infty}B_n^{(k)}(x)\frac{t^n}{n!}=\frac{L_{i_k}(1-e^{-t})}{e^t-1}e^{xt} \tag{10}$$

and

$$\sum_{n=0}^{\infty}G_n^{(k)}(x)\frac{t^n}{n!}=\frac{2L_{i_k}(1-e^{-t})}{e^t+1}e^{xt}. \tag{11}$$

The second kind weight Stirling numbers $\mathcal{S}_2(n,m,x)$ are defined by Carlitz in [5] as

$$\sum_{n=0}^{\infty}\mathcal{S}_2(n,m,x)\frac{t^n}{n!}=\frac{(e^t-1)^m}{m!}e^{xt}. \tag{12}$$

The λ-Stirling numbers of the second kind $\mathcal{S}_2^{\lambda}(n,m)$ are defined by [18]

$$\frac{(\lambda e^t-1)^m}{m!}=\sum_{n=m}^{\infty}\mathcal{S}_2^{\lambda}(n,m)\frac{t^n}{n!}. \tag{13}$$

Note that when $\lambda = 1$, we have $\mathcal{S}_2^1(n,m) = \mathcal{S}_2(n,m)$ ordinary Stirling numbers of the second kind.

The q-parameter poly-Bernoulli numbers are defined by Cenkci-Kamatsu in [7] by following generating functions:

$$\sum_{n=0}^{\infty} \mathcal{B}_{n,q}^{(k)} \frac{t^n}{n!} = \frac{qL_{i_k}(\frac{1-e^{-qt}}{q})}{1-e^{-qt}}. \tag{14}$$

Kaneko in [13] defined poly-Bernoulli numbers in the first time. He proved symmetry identity poly-Bernoulli numbers $B_n^{(-k)} = B_k^{(-n)}$. S. Peregrino in [22] proved the closed formulae for the poly-Bernoulli numbers.

Bayad and Hamahata in [4] introduced and investigated the poly-Bernoulli polynomials and gave some relations for these polynomials. Hamahata in [9] defined poly-Euler polynomials and gave the some recurrence relations.

Kim *et al.* in [15] defined poly-Genocchi polynomials and gave the some properties for the poly-Bernoulli numbers between the poly-Genocchi polynomials.

Jolany *et al.* in ([10], [11]) introduced and investigated the generalized poly-Bernoulli polynomials with the parameters a, b and c.

Cenkci and Komatsu in [7], Cenkci and P. T. Young in [6] are defined q-parameter poly-Bernoulli numbers and proved relation between the second kind Cauchy numbers and the first kind Cauchy numbers.

In this work, we give some relations between poly-Bernoulli numbers and poly-Genocchi numbers. Also, we define modified poly-Bernoulli polynomials and prove some relations for these polynomials. We introduce and investigate the poly-Genocchi polynomials. We prove the relation between the q-parameter poly-Genocchi numbers and the Stirling numbers of the second kind.

2. q-Parameter Poly-Genocchi Polynomials

In this section, we give some relations for the q-parameter poly-Genocchi polynomials. Also, we give some relation between q-parameter poly-Genocchi polynomials, q-parameter poly-Bernoulli polynomials, q-parameter poly-Genocchi numbers and the Stirling numbers of the second kind.

Definition 1. *The q parameter poly-Genocchi numbers and polynomials are defined the following generating functions respectively,*

$$\sum_{n=0}^{\infty} \mathcal{G}_{n,q}^{(k)} \frac{t^n}{n!} = \frac{2qL_{i_k}\left(\frac{1-e^{-qt}}{q}\right)}{e^{qt}+1} \tag{15}$$

and

$$\sum_{n=0}^{\infty} \mathcal{G}_{n,q}^{(k)}(x) \frac{t^n}{n!} = \frac{2qL_{i_k}\left(\frac{1-e^{-qt}}{q}\right)}{e^{qt}+1} e^{xt}, \tag{16}$$

where n, k integer, $n \geq 0$, $k \geq 1$ and $q \neq 0$ real parameter.

For $q = 1$, $k = 1$. Using (15) and (16), we have classical Genocchi numbers and Genocchi polynomials

$$\mathcal{G}_{n,1}^{(1)} = G_n, \ \ \mathcal{G}_{n,1}^{(1)}(x) = G_n(x).$$

Theorem 1. *The following relation holds true:*

$$\sum_{m=0}^{n}\binom{n}{m}\mathcal{G}_{n-m,q}^{(k)}q^{m}+\mathcal{G}_{n,q}^{(k)}=2\left\{\sum_{m=0}^{\infty}\frac{(-1)^{m+n+1}}{(m+1)^{k}}q^{n-m}(m+1)!\mathcal{S}_{2}(n,m+1)\right\}. \tag{17}$$

Proof. Using (15) and (6), we write as

$$\sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{(k)}\frac{t^{n}}{n!}=\frac{2qL_{i_k}\left(\frac{1-e^{-qt}}{q}\right)}{e^{qt}+1}$$

$$\begin{aligned}
e^{qt}\sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{(k)}\frac{t^{n}}{n!}+\sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{(k)}\frac{t^{n}}{n!} &= 2qL_{i_k}\left(\frac{1-e^{-qt}}{q}\right)\\
&= 2q\sum_{m=1}^{\infty}\frac{\left(\frac{1-e^{-qt}}{q}\right)^{m}}{m^{k}}\\
&= 2q\sum_{m=0}^{\infty}\frac{1}{(m+1)^{k}}(-1)^{m+1}\frac{1}{q^{m+1}}\left(e^{-qt}-1\right)^{m+1}\\
&= 2\sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{(m+1)^{k}}\frac{1}{q^{m}}\sum_{n=0}^{\infty}(m+1)!\mathcal{S}_{2}(n,m+1)\frac{(-qt)^{n}}{n!}.
\end{aligned}$$

$$\sum_{m=0}^{\infty}q^{m}\frac{t^{m}}{m!}\sum_{l=0}^{\infty}G_{l,q}^{(k)}\frac{t^{l}}{l!}+\sum_{n=0}^{\infty}G_{n,q}^{(k)}\frac{t^{n}}{n!}=\sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\binom{n}{m}G_{n-m,q}^{(k)}q^{m}+G_{n,q}^{(k)}\right\}\frac{t^{n}}{n!}$$

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\binom{n}{m}\mathcal{G}_{n-m,q}^{(k)}q^{m}+\mathcal{G}_{n,q}^{(k)}\right\}\frac{t^{n}}{n!}\\
=\ &\sum_{n=0}^{\infty}\left\{2\sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{(m+1)^{k}}\frac{1}{q^{m}}\sum_{n=0}^{\infty}(m+1)!\mathcal{S}_{2}(n,m+1)\frac{(-qt)^{n}}{n!}\right\}\frac{t^{n}}{n!}.
\end{aligned}$$

Comparing the coefficient of $\frac{t^n}{n!}$, we have (17). ∎

Theorem 2. *The following relation holds true*

$$\sum_{m=0}^{n}\binom{n}{m}\mathcal{G}_{n-m,q}^{(k)}q^{m}+\mathcal{G}_{n,q}^{(k)}=2\sum_{m=0}^{\infty}\frac{1}{(m+1)^{k}}\frac{(-1)^{j}}{q^{m}}\sum_{j=0}^{m+1}\binom{m+1}{j}(-qj)^{n}. \tag{18}$$

Proof. From (15), we write as

$$\begin{aligned}
e^{qt}\sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{(k)}\frac{t^{n}}{n!}+\sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{(k)}\frac{t^{n}}{n!} &= 2qL_{i_k}\left(\frac{1-e^{-qt}}{q}\right)\\
&= 2q\sum_{m=0}^{\infty}\frac{1}{(m+1)^{k}}(-1)^{m+1}\frac{1}{q^{m+1}}\sum_{j=0}^{m+1}\binom{m+1}{j}e^{-qtj}(-1)^{m+1-j}\\
&= \sum_{n=0}^{\infty}\left\{2\sum_{m=0}^{\infty}\frac{1}{(m+1)^{k}}\frac{(-1)^{j}}{q^{m}}\sum_{j=0}^{m+1}\binom{m+1}{j}(-qj)^{n}\right\}\frac{t^{n}}{n!}.
\end{aligned}$$

Comparing the coefficient of $\frac{t^n}{n!}$, we have (18). ∎

Theorem 3. *The following relation holds true*

$$2\mathcal{B}_{n,q}^{(k-1)} = \sum_{m=0}^{n} \binom{n}{m} (2q)^{n-m} \left[\mathcal{G}_{m,q}^{(k)} + \mathcal{G}_{m+1,q}^{(k)}\right] + \sum_{m=0}^{n} \binom{n}{m} q^{n-m}\mathcal{G}_{m+1,q}^{(k)}. \tag{19}$$

Proof. From

$$L_{i_k}^{!}(t) = \frac{1}{t}L_{i_{k-1}}(t),$$

we write

$$L_{i_k}^{!}\left(\frac{1-e^{-qt}}{q}\right) = \frac{qe^{-qt}}{1-e^{-qt}}L_{i_{k-1}}\left(\frac{1-e^{-qt}}{q}\right).$$

$$\sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{(k)}\frac{t^n}{n!} = \frac{2qL_{i_k}\left(\frac{1-e^{-qt}}{q}\right)}{e^{qt}+1}$$

$$2qL_{i_k}\left(\frac{1-e^{-qt}}{q}\right) = e^{qt}\sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{(k)}\frac{t^n}{n!} + \sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{(k)}\frac{t^n}{n!}. \tag{20}$$

Differentiating to the both sides of the Eq. (20), with respect to t, yields to the following equality

$$2q^2\frac{e^{-qt}}{1-e^{-qt}}L_{i_{k-1}}\left(\frac{1-e^{-qt}}{q}\right) = qe^{qt}\sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{(k)}\frac{t^n}{n!}+e^{qt}\sum_{n=1}^{\infty}\mathcal{G}_{n,q}^{(k)}\frac{t^{n-1}}{(n-1)!}+\sum_{n=1}^{\infty}\mathcal{G}_{n,q}^{(k)}\frac{t^{n-1}}{(n-1)!}$$

$$2q\frac{q}{1-e^{-qt}}L_{i_{k-1}}\left(\frac{1-e^{-qt}}{q}\right) = qe^{2qt}\sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{(k)}\frac{t^n}{n!}+e^{2qt}\sum_{n=1}^{\infty}\mathcal{G}_{n,q}^{(k)}\frac{t^{n-1}}{(n-1)!}+e^{qt}\sum_{n=1}^{\infty}\mathcal{G}_{n,q}^{(k)}\frac{t^{n-1}}{(n-1)!}$$

$$2q\sum_{n=0}^{\infty}\mathcal{B}_{n,q}^{(k-1)}\frac{t^n}{n!} = q\sum_{n=0}^{\infty}(2q)^n\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{(k)}\frac{t^n}{n!}+\sum_{n=0}^{\infty}(2q)^n\frac{t^n}{n!}\sum_{m=0}^{\infty}\mathcal{G}_{m+1,q}^{(k)}\frac{t^m}{m!}+\sum_{n=0}^{\infty}q^n\frac{t^n}{n!}\sum_{m=0}^{\infty}\mathcal{G}_{m+1,q}^{(k)}\frac{t^m}{m!}.$$

Comparing the coefficients of both sides, we have

$$2\mathcal{B}_{n,q}^{(k-1)} = \sum_{m=0}^{n} \binom{n}{m} (2q)^{n-m} \left[\mathcal{G}_{m,q}^{(k)} + \mathcal{G}_{m+1,q}^{(k)}\right] + \sum_{m=0}^{n} \binom{n}{m} q^{n-m}\mathcal{G}_{m+1,q}^{(k)}.$$

∎

Theorem 4. *The following relation holds true:*

$$2\left\{\mathcal{B}_{n,q}^{(k)} - \sum_{m=0}^{n}\binom{n}{m}\mathcal{B}_{n-m,q}^{(k)}(-q)^m\right\} = \sum_{m=0}^{n}\binom{n}{m}\mathcal{G}_{n-m,q}^{(k)}q^m + \mathcal{G}_{n,q}^{(k)} \tag{21}$$

Proof. From (14) and (15),

$$2\frac{L_{i_k}\left(\frac{1-e^{-qt}}{q}\right)}{\frac{1-e^{-qt}}{q}}\frac{1-e^{-qt}}{q} = 2\frac{L_{i_k}\left(\frac{1-e^{-qt}}{q}\right)}{\frac{e^{qt}+1}{q}}\frac{e^{qt}+1}{q}$$

$$\frac{2}{q}\sum_{n=0}^{\infty}\mathcal{B}_{n,q}^{(k)}\frac{t^n}{n!}\left(1-e^{-qt}\right) = \frac{1}{q}\sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{(k)}\frac{t^n}{n!}\left(e^{qt}+1\right)$$

$$2\left\{\sum_{n=0}^{\infty}\mathcal{B}_{n,q}^{(k)}\frac{t^n}{n!} - \sum_{n=0}^{\infty}\mathcal{B}_{n,q}^{(k)}\frac{t^n}{n!}\sum_{n=0}^{\infty}(-q)^n\frac{t^n}{n!}\right\} = \sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{(k)}\frac{t^n}{n!}\sum_{m=0}^{\infty}q^m\frac{t^m}{m!} + \sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{(k)}\frac{t^n}{n!}.$$

By Cauchy product and comparing the coefficient of $\frac{t^n}{n!}$, we have (21). ∎

Corollary 5. *TThe following relation holds true:*

$$2\mathcal{B}_{n,q}^{(k)}(x)=\sum_{m=0}^{n}\binom{n}{m}\left\{\mathcal{G}_{n-m,q}^{(k)}(x)q^{m}+2\left(-q\right)^{m}\mathcal{B}_{n-m,q}^{(k)}(x)\right\}+\mathcal{G}_{n,q}^{(k)}(x).$$

Theorem 6. *The following equation holds true:*

$$\mathcal{G}_{n,q}^{(-k)}=2\sum_{i=0}^{\infty}\left(-1\right)^{i}\sum_{j=0}^{\infty}\left(j!\right)^{2}\sum_{l=0}^{k}\binom{k}{l}\mathcal{S}_{2}^{q^{-1}}\left(l,j\right)\left[\mathcal{S}_{2}\left(n,j,1+i\right)-\mathcal{S}_{2}\left(n,j,\frac{x}{q}\right)\right]q^{n} \tag{22}$$

where $i,\ j,\ l,\ k\in\mathbb{N}_0$ *and* $x\in\mathbb{R}$.

Proof. From (15),

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{(-k)}\frac{t^{n}}{n!} &= \frac{2qL_{i_k}\left(\frac{1-e^{-qt}}{q}\right)}{e^{qt}+1}=\frac{2q}{e^{qt}+1}\sum_{m=1}^{\infty}\left(\frac{1-e^{-qt}}{q}\right)^{m}m^{k}\\ &= \frac{2q}{e^{qt}+1}\sum_{m=0}^{\infty}\left(m+1\right)^{k}\left(q^{-1}\left(1-e^{-qt}\right)\right)^{m+1}.\end{aligned}$$

From here,

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{(-k)}\frac{t^{n}}{n!}\frac{u^{k}}{k!} &= \frac{2q}{e^{qt}+1}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\left(m+1\right)^{k}\left(q^{-1}\left(1-e^{-qt}\right)\right)^{m+1}\frac{t^{k}}{k!}\\ &= \frac{2q}{e^{qt}+1}q^{-1}\left(1-e^{-qt}\right)e^{u}\sum_{m=0}^{\infty}\left[q^{-1}\left(1-e^{-qt}\right)e^{u}\right]^{m}\\ &= \frac{2\left(1-e^{-qt}\right)e^{u}}{e^{qt}+1}\frac{qe^{qt}}{qe^{qt}-e^{u+qt}+e^{u}}\end{aligned}$$

$$=2\sum_{i=1}^{\infty}\left(-1\right)^{i}\left\{\sum_{j=0}^{\infty}e^{u}\left(q^{-1}u-1\right)^{j}e^{qt(1+i)}\left(e^{qt}-1\right)^{j}-\sum_{j=0}^{\infty}e^{u}\left(q^{-1}e^{u}-1\right)^{j}e^{qt}\left(e^{qt}-1\right)^{j}\right\}.$$

Using (12) and (13), we write as

$$\begin{aligned}&= 2\sum_{i=0}^{\infty}\left(-1\right)^{i}\sum_{j=0}^{\infty}\left\{\sum_{m=0}^{\infty}\frac{u^{m}}{m!}j!\sum_{l=0}^{\infty}\mathcal{S}_{2}^{q^{-1}}\left(l,j\right)\frac{u^{l}}{l!}j!\sum_{n=0}^{\infty}\mathcal{S}_{2}\left(n,j,i+1\right)\frac{q^{n}t^{n}}{n!}\right.\\ &\qquad\left.-\sum_{m=0}^{\infty}\frac{u^{m}}{m!}j!\sum_{l=0}^{\infty}\mathcal{S}_{2}^{q^{-1}}\left(l,j\right)\frac{u^{l}}{l!}j!\sum_{n=0}^{\infty}\mathcal{S}_{2}\left(n,j,\frac{x}{q}\right)q^{n}\frac{t^{n}}{n!}\right\}\end{aligned}$$

$$=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\left\{2\sum_{i=0}^{\infty}\left(-1\right)^{i}\sum_{j=0}^{\infty}\left(j!\right)^{2}\sum_{l=0}^{k}\binom{k}{l}\mathcal{S}_{2}^{q^{-1}}\left(l,j\right)\left[\mathcal{S}_{2}\left(n,j,1+i\right)-\mathcal{S}_{2}\left(n,j,\frac{x}{q}\right)\right]q^{n}\right\}\frac{t^{n}}{n!}\frac{u^{k}}{k!}.$$

Comparing the coefficient of $\frac{t^n}{n!}$, we have closed relation in (22). ∎

Theorem 7. *The following relation holds true:*

$$\mathcal{G}_{n,q}^{(k)}(x)=2\sum_{l=0}^{n}\binom{n}{l}\sum_{m=0}^{\infty}\frac{1}{\left(m+1\right)^{k}}\sum_{i=0}^{\infty}\left(-1\right)^{i+l+r+m}q^{2l-m-n}\left(m+1\right)!\mathcal{S}_{2}\left(n,m+1,-\frac{x}{q}\right). \tag{23}$$

Proof. From (16)

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{(k)}(x)\frac{t^n}{n!} &= \frac{2q}{e^{qt}+1}\sum_{m=1}^{\infty}\frac{\left(\frac{1-e^{-qt}}{q}\right)^m}{m^k}e^{xt}\\ &= \frac{2q}{e^{qt}+1}\sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\left(\frac{1-e^{-qt}}{q}\right)^{m+1}e^{xt}\\ &= \frac{2q}{e^{qt}+1}\sum_{m=0}^{\infty}\frac{1}{(m+1)^k}q^{-m-1}(-1)^{m+1}\left(e^{-qt}-1\right)e^{xt}\\ &= \frac{2}{e^{qt}+1}\sum_{m=0}^{\infty}\frac{1}{(m+1)^k}q^{-m}(-1)^{m+1}\left(e^{-qt}-1\right)^{m+1}e^{xt}\end{aligned}$$

$$\begin{aligned}&= 2\sum_{i=0}^{\infty}(-1)^i e^{iqt}\sum_{m=0}^{\infty}\frac{1}{(m+1)^k}q^{-m}(-1)^{m+1}\left(e^{-qt}-1\right)^{m+1}e^{xt}\\ &= 2\sum_{i=0}^{\infty}(-1)^i\sum_{l=0}^{\infty}(iq)^l\frac{t^l}{l!}\sum_{m=0}^{\infty}\frac{1}{(m+1)^k}q^{-m}(-1)^{m+1}(m+1)!\sum_{n=0}^{\infty}\mathcal{S}_2\left(n,m+1,-\frac{x}{q}\right)\left(\frac{-t}{q}\right)^n\frac{1}{n!}\end{aligned}$$

$$=\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}2\sum_{i=0}^{\infty}(-1)^{i+l+r+m}\sum_{m=0}^{\infty}\frac{1}{(m+1)^k}q^{l-m-r}(m+1)!\mathcal{S}_2\left(n,m+1,-\frac{x}{q}\right)\frac{t^n}{n!}.$$

Cauchy product and comparing the coefficient of $\frac{t^n}{n!}$, we have (23). ∎

Theorem 8. *The following relation holds true:*

$$\mathcal{G}_{n,q}^{(k)}(x)=2q\sum_{i=0}^{\infty}(-1)^{m+1-j}\sum_{m=0}^{\infty}\frac{(-q)^{m+1}}{(m+1)^k}\sum_{j=0}^{m+1}\binom{m+1}{j}(x-qj+qi)^n. \tag{24}$$

Proof. From (16):

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{(k)}(x)\frac{t^n}{n!} &= \frac{2q}{e^{qt}+1}\sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{(m+1)^k}q^{-m-1}\left(e^{-qt}-1\right)^{m+1}e^{xt}\\ &= \frac{2q}{e^{qt}+1}\sum_{m=0}^{\infty}\frac{(-q)^{-m-1}}{(m+1)^k}\sum_{j=0}^{m+1}\binom{m+1}{j}e^{t(x-qj)}\\ &= 2q\sum_{i=0}^{\infty}(-1)^i\sum_{m=0}^{\infty}\frac{(-q)^{-m-1}}{(m+1)^k}\sum_{j=0}^{m+1}\binom{m+1}{j}e^{t(x-qj+qi)}\\ &= 2\sum_{n=0}^{\infty}\left\{q\sum_{i=0}^{\infty}(-1)^i\sum_{m=0}^{\infty}\frac{(-q)^{-m-1}}{(m+1)^k}\sum_{j=0}^{m+1}\binom{m+1}{j}(x-qj+qi)^n\right\}\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficient of $\frac{t^n}{n!}$, we have (24). ∎

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