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QUASI-ALMOST CONVERGENCE OF SEQUENCES OF SETS

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Abstract. In this paper, we defined concepts of Wijsman quasi-almost convergence and Wijsman quasi-almost statistically convergence. Also we give the concepts of Wijsman quasi-strongly almost convergence and Wijsman quasi q-strongly almost convergence. Then, we study relationship among these concepts. Furthermore, we investigate relationship between these concepts and some convergence types given earlier for consequences of sets, as well.

1. Introduction

The concept of statistical convergence was first introduced by Fast [9] and also independently by Buck [19] and Schoenberg [11] for real and complex sequences. Further this concept was studied by Šalát [22], Fridy [14], Et and Şengül [17, 18] and many others.

Connor [15] gave the relationships between the concepts of statistical convergence and strongly p-Cesàro convergence of sequences.

The idea of almost convergence was introduced by Lorentz [6]. Maddox [12] and (independently) Freedman [1] gave the concept of strong almost convergence. Similar concepts can be seen in [2, 13].

There are different convergence notions for sequence of sets. One of them handled in this paper is the concept of Wijsman convergence (see, [5, 7, 8, 10, 16, 20, 21]). The concepts of Wijsman statistical convergence, Wijsman almost convergence and Wijsman Cesàro summability were introduced by Nuray and Rhoades [5].

The idea of quasi-almost convergence in a normed space was introduced by Hajduković [3]. Then, Nuray [4] studied concepts of quasi-invariant convergence and quasi-invariant statistical convergence in a normed space.

2. Definitions and Notations

Now, we recall the basic definitions and concepts (See [5, 7, 8, 16]).

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Let X be any non-empty set and $\mathbb{N}$ be the set of natural numbers. The function

$$f : \mathbb{N} \to P(X)$$

is defined by $f(k) = A_k \in P(X)$ for each $k \in \mathbb{N}$, where $P(X)$ is power set of X.

The sequence $\{A_k\} = (A_1, A_2, \ldots)$, which is the range's elements of f, is said to be *sequences of sets*.

Let (X, ρ) be a metric space. For any point $x \in X$ and any non-empty subset A of X, we define the distance from x to A by

$$d(x, A) = \inf_{a \in A} \rho(x, a).$$

Throughout the paper we take (X, ρ) as a metric space and A, A_k as any non-empty closed subsets of X.

A sequence $\{A_k\}$ is said to be *Wijsman convergent to* A if for each $x \in X$,

$$\lim_{k \to \infty} d(x, A_k) = d(x, A)$$

and denoted by $A_k \stackrel{W}{\to} A$ or $W - \lim A_k = A$.

A sequence $\{A_k\}$ is said to be *bounded* if for each $x \in X$,

$$\sup_k \big\{ d(x, A_k) \big\} < \infty.$$

The set of all bounded sequences of sets is denoted by L_∞.

A sequence $\{A_k\}$ is *Wijsman statistically convergent to* A if for each $x \in X$ and every $\varepsilon > 0$,

$$\lim_{n \to \infty} \frac{1}{n} \Big| \big\{ k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon \big\} \Big| = 0$$

and it is denoted by $st - \lim_W A_k = A$.

A sequence $\{A_k\}$ is *Wijsman Cesàro summable to* A if for each $x \in X$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} d(x, A_k) = d(x, A).$$

A sequence $\{A_k\}$ is *Wijsman strongly Cesàro summable to* A if for each $x \in X$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} |d(x, A_k) - d(x, A)| = 0.$$

A sequence $\{A_k\}$ is *Wijsman strongly p-Cesàro summable to* A if for each $x \in X$ and $0 < p < \infty$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} |d(x, A_k) - d(x, A)|^p = 0.$$

A sequence $\{A_k\}$ is *Wijsman almost convergent to* A if for each $x \in X$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=m+1}^{m+n} d(x, A_k) = d(x, A),$$

uniformly in $m = 0, 1, 2, \ldots$ or

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} d(x, A_{k+m}) = d(x, A).$$

A sequence $\{A_k\}$ is *Wijsman strongly almost convergent to A* if for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=m+1}^{m+n} |d(x, A_k) - d(x, A)| = 0,$$

uniformly in m.

A sequence $\{A_k\}$ is *Wijsman strongly p-almost convergent to A* if for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=m+1}^{m+n} |d(x, A_k) - d(x, A)|^p = 0,$$

uniformly in m.

A sequence $\{A_k\}$ is *Wijsman almost statistically convergent to A* if for each $x \in X$ and $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \Big| \big\{ k \le n : |d(x, A_{k+i}) - d(x, A)| \ge \varepsilon \big\} \Big| = 0,$$

uniformly in i.

3. Main Results

In this section, we defined concepts of Wijsman quasi-almost convergence and Wijsman quasi-almost statistically convergence. Also we give the concepts of Wijsman quasi-strongly almost convergence and Wijsman quasi q-strongly almost convergence. Then, we study relationship among these concepts. Furthermore, we investigate relationship between these concepts and some convergences types given earlier for consequences of sets, as well.

Definition 3.1. *A sequence $\{A_k\} \in L_\infty$ is Wijsman quasi-almost convergent to A if for each $x \in X$,*

$$\left| \frac{1}{p} \sum_{k=np}^{np+p-1} d_x(A_k) - d_x(A) \right| \longrightarrow 0 \quad (as\ p \to \infty), \tag{3.1}$$

uniformly in $n = 0, 1, 2, \ldots$ where $d_x(A_k) = d(x, A_k)$ and $d_x(A) = d(x, A)$. In this case, we will write $WQF - \lim A_k = A$ or $A_k \overset{WQF}{\longrightarrow} A$.

Example 3.2. *Let we define a sequence $\{A_k\}$ as follows:*

$$A_k := \begin{cases} \{1\} & , \quad \textit{if } k \ge 1 \textit{ and } k \textit{ is square integer,} \\ \{0\} & , \quad \textit{otherwise.} \end{cases}$$

This sequence is not Wijsman convergent. But since for each $x \in X$

$$\lim_{p\to\infty} \left| \frac{1}{p} \sum_{k=np}^{np+p-1} d_x(A_k) - d_x(\{0\}) \right| = 0$$

uniformly in n, this sequence is Wijsman quasi-almost convergent to the set $A = \{0\}$.

Theorem 3.3. *If a sequence $\{A_k\} \in L_\infty$ is Wijsman almost convergent to A, then $\{A_k\}$ is Wijsman quasi-almost convergent to A.*

Proof. Suppose that the sequence $\{A_k\}$ is Wijsman almost convergent to A. Then, for each $x \in X$ and every $\varepsilon > 0$ there exists an integer $p_0 > 0$ such that for all $p > p_0$

$$\left| \frac{1}{p} \sum_{k=0}^{p-1} d_x(A_{k+m}) - d_x(A) \right| < \varepsilon,$$

uniformly in m. If m is taken as $m = np$, then we have

$$\left| \frac{1}{p} \sum_{k=0}^{p-1} d_x(A_{k+np}) - d_x(A) \right| = \left| \frac{1}{p} \sum_{k=np}^{np+p-1} d_x(A_k) - d_x(A) \right| < \varepsilon,$$

uniformly in n. Since $\varepsilon > 0$ is an arbitrary, the limit is taken for $p \to \infty$ we can write

$$\left| \frac{1}{p} \sum_{k=np}^{np+p-1} d_x(A_k) - d_x(A) \right| \longrightarrow 0$$

uniformly in n. That is, $A_k \stackrel{WQF}{\longrightarrow} A$. □

Theorem 3.4. *If a sequence $\{A_k\} \in L_\infty$ is Wijsman quasi-almost convergent to A, then $\{A_k\}$ is Wijsman Cesàro summable to A.*

Proof. Assume that the sequence $\{A_k\} \in L_\infty$ is Wijsman quasi-almost convergent to A. Then, (3.1) is true which for $n = 0$ implies for every $\varepsilon > 0$ and each $x \in X$,

$$\left| \frac{1}{p} \sum_{k=0}^{p-1} d_x(A_k) - d_x(A) \right| \longrightarrow 0 \quad (as\ p \to \infty);$$

so, $\{A_k\}$ is Wijsman Cesàro summable to A. □

Definition 3.5. *A sequence $\{A_k\}$ is Wijsman quasi-almost statistically convergent to A if for each $x \in X$ and every $\varepsilon > 0$*

$$\lim_{p\to\infty} \frac{1}{p} \left| \left\{ k \le p : \left| d_x(A_{k+np}) - d_x(A) \right| \ge \varepsilon \right\} \right| = 0,$$

uniformly in n. In this case, we will write $WQS - \lim A_k = A$ or $A_k \stackrel{WQS}{\longrightarrow} A$.

Theorem 3.6. *If a sequence $\{A_k\}$ is Wijsman almost statistically convergent to A, then $\{A_k\}$ is Wijsman quasi-almost statistically convergent to A.*

Proof. Suppose that the sequence $\{A_k\}$ is Wijsman almost statistically convergent to A. Then, for every $\varepsilon, \delta > 0$ and for each $x \in X$ there exists an integer $p_0 > 0$ such that for all $p > p_0$

$$\frac{1}{p} \left| \left\{ k \le p : \left| d_x(A_{k+m}) - d_x(A) \right| \ge \varepsilon \right\} \right| < \delta,$$

uniformly in m. If m is taken as $m = np$, then we have

$$\frac{1}{p} \left| \left\{ k \le p : \left| d_x(A_{k+np}) - d_x(A) \right| \ge \varepsilon \right\} \right| < \delta,$$

uniformly in n. Since $\delta > 0$ is an arbitrary, we have

$$\lim_{p\to\infty} \frac{1}{p} \left| \left\{ k \le p : \left| d_x(A_{k+np}) - d_x(A) \right| \ge \varepsilon \right\} \right| = 0,$$

uniformly in n which means that $\{A_k\}$ is Wijsman quasi-almost statistically convergent to A. □

Definition 3.7. *A sequence* $\{A_k\} \in L_\infty$ *is Wijsman quasi-strongly almost convergent to* A *if for each* $x \in X$,

$$\frac{1}{p}\sum_{k=np}^{np+p-1} \left|d_x(A_k) - d_x(A)\right| \longrightarrow 0$$

uniformly in n. *In this case, we will write* $[WQF] - \lim A_k = A$ *or* $A_k \stackrel{[WQF]}{\longrightarrow} A$.

Definition 3.8. *A sequence* $\{A_k\} \in L_\infty$ *is Wijsman quasi* q*-strongly almost convergent to* A *if for each* $x \in X$ *and* $0 < q < \infty$,

$$\frac{1}{p}\sum_{k=np}^{np+p-1} \left|d_x(A_k) - d_x(A)\right|^q \longrightarrow 0, \tag{3.2}$$

uniformly in n. *In this case, we will write* $[WQF]^q - \lim A_k = A$ *or* $A_k \stackrel{[WQF]^q}{\longrightarrow} A$.

Theorem 3.9. *Let* $0 < q < \infty$. *Then, we have following assertions:*

i. *If a sequence* $\{A_k\}$ *is Wijsman quasi* q*-strongly almost convergent to* A, *then the sequence* $\{A_k\}$ *is Wijsman quasi-almost statistically convergent to* A.
ii. *If a sequence* $\{A_k\} \in L_\infty$ *and Wijsman quasi-almost statistically convergent to* A, *then the sequence* $\{A_k\}$ *is Wijsman quasi* q*-strongly almost convergent to* A.

Proof. (i) Let $\varepsilon > 0$ be given. Then, for each $x \in X$ following inequality is proved

$$\sum_{k=np}^{np+p-1} \left|d_x(A_k) - d_x(A)\right|^q \geq \varepsilon^q \Big|\big\{k \leq p : |d_x(A_{k+np}) - d_x(A)| \geq \varepsilon\big\}\Big|, \tag{3.3}$$

uniformly in n. Since the sequence $\{A_k\}$ is Wijsman quasi q-strongly almost convergent to A; if the both side of inequality (3.3) are multipled by $\frac{1}{p}$ and after that the limit is taken for $p \to \infty$, then we have

$$\lim_{p\to\infty} \frac{1}{p}\sum_{k=np}^{np+p-1} \left|d_x(A_k) - d_x(A)\right|^q \geq \varepsilon^q \lim_{p\to\infty} \frac{1}{p}\Big|\big\{k \leq p : |d_x(A_{k+np}) - d_x(A)| \geq \varepsilon\big\}\Big|$$

$$0 \geq \varepsilon^q \lim_{p\to\infty} \frac{1}{p}\Big|\big\{k \leq p : |d_x(A_{k+np}) - d_x(A)| \geq \varepsilon\big\}\Big|.$$

Hence, we handle

$$\lim_{p\to\infty} \frac{1}{p}\Big|\big\{k \leq p : |d_x(A_{k+np}) - d_x(A)| \geq \varepsilon\big\}\Big| = 0,$$

uniformly in n.

(ii) Since $\{A_k\}$ is bounded, we can write

$$\sup_k\big\{d_x(A_k)\big\} + d_x(A) = M, \quad (0 < M < \infty),$$

for each $x \in X$.

If $\{A_k\}$ is Wijsman quasi-almost statistically convergent to A, then for a given $\varepsilon > 0$ a number $N_\varepsilon \in \mathbb{N}$ can be chosen such that for all $p > N_\varepsilon$ and each $x \in X$

$$\frac{1}{p}\left|\left\{k \leq p : |d_x(A_{k+np}) - d_x(A)| \geq \left(\frac{\varepsilon}{2}\right)^{1/q}\right\}\right| < \frac{\varepsilon}{2M^q}$$

uniformly in n. Let take the set

$$T_p = \left\{k \leq p : |d_x(A_{k+np}) - d_x(A)| \geq \left(\frac{\varepsilon}{2}\right)^{1/q}\right\}.$$

Thus, for each $x \in X$ we have

$$\begin{aligned}
\frac{1}{p}\sum_{k=np}^{np+p-1}\left|d_x(A_k) - d_x(A)\right|^q &= \frac{1}{p}\Bigg(\sum_{\substack{k\leq p\\ k\in T_p}} |d_x(A_{k+np}) - d_x(A)|^q \\
&\quad + \sum_{\substack{k\leq p\\ k\notin T_p}} |d_x(A_{k+np}) - d_x(A)|^q\Bigg) \\
&< \frac{1}{p}p\frac{\varepsilon}{2M^q}M^q + \frac{1}{p}p\frac{\varepsilon}{2} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

uniformly in n. So, the proof is completed. □

Theorem 3.10. *If the sequence $\{A_k\}$ is Wijsman quasi q-strongly almost convergence to A, then $\{A_k\}$ is Wijsman strongly q-Cesàro summable to A.*

Proof. Suppose that the sequence $\{A_k\} \in L_\infty$ is Wijsman quasi q-strongly almost convergent to A. Then, (3.2) is true which for $n = 0$ implies for every $\varepsilon > 0$ and each $x \in X$,

$$\frac{1}{p}\sum_{k=0}^{p-1}\left|d_x(A_k) - d_x(A)\right|^q \longrightarrow 0 \quad (as\ p \to \infty);$$

so, $\{A_k\}$ is Wijsman strongly q-Cesàro summable to A. □

Theorem 3.11. *If a sequence $\{A_k\}$ is Wijsman quasi q-strongly almost convergence to A, then the sequence $\{A_k\}$ is Wijsman statistically convergent to A.*

Proof. Assume that the sequence $\{A_k\}$ is Wijsman quasi q-strongly almost convergence to A. Then, by Theorem 3.10, the sequence $\{A_k\}$ is Wijsman strongly q-*Cesàro* summable to A. For each $x \in X$ and every $\varepsilon > 0$, we can write

$$\sum_{k=0}^{p-1}\left|d_x(A_k) - d_x(A)\right|^q \geq \varepsilon^q\left|\left\{k \leq p : |d_x(A_k) - d_x(A)| \geq \varepsilon\right\}\right|. \tag{3.4}$$

Since the sequence $\{A_k\}$ is Wijsman strongly q-*Cesàro* summable to A; if the both sides of inequality (3.4) are multipled by $\frac{1}{p}$ and after that the limit is taken for $p \to \infty$, left side of the inequality (3.4) is equal to 0. Hence, we handle

$$\lim_{p\to\infty}\frac{1}{p}\left|\left\{k \leq p : |d_x(A_k) - d_x(A)| \geq \varepsilon\right\}\right| = 0.$$

The proof of theorem is completed. □

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