JOURNAL OF INEQUALITIES AND SPECIAL FUNCTIONS
ISSN: 2217-4303, URL: WWW.ILIRIAS.COM/JIASF
VOLUME 8 ISSUE 5(2017), PAGES 40-45.

BEST PROXIMITY POINT THEOREMS FOR WEAKLY CONTRACTIVE MAPPING BY P-CENTER

MOHAMMAD REZA HADDADI

ABSTRACT. Let A and B be two nonempty subsets of a metric space (X, d) and $T : A \to B$ be a nonself mapping. A point $x \in A$ is said to a best proximity point for T, if $d(x, Tx) = dist(A, B)$. Some authors provided sufficient conditions which warrant the existence and uniqueness of the best proximity point for some types of contractive mappings such that P-property play an important role. In this note, we show that P-property is very strong condition. The presented results extend and generalize some known of best proximity point theorems such that some conditions, for instance P-property, had been omitted. Examples are given to usability of our main results.

1. INTRODUCTION

Let (X, d) be a metric space and A and B be nonempty subsets of X. Put

$$\begin{aligned} A_0 &= \{x \in A : d(x, y) = dist(A, B) \ for \ some \ y \in B\}, \\ B_0 &= \{x \in B : d(x, y) = dist(A, B) \ for \ some \ y \in A\}. \end{aligned}$$

A pair $(x_0, y_0) \in A \times B$ is said to be a best proximity pair if $d(x_0, y_0) = dist(A, B)$,that $dist(A, B)$ is distance of A and B. Best proximity pair evolves as a generalization of the concept of best approximation.

We can find the best proximity points of the set A, by considering a map $T : A \to B$. We say that the point $x \in A$ is a best proximity point of the mapping T, if $d(x, Tx) = dist(A, B)$ and we denote the set of all best proximity points of T by $P_T(A)$, that is

$$P_T(A) := \{x \in A : d(x, Tx) = dist(A, B)\}.$$

A best proximity point theorem for contractive mappings has been detailed in Basha [7]. Eldred [3] have elicited a best proximity point theorem for relatively nonexpansive mappings, an alternative treatment to which has been focused in Sankar and Veeramani [9]. A best proximity point theorem for contraction has been obtained in Basha [8]. Sankar in [10] have discussed best proximity point theorems for contractive non-self-mappings and Abkar and Gabeleh continued it in [1, 5]. Best proximity point theorems for various variants of contractions have been explored

2010 *Mathematics Subject Classification.* 47H10, 47H09, 46M05.

Key words and phrases. Best proximity pair, best proximity point, weakly contractive map, fixed point, P-property.

Submitted October 1, 2016. Published September 21, 2017.
Communicated by Valmir Krasniqi.

Eldred and Veeramani [4], Haddadi and Moshtaghioun [6]. Particularly, the following generalization of Banach contraction principle is due to weakly contractive map that is introduced by Alber and Guerre-Delabriere [2] and in this paper we focus on it.

Definition 1.1. *Let A and B be nonempty subsets of a metric space X with $A_0 \neq \emptyset$. The pair (A, B) is said to have P-property if*

$$\begin{cases} d(x_1, y_1) = dist(A, B) \\ \\ d(x_2, y_2) = dist(A, B) \end{cases} \Rightarrow d(x_1, x_2) = d(y_1, y_2).$$

In [10], V. Sankar Raj introduced P-property and studied the existence of best proximity point for weakly contractive mapping under P-property. Also P-property plays an important role in many papers, for instance [1,**??**,5,10,11].

In this paper, we show that for weakly contractive map $T : A \to B$ such that $T(A_0) \subseteq B_0$ with P-property, every best proximity pair is best proximity point (i.e. $A_0 = P_T(A)$). Hence if $A_0 \neq \emptyset$, then $P_T(A) \neq \emptyset$, and if best proximity pair is unique, best proximity point will be also unique.

Also, the presented results extend and generalize some known results from best proximity point theorems such that the P-property, $A_0 \neq \emptyset$ and $T(A_0) \subseteq B_0$ conditions have been omitted.

Definition 1.2. *([10]). Let (A, B) be a pair of two nonempty subsets of a complete metric space X. A map $T : A \to B$ is said to be a weakly contractive mapping if*

$$d(Tx, Ty) \leq d(x, y) - \psi(d(x, y)), \ \forall x, y \in A,$$

where $\psi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that ψ is positive on $(0, \infty)$, $\psi(0) = 0$ and $lim_{t\to\infty}\psi(t) = \infty$. If A is bounded, then the infinity condition can be omitted.

Theorem 1.3. *[10, Theorem 3.1.] Let (A, B) be a pair of two nonempty closed subsets of a complete metric space X such that A_0 is nonempty. Let $T : A \to B$ be a weakly contractive mapping such that $T(A_0) \subseteq B_0$. Assume that the pair (A, B) has the P-property. Then there exists a unique $x_0 \in A$ such that $d(x_0, Tx_0) = dist(A, B)$.*

2. Main Results

We start with the following new definition.

Definition 2.1. *Let (A, B) be a pair of two nonempty closed subsets of a complete metric space X such that A_0 is nonempty. We said that $T : A \to B$ is a P-center map if for every $b \in B$ there is $a \in A_0$ with $d(a, b) = dist(A, B)$ such that*

$$d(Tx, a) \leq d(x, b) \ (x \in X).$$

Proposition 2.2. *Let (A, B) be a pair of two nonempty closed subsets of a complete metric space X such that A_0 is nonempty and $T : A \to B$ be a P-center. Then $P_T(A) = A_0$.*

Proof. Let $x \in A_0$, then there is $y \in B_0$ such that $d(x, y) = dist(A, B)$. Since T is P-center, $d(Tx, x) \leq d(x, y)$. Hence $d(Tx, x) = dist(A, B)$ and so $x \in P_T(A)$. □

Theorem 2.3. *Let (A, B) be a pair of two nonempty closed subsets of a complete metric space X such that A_0 is nonempty. Suppose $T : A \cup B \to A \cup B$ such that $T(A) \subseteq B$ and $T(B) \subseteq A$. Let $T(A_0) \subseteq B_0$ and $T_{|_A}$ is a weakly contractive mapping. Assume that the pair (A, B) has the P-property. Then T is P-center.*

Proof. By Theorem 1.3 there is $a \in A$ such that $d(a, Ta) = dist(A, B)$. Since T is weakly contractive map

$$d(T^2a, Ta) \leq d(Ta, a),$$

and so

$$d(T^2a, Ta) = d(Ta, a) = dist(A, B).$$

By the P-property of (A, B) we have $d(T^2a, a) = 0$ and so $T^2a = a$. Therefore for every $x \in A$ we have

$$d(Tx, a) = d(Tx, T^2a) \leq d(x, Ta),$$

i.e. T is P-center. □

Corollary 2.4. *Let (A, B) be a pair of two nonempty closed subsets of a complete metric space X such that A_0 is nonempty. Suppose $T : A \cup B \to A \cup B$ such that $T(A) \subseteq B$ and $T(B) \subseteq A$. Let $T(A_0) \subseteq B_0$ and $T_{|_A}$ is a weakly contractive mapping. Assume that the pair (A, B) has the P-property. Then $P_T(A) = A_0$ and are singleton.*

Proof. By Theorem 1.3 $P_T(A)$ is singleton and also by Proposition 2.2 and Theorem 2.3 $P_T(A) = A_0$. □

In the following, we introduce cyclic weakly contractive mapping and give new conditions for existence and uniqueness of best proximity point.

Definition 2.5. *Let (A, B) be a pair of two nonempty subsets of a complete metric space X. A map $T : A \cup B \to A \cup B$ is said to be a cyclic weakly contractive mapping if $T(A) \subset B$, $T(B) \subset A$ and*

$$d(Tx, Ty) \leq d(x, y) - \psi(d(x, y)), \ \forall x, y \in A \cup B,$$

where $\psi : [0, \infty) \to [0, \infty)$ is a function such that ψ is continuous and positive on $(0, d) \cup (d, \infty)$ and $\psi(0) = \psi(d) = 0$ that $d := dist(A, B)$.

Theorem 2.6. *Let (A, B) be a pair of two nonempty closed subsets of a complete metric space X such that $diam(A) < dist(A, B)$. Suppose $T : A \cup B \to A \cup B$ be a cyclic weakly contractive map. Then for $x_0 \in A$ and $x_{n+1} = Tx_n$, $\{x_{2n}\}$ converges to x and $d(x, Tx) = dist(A, B)$.*

Proof. Fix $x \in A \cup B$ and define a sequence $\{x_n\}$ in $A \cup B$ by $x_n = T^n x$, $n \in \mathbb{N}_0$. Set We divide the proof into 4 steps:

Step 1. $\lim_{n\to\infty} d(x_n, x_{n+1}) = dist(A, B)$. Note

$$d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1}) \leq d(x_n, x_{n+1}) - \psi(d(x_n, x_{n+1})).$$

Hence $\{d(x_n, x_{n+1})\}$ is monotonic decreasing and bounded below. Therefore $\lim_{n\to\infty} d(x_n, x_{n+1})$ exists. Let $\lim_{n\to\infty} d(x_n, x_{n+1}) = \delta \geq dist(A, B)$. Assume that $\delta > dist(A, B)$. By the continuity of ψ,

$$\delta = \lim_{n\to\infty} d(x_{n+1}, x_{n+2}) \leq \lim_{n\to\infty}[d(x_n, x_{n+1}) - \psi(d(x_n, x_{n+1}))] \leq \delta - \psi(\delta) < \delta,$$

so $\delta = dist(A, B)$.

Step 2. $\lim_{n\to\infty} d(x_{2n}, x_{2n+2}) = 0$. Note

$$\begin{aligned} d(x_{2n}, x_{2n+2}) &= d(Tx_{2n-1}, Tx_{2n+1}) \leq d(x_{2n-1}, x_{2n+1}) - \psi(d(x_{2n-1}, x_{2n+1})) \\ &\leq d(x_{2n-1}, x_{2n+1}) \leq d(x_{2n-2}, x_{2n}) - \psi(d(x_{2n-2}, x_{2n})) \end{aligned}$$

Hence $\{d(x_{2n}, x_{2n+2})\}$ is decreasing and bounded below. Hence $\lim_{n\to\infty} d(x_{2n}, x_{2n+2})$ exists. Let $\lim_{n\to\infty} d(x_{2n}, x_{2n+2}) = \delta$. It is clear that $0 \leq \delta < dist(A, B)$. Assume that $\delta > 0$. By the continuity of ψ,

$$\begin{aligned} \delta = \lim_{n\to\infty} d(x_{2n}, x_{2n+2}) &\leq \lim_{n\to\infty} d(x_{2n-1}, x_{2n+1}) \\ &\leq \lim_{n\to\infty} [d(x_{2n-2}, x_{2n}) - \psi(d(x_{2n-2}, x_{2n}))] \\ &\leq \delta - \psi(\delta) < \delta, \end{aligned}$$

so $\delta = 0$.

Step 3. $\{x_{2n}\}$ is Cauchy sequence. Assume that $\{x_{2n}\}$ is not Cauchy. Then there exist $\varepsilon > 0$ and integers $2m_k, 2n_k \in \mathbb{N}$ such that $2m_k > 2n_k \geq k$ and $d(x_{2n_k}, x_{2m_k}) \geq \varepsilon$ for $k = 0, 1, 2, \cdots$. Also, choosing m_k as small as possible, it may be assumed that

$$d(x_{2n_k}, x_{2m_k-2}) < \varepsilon.$$

Hence for each $k \in \mathbb{N}$, we have

$$\begin{aligned} \varepsilon \leq d(x_{2n_k}, x_{2m_k}) &\leq d(x_{2n_k}, x_{2m_k-2}) + d(x_{2m_k-2}, x_{2m_k}) \\ &\leq \varepsilon + d(x_{2m_k-2}, x_{2m_k}) \end{aligned}$$

and since $d(x_{2m_k-2}, x_{2m_k}) \to 0$, hence $\lim_{k\to\infty} d(x_{2n_k}, x_{2m_k}) = \varepsilon$. Observe that

$$d(x_{2n_k}, x_{2m_k}) \leq d(x_{2n_k}, x_{2n_k+2}) + d(x_{2n_k+2}, x_{2m_k+2}) + d(x_{2n_k+2}, x_{2m_k})$$

Letting $k \to \infty$, we obtain

$$\varepsilon = \lim_{k\to\infty} d(x_{2n_k+2}, x_{2m_k+2}) \leq \lim_{k\to\infty} d(x_{2n_k+1}, x_{2m_k+1}).$$

On the other hand

$$\lim_{k\to\infty} d(x_{2n_k+1}, x_{2m_k+1}) \leq \lim_{k\to\infty} [d(x_{2n_k}, x_{2m_k}) - \psi(d(x_{2n_k}, x_{2m_k}))].$$

So by and using the upper semicontinuity of ψ from the right we have

$$\varepsilon \leq \varepsilon - \psi(\varepsilon) < \varepsilon.$$

which is a contradiction. Hence $\{x_{2n}\}$ is a Cauchy sequence in A.

Step 4. Existence of best proximity pair. Because $\{x_{2n}\}$ is Cauchy, X is complete and A is closed, $\lim_{n\to\infty} x_{2n} = x \in A$. Now

$$dist(A, B) \leq d(x, x_{2n-1}) \leq d(x, x_{2n}) + d(x_{2n}, x_{2n-1}).$$

Thus, by step 1 we have $d(x, x_{2n-1})$ converges to $dist(A, B)$. Since

$$dist(A, B) \leq d(x_{2n}, Tx) \leq d(x_{2n-1}, x) - \psi(d(x_{2n-1}, x)),$$

therefore by upper semicontinuity of ψ we have

$$\begin{aligned} dist(A,B) &\leq \lim_{n\to\infty} d(x_{2n},Tx) \\ &\leq \lim_{n\to\infty}[d(x_{2n-1},x)-\psi(d(x_{2n-1},x))] \\ &= dist(A,B)-\psi(dist(A,B))=dist(A,B). \end{aligned}$$

so $d(x,Tx)=dist(A,B)$. □

Remark. *Under the cyclic weakly contractive mappings in Theorem 2.6, P-property, $A_0 \neq \emptyset$ and $T(A_0)\subseteq B_0$ conditions had been omitted with respect to Theorem 1.3.*

Theorem 2.7. *Let (A,B) be a pair of two nonempty closed covex subsets of a uniformly convex Banach space X. such that $diam(A) < dist(A,B)$. Suppose $T: A\cup B \to A\cup B$ is a cyclic weakly contractive map. Then there exist a unique $x\in A$ such that $\|x-Tx\| = dist(A,B)$. Further, if $x_0\in A$ and $x_{n+1}=Tx_n$, then $\{x_{2n}\}$ converges to the best proximity point.*

Proof. By Theorem 2.6 $P_T(A,B)\neq\emptyset$. Suppose $x,y\in P_T(A,B)$ such that $x\neq y$. Since $\|x-Tx\| = dist(A,B)$ and $\|y-Ty\| = dist(A,B)$ where necessarily uniformly convexity of X, $T^2x=x$ and $T^2y=y$. Since $x\neq y$, we have $dist(A,B)<\|Tx-y\|$ and so $\psi(\|Tx-y\|)>0$. Therefore $\|x-Ty\| = \|T^2x-Ty\| \leq \|Tx-y\|-\psi(\|Tx-y\|) < \|Tx-y\|$. Similarly $\|Tx-y\| < \|x-Ty\|$ that it is a contradiction. Therefore $x=y$. □

Example 1 Let A and B be subsets of $\mathbb{R}^2$ defined by

$$A=\{(x,0): x\geq 1\},\ B=\{(0,y): y\geq 1\}.$$

Suppose $T(x,y)=(\sqrt{y},\sqrt{x})$ and

$$\psi(t)=\begin{cases} \sqrt{t} & 0\leq t<\sqrt{2} \\ \frac{\sqrt{t-\sqrt{2}}}{2} & t\geq\sqrt{2}. \end{cases}$$

Then T is cyclic weakly contractive on $A\cup B$ and $\|(0,1)-T((1,0))\| = dist(A,B)$.

Proof. Here $dist(A,B)=\sqrt{2}$. For $(x,0)\in A$ and $(0,y)\in B$ we have

$$\begin{aligned} \|T(x,0)-T(0,y)\| &= \|(0,\sqrt{x})-(\sqrt{y},0)\| \\ &= \|(\sqrt{y},\sqrt{x})\| \\ &= \sqrt{x+y} \leq \sqrt{x^2+y^2}-\frac{\sqrt{\sqrt{x^2+y^2}-\sqrt{2}}}{2} \\ &= \|(x,0)-(0,y)\|-\psi(\|(x,0)-(0,y)\|). \end{aligned}$$

and for $(x,0),(y,0)\in A$ we have

$$\begin{aligned} \|T(x,0)-T(y,0)\| &= \|(0,\sqrt{x}-\sqrt{y})\| \\ &= |\sqrt{x}-\sqrt{y}| \\ &\leq \sqrt{x^2+y^2}-\sqrt{\sqrt{x^2+y^2}} \\ &= \|(x,0)-(y,0)\|-\psi(\|(x,0)-(y,0)\|). \end{aligned}$$

Then T is cyclic weakly contractive on $A\cup B$ and $\|(0,1)-T((1,0))\| = \sqrt{2} = dist(A,B)$. □

Acknowledgments. The authors would like to thank the anonymous referee for his/her comments that helped us improve this article.

References

[1] A. Abkar, M. Gabeleh, Best proximity points of non-self mappings, TOP, **21** (2013), 287-295.
[2] Ya.I. Alber, S. Guerre-Delabriere, Principle of weakly contractive maps in Hilbert spaces, in: New Results in Operator Theory and its Applications, **98** (1997), 7-22. 2012-231
[3] A. A.Eldred, W. A. Kirk, P. Veeramani, Proximinal normal structure and relatively nonexpansive mappings, Studia Math **171**(3)(2005)283-293.
[4] A. A. Eldred and P. Veeramani, Existence and convergence of best proximity points, J. Math. Anal. Appl. **323**(2006), 1001-1006.
[5] M. Gabeleh, Proximal Weakly Contractive and Proximal Nonexpansive Non-self-Mappings in Metric and Banach Spaces, J. Optim. Theory Appl. , 2012.
[6] M. R. Haddadi and S. M. Moshtaghioun, Some Results on the Best Proximity Pair, Abstract and Applied Analysis, **2011** (2011), ID 158430, 9 pages.
[7] S. Sadiq Basha, Best proximity points: global optimal approximate solution. J Global Optim. doi:10.1007/s10898-009-9521-0(2010)
[8] S. Sadiq Basha, Extensions of Banach contraction principle. Numer Funct Anal Optim **31**(2010), 569-576.
[9] Raj V. Sankar ,P. Veeramani, Best proximity pair theorems for relatively nonexpansive mappings, Appl Gen Topol **10**(1)(2009), 21-28
[10] Raj V. Sankar, A best proximity point theorem for weakly contractive non-self-mappings, Nonlinear Analysis **74** (2011), 4804-4808.
[11] J. Zhang, Y. Su and Q. Cheng, Best proximity point theorems for generalized contractions in partially ordered metric spaces, Fixed Point Theory and Applications 83(2013), 2013.

Mohammad Reza Haddadi
Faculty of Mathematics, Ayatollah Boroujerdi University, Boroujerd, Iran
E-mail address: haddadi@abru.ac.ir