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DYNAMICAL SYSTEMS AND VARIATIONAL INEQUALITIES

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Abstract. Dynamical systems arise naturally in numerous applied and theoretical fields including celestial mechanics, financial forecasting, environmental applications, neuroscience, brain modeling. It is known that the variational inequalities are equivalent to the fixed point problems. This alternative equivalent formulation is used to consider some projected dynamical system associated with variational inequalities. These dynamical systems are used to suggest some new and efficient implicit methods for solving variational inequalities and related optimization problems. The convergence analysis of the new proposed methods is considered under some mild conditions. Our methods of analysis is very simple as compared with other techniques. Several special cases are also discussed as applications of our main results. These results can be viewed as significant refinements of the previously known results.

1. Introduction

Variational inequality theory contains a wealth of new ideas and techniques. Variational inequality theory, which was introduced and considered in early sixties, can be viewed as a natural extension and generalization of the variational principles. It is amazing that a wide class of unrelated problems, which arise in various different branches of pure and applied sciences, can be studied in the general and unified framework of variational inequalities. For the applications, motivation, numerical results and other aspects of variational inequalities, see [1-25] and the references therein.

Related to variational inequalities, we have problem of dynamical systems. The dynamical system approach has been used to study the existence and stability of the solution of variational inequalities. In 1990's, it was shown that the variational inequalities can be reformulated in terms of dynamical system. For the applications and numerical methods for the dynamical systems, see [1, 2, 9, 5, 6, 7, 8, 13, 14, 15, 16, 17, 23, 24, 25].

In this paper, we consider the dynamical system using the equivalent fixed point formulation. We have shown that the dynamical system can be used to suggest

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some implicit and explicit iterative methods for solving variational inequalities. To the best of knowledge, this approach has not been considered previously. The convergence criteria of the proposed implicit methods is discussed under some mild conditions. Some special cases are also discussed. Results proved in this paper continue to hold for the special cases.

2. Formulations and basic facts

Let H be a real Hilbert space, whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ respectively. First of all, we recall some basic concepts and results, which are needed in the derivation of our results.

Definition 2.1. *Let K be any set in H. The set K is said to be convex set, if*

$$(1-t)u+tv \in K, \quad \forall u,v \in K, t \in [0,1].$$

Definition 2.2. *The function $F : K \to H$ is said to be a convex function, if*

$$F((1-t)u+tv) \leq (1-t)F(u)+tF(v), \quad \forall u,v \in K, t \in [0,1].$$

If the function F is differentiable, then we have the following well known result.

Theorem 2.1. *Let K be a closed and convex set in H. Let F be a differentiable convex function. Then $u \in K$ is the minimum of the differentiable convex $F(u)$ if and only if, $u \in K$ satisfies*

$$\langle F'(u), v-u\rangle \geq 0, \quad \forall v \in K, \tag{2.1}$$

where F' is the Frechet derivative of F at $u \in K$.

The inequality (2.1). is called the variational inequality. This shows that the variational inequalities arise naturally in connection with the minimization of the convex functional subject to certain constraint. This result motivated us to study a more general variational inequality of which (2.1) is a special case.

To be more precise, let $T : H \to H$ be a continuous nonlinear operator. We consider the problem of finding $u \in K$, such that

$$\langle Tu, v-u\rangle \geq 0, \quad \forall v \in K, \tag{2.2}$$

which is called the variational inequalities, introduced and studied by Stampacchia [22]. A wide class of problems arising in pure and applied sciences can be studied via strongly mixed variational inequalities (2.2). For the applications, numerical methods, dynamical systems and other aspects of variational inequalities and related problems, see [1-25] a and the references therein.

To obtain the main results, we recall some well-known concepts and results.

Lemma 2.2. *$\forall u,v \in H$, we have*

$$2\langle u,v\rangle = \|u+v\|^2 - \|u\|^2 - \|v\|^2, \tag{2.3}$$

Definition 2.3. *An operator $T : H \to H$ is said to be:*

(1) *Strongly monotone, if there exist a constant $\alpha > 0$, such that*

$$\langle Tu - Tv, u-v\rangle \geq \alpha\|u-v\|^2, \quad \forall u,v \in H.$$

(2) *Lipschitz continuous, if there exist a constant* $\beta > 0$, *such that*

$$\|Tu - Tv\| \leq \beta \|u - v\|, \quad \forall u, v \in H.$$

(3) *Monotone, if*

$$\langle Tu - Tv, u - v \rangle \geq 0, \quad \forall u, v \in H.$$

Remark. *If the operator* T *is both strongly monotone with constant* $\alpha > 0$ *and Lipschitz continuous with constant* $\beta > 0$, *then* $\alpha \leq \beta$.

The projection operator has the following characterization.

Lemma 2.3. *Let* K *be a closed and convex set in* H. *Then, for a given* $z \in H$, $u \in K$ *satisfies the inequality*

$$\langle u - z, v - u \rangle \geq 0, \quad \forall v \in K, \tag{2.4}$$

if and only if

$$u = P_K(z),$$

where P_K *is the projection operator.*

It is well known that the projection operator P_K is nonexpansive.

3. Main results

In this section, we consider the projected dynamical systems associated with variational inequalities. We investigate the convergence analysis of these new methods involving only the monotonicity of the operator. For this purpose, we need the following equivalent result between variational inequalities and the fixed point problem.

Lemma 3.1. [14] *The function* $u \in K$ *is a solution of the variational inequalities* (2.2) *if and only if* $u \in K$ *satisfies the relation*

$$u = P_K(u - \rho Tu), \tag{3.1}$$

where P_K *is the projection operator and* $\rho > 0$ *is a constant.*

Lemma 3.1 implies that the variational inequalities (2.2) is equivalent to the fixed point problem (3.1). This alternative equivalent formulation is very useful from the numerical and theoretical point of view.

We now define the residue vector $R(u)$ by the relation

$$R(u) = u - P_K[u - \rho Tu]. \tag{3.2}$$

Invoking Lemma 3.1, one can easily conclude that $u \in K$ is a solution of (2.2) if and only if $u \in K$ is a zero of the equation

$$R(u) = 0. \tag{3.3}$$

We now consider a projected dynamical system associated with the variational inequalities. Using the equivalent formulation (3.1), we suggest a class of projected dynamical systems as

$$\frac{du}{dt} = \lambda P_K[u - \rho Tu] - u\}, \quad u(t_0) = u_0 \in K, \tag{3.4}$$

where λ is a parameter. The system of type (4.1) is called the projected dynamical system associated with variational inequalities (2.2). Here the right hand is related to the resolvent and is discontinuous on the boundary. From the definition, it is clear that the solution of the dynamical system always stays in H. This implies that the qualitative results such as the existence, uniqueness and continuous dependence of the solution of (4.1) can be studied. These projected dynamical systems associated with the variational inequalities (2.12), which have been studied extensively. We use the projected dynamical system (4.1) to suggest some iterative for solving variational inequalities (2.2). These methods can be viewed in the sense of Koperlevich [11] and Noor [14] involving the double resolvent operator.

For simplicity, we take $\lambda = 1$. Thus the dynamical system (4.1) becomes

$$\frac{du}{dt} + u = P_K[u - \rho Tu], \quad u(t_0) = \alpha. \tag{3.5}$$

We construct the implicit iterative method using the forward difference scheme. Discretizing (3.5), we have

$$\frac{u_{n+1} - u_n}{h} + u_{n+1} = P_K[u_n - \rho Tu_{n+1}], \tag{3.6}$$

where h is the step size. Now, using (3.6), we can suggest the following implicit iterative method for solving the variational inequality (2.2).

Algorithm 3.1. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = P_K\left[u_n - \rho Tu_{n+1} - \frac{u_{n+1} - u_n}{h}\right], \quad n = 0, 1, 2, \ldots.$$

This is a new implicit method and is quite different from the implicit method of [11]. Using Lemma 2.3, Algorithm 3.1 can be rewritten in the equivalent form as:

Algorithm 3.2. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\langle \rho Tu_{n+1} + \{\frac{(1+h)u_{n+1} - (1+h)u_n}{h}\}, v - u_{n+1}\rangle \geq 0, \quad \forall v \in K, \quad n = 0, 1, 2, \ldots. \tag{3.7}$$

We now study the convergence analysis of algorithm 3.1 under some mild conditions.

Theorem 3.2. *Let $u \in K$ be a solution of variational inequality (2.2). Let u_{n+1} be the approximate solution obtained from (3.6). If T is monotone, then*

$$\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - \|u_n - u_{n+1}\|^2. \tag{3.8}$$

Proof. Let $u \in K$ be a solution of (2.2). Then

$$\langle Tv, v - u\rangle \geq 0, \quad \forall v \in K, \tag{3.9}$$

since T is a monotone operator.
Set $v = u_{n+1}$ in (3.9), to have

$$\langle Tu_{n+1}, u_{n+1} - u\rangle \geq 0. \tag{3.10}$$

Take $v = u$ in algorithm 3.7, we have

$$\langle \rho Tu_{n+1} + \{\frac{(1+h)u_{n+1} - (1+h)u_n}{h}\}, u - u_{n+1}\rangle \geq 0. \tag{3.11}$$

From (3.10) and (3.11), we have

$$\langle (1+h)u_{n+1} - (1+h)u_n, u - u_{n+1}\rangle \geq 0. \tag{3.12}$$

Using Lemma 2.2 and from (3.12), we obtain

$$\|u_{n+1} - u\|^2 \leq \|u - u_n\|^2 - \|u_{n+1} - u_n\|^2. \tag{3.13}$$

the required result. □

Theorem 3.3. *Let $u \in K$ be the solution of variational inequality (2.2). Let u_{n+1} be the approximate solution obtained from (3.6). If T is a monotone operator, then u_{n+1} converges to $u \in H$ satisfying (2.2).*

Proof. Let T be a monotone operator. Then, from (3.8), it follows that the sequence $\{u_i\}_{i=1}^{\infty}$ is a bounded sequence and

$$\sum_{i=1}^{\infty} \|u_n - u_{n+1}\|^2 \leq \|u - u_0\|^2,$$

which implies that

$$\lim_{n\to\infty} \|u_{n+1} - u_n\|^2 = 0. \tag{3.14}$$

Since sequence $\{u_i\}_{i=1}^{\infty}$ is bounded, so there exists a cluster point $\hat{u}$ to which the subsequence $\{u_{ik}\}_{k=1}^{\infty}$ converges. Taking limit in (3.7) and using (3.14), it follows that $\hat{u} \in K$ satisfies

$$\langle T\hat{u}, v - \hat{u}\rangle \geq 0, \quad \forall v \in K,$$

and

$$\|u_{n+1} - u\|^2 \leq \|u - u_n\|^2.$$

Using this inequality, one can show that the cluster point $\hat{u}$ is unique and

$$\lim_{n\to\infty} u_{n+1} = \hat{u}.$$

□

We now suggest an other implicit iterative method for solving (2.5). Discretizing (3.5), we have

$$\frac{u_{n+1} - u_n}{h} + u_{n+1} = P_K[u_{n+1} - \rho T u_{n+1}], \tag{3.15}$$

where h is the step size.
This formulation enable us to suggest the following iterative method.

Algorithm 3.3. *For a given $u_0 \in K$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = P_K\left[u_{n+1} - \rho T u_{n+1} - \frac{u_{n+1} - u_n}{h}\right], \quad n = 0, 1, 2, \ldots.$$

Algorithm 3.4. *For a given $u_0 \in K$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = P_K\left[u_{n+1} - \rho T u_{n+1} - \frac{u_{n+1} - u_n}{h}\right], \quad n = 0, 1, 2, \ldots.$$

Using lemma 2.3, algorithm 3.6 can be rewritten in the equivalent form as:

Algorithm 3.5. *For a given $u_0 \in K$, compute u_{n+1} by the iterative scheme*

$$\langle \rho T u_{n+1} + \{\frac{u_{n+1} - u_n}{h}\}, v - u_{n+1}\rangle \geq 0, \quad \forall v \in K. \tag{3.16}$$

Again using the dynamical systems, we can suggested some iterative methods for solving the variational inequalities and related optimization problems.

Algorithm 3.6. *For a given $u_0 \in K$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = P_K\left[\frac{(h+1)u_n - u_{n+1}}{h} - \rho T u_n\right], \quad n = 0, 1, 2, \ldots,$$

which can be written in the equivalent form as

Algorithm 3.7. *For a given $u_0 \in K$, compute u_{n+1} by the iterative scheme*

$$\langle \rho T u_n + \{\frac{h+1}{h}(u_{n+1} - u_n)\}, v - u_{n+1}\rangle \geq 0, \quad \forall v \in K. \tag{3.17}$$

In a similar way, technique of the projected dynamical systems can be used to suggest a number of iterative methods for solving the variational inequalities and related problems.

4. Remarks and open problems

We would like to emphasize that equivalence between the variational inequalities and fixed point problem can be used to suggest the second order dynamical system. To be more precise,we consider the problem of finding u such that

$$\nu\frac{d^2u}{d^2t} + \gamma\frac{du}{dt} = \lambda P_K[u - \rho T u] - u\}, \quad u(t_0) = u_0, \quad \frac{du(t_0)}{dt} = \beta \tag{4.1}$$

is the second order dynamical systems associated with variational inequalities. where ν, λ, γ and β are constants. Using the discretization of the (4.1), we can suggest some inertial type iterative methods for solving the variational inequalities and related problems. For example, using forward finite difference schemes and $\lambda = -1$, we have the following iterative method.

Algorithm 4.1. *For a given u_0, find the approximate solution by the iterative scheme*

$$\nu\frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} + \gamma\frac{u_{n+1} - u_n}{h} + u_{n+1} = P_K[u_n - \rho T u_{n+1}], \quad n = 0, 1, 2, ...,$$

which can be written as

$$\langle \rho T u_{n+1} + u_{n+1} - \frac{(1-\gamma h)u_{n+1} + (h^2 - \gamma h - 2)u_n - u_{n_1}}{h^2}, v - u_{n+1}\rangle \geq 0, \quad \forall v \in K. \tag{4.2}$$

The implementation and convergence analysis of such type of inertial iterative methods need further efforts and is an interesting problem for future research. We have only tried to give some glimpse of the applications of the dynamical systems approach. This technique is used to suggest and analyze some inertial proximal methods for solving mixed variational inequalities, see [1].

Conclusion

In this paper we have introduced and considered variational inequalities. Using the equivalence between the variational inequalities and fixed point problems, we have considered some dynamical systems associated with variational inequalities. Using the finite difference schemes, we have shown that these dynamical systems can be used to suggest various implicit and explicit iterative methods for solving variational inequalities and related optimization problems. It is expected that the technique of the dynamical system may stimulate further research.

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