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SOME COMMON FIXED POINTS RESULTS ALONG WITH THE COMMON LIMIT IN THE RANGE OF PROPERTY FOR GENERALIZED CONTRACTIVE MAPPINGS

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ABSTRACT. In this work, we give some common fixed point results for two nonlinear mappings satisfying generalized contractive condition. The common limit in the range of property ($CLR-$property) which was introduced in [1] is used in proving our results. The presented results extend, generalize, and improve previous results.

1. INTRODUCTION

A large number of mathematicians studied in fixed point theorems for solving problems in applied mathematics and the greater number of other sciences. At the beginning, Banach proved a theorem, which is well known as Banach's Fixed Point Theorem to establish the existence of solutions for integral equations. Because of Banach's fixed point theorem is modesty, utility and applications, it has become a very favorite tools in solving the existence problems in many branches of mathematical analysis. Moreover, many authors have improved, extended and generalized Banach's fixed point theorem in various ways.

Khan *et al.* [2] presented a new category of contractive fixed point problems. In this study, the notion of altering distance function which is a control function that alters distance between two points in a metric space was introduced. Some of which are noted in [3]-[6] have been used this function and its extensions in several problems of fixed point theory.

Afterward, Jungck and Rhoades [7] introduced a new concept of weakly compatible maps to produce fixed point theorems in metric space for set valued noncontinuous functions.

In recently, Sintunavarat and Kumam [1] proposed a new concept of the common limit in the range of g property in fuzzy metric spaces. Theirs theorems are improve and generalize the main results of Mihet in (Mihet, 2010) and many fixed point theorems in fuzzy metric spaces.

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On the other hand, Huseyin *et al.* [12] introduced the notion of T-cyclic (α, β)-contraction and give some common fixed point results in metric space for this type of contractions.

2. Preliminaries

In this section, we will introduce the important definitions which are used in this paper.

Definition 2.1. [2] A function $\varphi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied:

(1) φ is nondecreasing and continuous,
(2) $\varphi(t) = 0$ if and only if $t = 0$.

Definition 2.2. [8] Let X be a nonempty set and $f, T : X \to X$. We say that a point $x \in X$ is a coincidence point of f and T if $f(x) = T(x)$.

Definition 2.3. [7] Let X be a nonempty set and $f, T : X \to X$. f and T is said to be weakly compatible if f and T commute at their coincidence points (i.e. $fTx = Tfx$ whenever $fx = Tx$). A point $y \in X$ is called a point of coincidence of f and T if there exists a point $x \in X$ such that $y = fx = Tx$.

Following the direction in [9], we denote by Ψ the family of all continuous functions $\psi : \mathbb{R}_+^4 \to \mathbb{R}_+$ such that:

(1) ψ is nondecreasing in each coordinate,
(2) $\psi(t, t, t, t) \leq t, \psi(t, 0, 0, t) \leq t$ and $\psi(0, 0, t, \frac{1}{2}t) \leq t$ for all $t > 0$,
(3) $\psi(t_1, t_2, t_3, t_4) = 0$ if and only if $t_1 = t_2 = t_3 = t_4 = 0$.

Definition 2.4. [1] Let (X, d) be a metric space. Two mappings $f : X \to X$ and $g : X \to X$ are said to satisfy the common limit in the range of T property if there exists sequences $\{x_n\}$ in X such that

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} T(x_n) = T(x), \text{ for some } x \in X.$$

In what follows, the common limit in the range of g property will be denoted by the (CLR_T) property. If T is an identity mapping, then f is said to satisfy (CLR_I) property

The aim of this work is to introduce the common fixed point results for two nonlinear mappings are satisfying generalized contractive condition by using the common limit in the range of property due to Sintunavarat and Kumam [1] which are improve the main results of Huseyin *et al.* [12].

3. The main results

First of all, we acquainted the following definitions which will be used completely in the proof of main results as follows.

Definition 3.1. Let (X, d) be a metric space and $f, T : X \to X$ two given mappings. We say that f is a (T, φ, ψ, η)-contractive mapping if

$$\varphi(d(fx, fy)) \leq \eta(M(x, y)), \tag{3.1}$$

for all $x, y \in X$, where

$$M(x,y) = \psi(d(Tx,Ty), d(Tx,fx), d(Ty,fy), \frac{1}{2}[d(Tx,fy) + d(Ty,fx)])$$

$\psi \in \Psi, \varphi$ is an altering distance function and $\eta : [0,\infty) \to [0,\infty)$ is a nondecreasing function and continuous from the right with the condition $\varphi(t) > \eta(t)$ for all $t > 0$.

Theorem 3.2. *Let (X,d) be a metric space and let f and T be self-mappings on X. Suppose that f is a (T,φ,ψ,η)-contractive mapping and f and T satisfy the (CLR_T) property. Then f and T have a unique point of coincidence in X. Moreover, if f and T are weakly compatible, then f and T have a unique common fixed point.*

Proof. Since f and T satisfy the (CLR_T) property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} T(x_n) = Tx, \text{ for some } x \in X.$$

Now, applying inequality (3.1), we get

$$\varphi(d(fx_n, fx)) \leq \eta(M(x_n,x)), \tag{3.2}$$

where

$$\begin{aligned} M(x_n,x) &= \psi(d(Tx_n,Tx), d(Tx_n,fx_n), d(Tx,fx), \frac{1}{2}[d(Tx_n,fx) + d(Tx,fx_n)]) \\ &\leq \psi(d(Tx_n,Tx), d(Tx_n,fx_n), d(Tx,fx), \frac{1}{2}\max[d(Tx,fx), d(Tx_n,fx) + d(Tx,fx_n)]). \end{aligned}$$

Suppose that $d(Tx,fx) > 0$, taking $n \to \infty$ in the inequality (3.2) and using the properties of φ, ψ, η and the above inequality we have

$$\begin{aligned} \varphi(d(Tx,fx)) &\leq \eta(\psi(0,0,d(Tx,fx), \frac{1}{2}d(Tx,fx)) \\ &\leq \eta(d(Tx,fx)) < \varphi(d(Tx,fx)). \end{aligned}$$

This is a contradiction. Therefore, $d(Tx,fx) = 0$ that is,

$$Tx = fx. \tag{3.3}$$

Thus, Tx is a point of coincidence for f and T. The uniqueness of the point of coincidence is a consequence of the conditions (3.1), and so we omit the details. By (3.3) and using the weak compatibility of f and T, we obtain

$$fTx = Tfx = TTx. \tag{3.4}$$

Uniqueness of the point of coincidence implies $Tx = fTx = TTx$. Consequently, Tx is a common fixed point of f and T. The unique of common fixed point of f and T follows from the conditions (3.1). □

Example 3.3. *Let $X = [0,5]$ be endowed with the usual metric $d(x,y) = |x-y|$ for all $x,y \in X$. Also, let $\varphi(t) = 5t$ and $\eta(t) = 3t$ for all $t > 0$, and $\psi(t_1,t_2,t_3,t_4) = \frac{1}{2}\max\{t_1,t_2,t_3,t_4\}$ for all $t_1,t_2,t_3,t_4 > 0$. Now, define the self-mappings f and T on X by*

$$fx = \sqrt{x^2 - 8x + 40} \qquad \textit{for all} \qquad x \in X$$

and

$$Tx = \frac{2x+5}{3} \qquad \textit{for all} \qquad x \in X.$$

Suppose that f is a (T, φ, ψ, η)-contractive mapping, we have

$$\begin{aligned}
\varphi(d(fx, fy)) &= 5|fx - fy| \\
&= 5|\sqrt{x^2 - 8x + 40} - \sqrt{y^2 - 8y + 40}| \\
&\leq 5\frac{1}{5}|x - y| = \frac{3}{2}\frac{2}{3}|x - y| = \frac{3}{2}|\frac{2x}{3} - \frac{2y}{3}| \\
&= \frac{3}{2}|\frac{2x+5}{3} - \frac{2y+5}{3}| = \frac{3}{2}(d(Tx, Ty)) \\
&\leq 3\frac{1}{2}\max\{d(Tx, Ty), d(Tx, fx), d(Ty, fy), \frac{1}{2}\max[d(Ty, fy), d(Tx, fy) + d(Ty, fx)]\} \\
&= 3M(x, y) = \eta(M(x, y)),
\end{aligned}$$

Therefore, f is a (T, φ, ψ, η)-contractive mapping by the mean-value theorem. Let $\{x_n\}$ be a sequence defined by $\{x_n\} = 5 - \frac{1}{n}$. Then $\{x_n\}$ be a sequence in X.

$$\begin{aligned}
\lim_{n\to\infty} f(5 - \frac{1}{n}) &= \lim_{n\to\infty}\sqrt{(5 - \frac{1}{n})^2 - 8(5 - \frac{1}{n}) + 40} \\
&= \sqrt{\lim_{n\to\infty}((5 - \frac{1}{n})^2 - 8(5 - \frac{1}{n}) + 40)} = 5 \\
&= \lim_{n\to\infty}\frac{2(5 - \frac{1}{n}) + 5}{3} \\
&= \lim_{n\to\infty} T(5 - \frac{1}{n}) = T(5).
\end{aligned}$$

Thus, f and T satisfy the (CLR_T) property. By Theorem 3.2,therefore f and T have a point of coincidence in X. Next, we will show that f and T are weakly compatible, we get

$$fTx = \sqrt{(\frac{2x+5}{3})^2 - 8(\frac{2x+5}{3}) + 40} = Tfx \qquad \textit{for all} \qquad x \in X.$$

Consequently, all conditions of Theorem 3.2 hold, and hence f and T have a unique common fixed point.

Corollary 3.4. *Let (X, d) be a metric space and let f and T be self-mappings on X. Suppose that f and T satisfy the (CLR_T) property and*

$$\varphi(d(fx, fy)) \leq \eta(M(x, y)), \tag{3.5}$$

for all $x, y \in X$, where φ is an altering distance function and $\eta : [0, \infty) \to [0, \infty)$ is a nondecreasing function and continuous from the right with the condition $\varphi(t) > \eta(t)$ for all $t > 0$ and

$$M(x, y) = \max\{d(Tx, Ty), d(Tx, fy), d(Ty, fy), \frac{1}{2}[d(Tx, fy) + d(Ty, fx)]\}.$$

Then f and T have a unique point of coincidence in X. Moreover, if f and T are weakly compatible, then f and T have a unique common fixed point.

We have the following corollary when we contained $T = I_X$ in Theorem 3.2.

Corollary 3.5. *Let (X, d) be a metric space and let $f : X \to X$. Suppose that f satisfy the (CLR_I) property and*

$$\varphi(d(fx, fy)) \le \eta(M_f(x, y)), \tag{3.6}$$

for all $x, y \in X$, where $\psi \in \Psi$, φ is an altering distance function and $\eta : [0, \infty) \to [0, \infty)$ is a nondecreasing function and continuous from the right with the condition $\varphi(t) > \eta(t)$ for all $t > 0$ and

$$M_f(x, y) = \psi(d(x, y), d(x, fy), d(y, fy), \frac{1}{2}[d(x, fy) + d(y, fx)]).$$

Then f has a unique fixed point.

When we choose $\eta(t) = \varphi(t) = \eta^1(t)$ in Corollary 3.5, we have the following corollary.

Corollary 3.6. *Let (X, d) be a metric space and $f : X \to X$. Suppose that f satisfy the (CLR_I) property and*

$$\varphi(d(fx, fy)) \le \varphi(M_f(x, y)) - \eta^1(M_f(x, y)), \tag{3.7}$$

for all $x, y \in X$, where $\psi \in \Psi$, φ is an altering distance function and $\eta^1 : [0, \infty) \to [0, \infty)$ is such that $\varphi(t) - \eta^1(t)$ is a nondecreasing function and $\eta^1(t)$ is continuous from the right, with the condition $\varphi(t) > \eta^1(t)$ for all $t > 0$. Then f has a unique fixed point.

When we take $\varphi(t) = t$ in Corollary 3.6, we have the following corollary.

Corollary 3.7. *Let (X, d) be a metric space and $f : X \to X$. Suppose that f satisfy the (CLR_I) property and*

$$\varphi(d(fx, fy)) \le M_f(x, y) - \eta^1(M_f(x, y)), \tag{3.8}$$

for all $x, y \in X$, where $\psi \in \Psi$ and $\eta^1 : [0, \infty) \to [0, \infty)$ is such that $t - \eta^1(t)$ is a nondecreasing function and $\eta^1(t)$ is continuous from the right, with the condition $\eta^1(t) > 0$ for all $t > 0$. Then f has a unique fixed point.

We denote by Φ the family of all continuous functions $\phi : \mathbb{R}_+^4 \to \mathbb{R}_+$ such that

(1) ϕ is nondecreasing in each coordinate,
(2) $\phi(t, t, t, t) \le t, \phi(t, \frac{t}{2}, t, 0) \le t$ and $\phi(0, \frac{t}{2}, 0, t) \le t$ for all $t > 0$,
(3) $\phi(t_1, t_2, t_3, t_4) = 0$ if and only if $t_1 = t_2 = t_3 = t_4 = 0$.

Definition 3.8. Let (X, d) be a metric space and let f and T be two given mappings. We say that f is a (T, φ, η, ϕ)-contractive mapping if

$$\varphi(d(fx, fy)) \le \eta(N(x, y)), \tag{3.9}$$

for all $x, y \in X$, where

$$N(x, y) = \phi(d(Tx, Ty), \frac{1}{2}d(Tx, fy), d(Ty, fx), \frac{[1 + d(Tx, fx)]d(Ty, fy)}{1 + d(Tx, Ty)}),$$

$\phi \in \Phi, \varphi$ is an altering distance function and $\eta : [0, \infty) \to [0, \infty)$ is a nondecreasing function and continuous from the right with the condition $\varphi(t) > \eta(t)$ for all $t > 0$.

Theorem 3.9. *Let (X, d) be a metric space and let f and T be self-mappings on X. Suppose that f is a (T, φ, η, ϕ)-contractive mapping and f and T satisfy the (CLR_T) property. Then f and T have a unique point of coincidence in X. Moreover, if f and T are weakly compatible, then f and T have a unique common fixed point.*

Proof. Since f and T satisfy the (CLR_T) property, there exists a sequence in such that $\{x_n\}$ in X such that

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} T(x_n) = Tx, \text{ for some } x \in X.$$

Now, applying (3.9), we get

$$\varphi(d(fx_n, fx)) \leq \eta(N(x_n, x)), \tag{3.10}$$

where

$$N(x, y) = \phi(d(Tx_n, Tx), \frac{1}{2}d(Tx_n, fx), d(Tx, fx_n), \frac{[1 + d(Tx_n, fx_n)]d(Tx, fx)}{1 + d(Tx_n, Tx)}).$$

Suppose that $d(Tx, fx) > 0$, taking $n \to \infty$ in the inequality (3.10) and using the properties of φ, η, ϕ and the previous inequality, we have

$$\begin{aligned}\varphi(d(Tx, fx)) &\leq \eta(\phi(0, \frac{1}{2}d(Tx, fx), 0, d(Tx, fx))) \\ &\leq \eta(d(Tx, fx)) < \varphi(d(Tx, fx)),\end{aligned}$$

This is a contradiction. Therefore, $d(Tx, fx) = 0$ that is,

$$Tx = fx. \tag{3.11}$$

Thus, Tx is a point of coincidence for f and T. The uniqueness of the point of coincidence is a consequence of the conditions (3.9), and so we omit the details. By (3.11) and using the weak compatibility of f and T, we obtain

$$fTx = Tfx = TTx. \tag{3.12}$$

Uniqueness of the point of coincidence implies $Tx = fTx = TTx$. Consequently, Tx is a common fixed point of f and T.The unique of common fixed point of f and T follows from the conditions (3.9). □

Example 3.10. *Let $X = [0, \infty)$ be endowed with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$. Also, let $\varphi(t) = t$ and $\eta(t) = \frac{t}{7}$ for all $t > 0$, and $\phi(t_1, t_2, t_3, t_4) = \frac{t_1+t_2+t_3+t_4}{5}$ for all $t_1, t_2, t_3, t_4 > 0$. Now, define the self-mappings f and T on X by*

$$fx = \frac{x}{35} \qquad \textit{for all} \qquad x \in X$$

and

$$Tx = x \qquad \textit{for all} \qquad x \in X.$$

Suppose that f is a a (T,φ,η,ϕ)-contractive mapping, we have

$$\begin{aligned}\varphi(d(fx,fy)) &= |fx-fy| = \frac{1}{35}|x-y|\\ &\leq \frac{1}{35}[|x-y|+\frac{1}{2}|x-\frac{y}{35}|+|y-\frac{x}{35}|+\frac{[1+|x-\frac{x}{35}|]|y-\frac{y}{35}|}{1+|x-y|}]\\ &= \frac{1}{7}[\frac{|x-y|+\frac{1}{2}|x-\frac{y}{35}|+|y-\frac{x}{35}|+\frac{[1+|x-\frac{x}{35}|]|y-\frac{y}{35}|}{1+|x-y|}}{5}]\\ &= \frac{1}{7}N(x,y) = \eta(N(x,y)).\end{aligned}$$

Then, there exists a sequence $\{x_n\} = \frac{1}{n}$ in X such that

$$\lim_{n\to\infty} f(\frac{1}{n}) = \lim_{n\to\infty}(\frac{1}{35n}) = 0 = \lim_{n\to\infty}(\frac{1}{n}) = \lim_{n\to\infty} T(\frac{1}{n}) = T(0).$$

Thus, f and T satisfy the (CLR_T) property. By Theorem 3.9,therefore f and T have a point of coincidence in X. Next, we will show that f and T are weakly compatible, we get

$$fTx = \frac{x}{35} = Tfx \qquad \text{for all} \qquad x \in X.$$

Consequently, all conditions of Theorem 3.9 hold, and hence f and T have a unique common fixed point.

Corollary 3.11. *Let (X,d) be a metric space and let f and T be self-mappings on X. Suppose that f and T satisfy the (CLR_T) property and*

$$\varphi(d(fx,fy)) \leq \eta(N(x,y)), \tag{3.13}$$

for all $x,y \in X$, where φ is an altering distance function and $\eta:[0,\infty)\to[0,\infty)$ is a nondecreasing function and continuous from the right with the condition $\varphi(t) > \eta(t)$ for all $t > 0$ and

$$N(x,y) = \max\{d(Tx,Ty), \frac{1}{2}d(Tx,fy), d(Ty,fy), \frac{[1+d(Tx,fx)]d(Ty,fy)}{1+d(Tx,Ty)}\}.$$

Then f and T have a unique point of coincidence in X. Moreover, if f and T are weakly compatible, then f and T have a unique common fixed point.

We have the following corollary if we take $T = I_X$ and $\eta(t) = \varphi(t) - \eta^1(t)$ in Theorem 3.9.

Corollary 3.12. *Let (X,d) be a metric space and $f: X \to X$. Suppose that f satisfy the (CLR_I) property and*

$$\varphi(d(fx,fy)) \leq \varphi((N_f(x,y) - \eta^1(N_f(x,y)), \tag{3.14}$$

for all $x,y \in X$, where $\phi \in \Phi$, φ is an altering distance function and $\eta^1:[0,\infty)\to[0,\infty)$ is such that $\varphi(t)-\eta^1(t)$ is a nondecreasing function and $\eta^1(t)$ continuous from the right with the condition $\varphi(t) > \eta^1(t)$ for all $t > 0$ and

$$N_f(x,y) = \phi(d(x,y), \frac{1}{2}d(x,fy), d(y,fx), \frac{[1+d(x,fx)]d(y,fy)}{1+d(x,y)}).$$

Then f has a unique fixed point.

If we choose $\varphi(t) = t$ in Corollary 3.12, we have the following corollary.

Corollary 3.13. *Let (X, d) be a metric space and $f : X \to X$. Suppose that f satisfy the (CLR_I) property and*

$$d(fx, fy) \leq (N_f(x, y) - \eta^1(N_f(x, y), \tag{3.15}$$

for all $x, y \in X$, where $\eta^1 : [0, \infty) \to [0, \infty)$ is such that $t - \eta^1(t)$ is a nondecreasing function and $\eta^1(t)$ continuous from the right with the condition $\eta^1(t) > 0$ for all $t > 0$. Then f has a unique fixed point.

Next, we will explain Lemma which is reduce to the (CLR_T) property. When we finished to proof, ours main results can reduce to main results of Huseyin *et al.* [12] and improvement.

Lemma 3.14. *Let (X, d) be a complete metric space and f and T be self-mappings on X such that $f(x) \subset T(x)$. Suppose that f is a (T, φ, ψ, η)-contractive mapping and $T(x)$ is a closed subset of X. Then f and T satisfy the (CLR_T) property.*

Proof. Let $x \in X$ and define the sequences x_n and y_n in X by

$$y_n = fx_n = Tx_{n+1}, n \in \mathbb{N} \cup \{0\}. \tag{3.16}$$

Therefore by (3.1) and using (3.16), we get

$$\varphi(d(y_n, yn + 1)) = \varphi(d(fx_n, fx_{n+1})) \leq \eta(M(x_n, x_{n+1})) < \varphi(M(x_n, x_{n+1})), \tag{3.17}$$

Since φ is nondecreasing, we have

$$d(y_n, y_{n+1}) < M(x_n, x_{n+1}), \tag{3.18}$$

where

$$\begin{aligned} M(x_n, x_{n+1}) &= \psi(d(Tx_n, Tx_{n+1}), d(Tx_n, fx_n), d(Tx_{n+1}, fx_{n+1}), \frac{1}{2}[d(Tx_n, fx_{n+1}) + d(Tx_{n+1}, fx_n)]) \\ &= \psi(d(y_{n-1}, y_n), d(y_{n-1}, y_n), d(y_n, y_{n+1}), \frac{1}{2}[d(y_{n-1}, y_{n+1}) + d(y_n, y_n)]) \\ &\leq (d(y_{n-1}, y_n), d(y_{n-1}, y_n), d(y_n, y_{n+1}), \frac{1}{2}[d(y_{n-1}, y_n) + d(y_n, y_{n+1})]). \end{aligned} \tag{3.19}$$

Thus, from (3.18), we obtain

$$\begin{aligned} d(y_n, y_{n+1}) &< M(x_n, x_{n+1}) \\ &\leq \psi(d(y_{n-1}, y_n), d(y_{n-1}, y_n), d(y_n, y_{n+1}), \frac{1}{2}[d(y_{n-1}, y_n) + d(y_n, y_{n+1})]). \end{aligned}$$

If $d(y_{n-1}, y_n) \leq d(y_n, y_{n+1})$ for some $n \in N$, then

$$\begin{aligned} d(y_n, y_{n+1}) &< \psi(d(y_{n-1}, y_n), d(y_{n-1}, y_n), d(y_n, y_{n+1}), \frac{1}{2}[d(y_{n-1}, y_n) + d(y_n, y_{n+1})]) \\ &\leq \psi(d(y_n, y_{n+1}), d(y_n, y_{n+1}), d(y_n, y_{n+1}), d(y_n, y_{n+1})) \\ &\leq d(y_n, y_{n+1}), \end{aligned}$$

which is a contradiction, and hence $d(y_n, y_{n+1}) < d(y_{n-1}, y_n)$ for all $n \in N$. Therefore, the sequence $d(y_n, y_{n+1})$ is decreasing and bounded below. Thus, there exists

$r \geq 0$ such that $\lim_{n\to\infty} d(y_n, y_{n+1}) = r$. Assume $r > 0$. Also, from (3.17), (3.19) and using the properties of ψ, we deduce

$$\begin{aligned}
\varphi(d(y_n, y_{n+1}) &\leq \eta(M(x_n, x_{n+1})) \\
&\leq \eta(\psi(d(y_{n-1}, y_n), d(y_{n-1}, y_n), d(y_n, y_{n+1}), \frac{1}{2}[d(y_{n-1}, y_n) + d(y_n, y_{n+1})])) \\
&\leq \eta(\psi(d(y_{n-1}, y_n), d(y_{n-1}, y_n), d(y_{n-1}, y_n))) \\
&\leq \eta(d(y_{n-1}, y_n)). \qquad (3.20)
\end{aligned}$$

Considering the properties of φ and η, letting $n \to \infty$ in (3.20), we get

$$\begin{aligned}
\varphi(r) &= \lim_{n\to\infty} \varphi(d(y_n, y_{n+1}) \\
&\leq \lim_{n\to\infty} \eta(d(y_{n-1}, y_n) = \eta(r) < \varphi(r),
\end{aligned}$$

which implies $r = 0$ and so

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0. \qquad (3.21)$$

Now, we prove that $\{y_n\}$ is a Cauchy sequence. Suppose, to the contrary, that $\{y_n\}$ is not a Cauchy sequence. Then there exists an $\varepsilon > 0$ for which we can find two subsequences $\{y_{m_k}\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ such that m_k is the smallest index for which $m_k > n_k > k$ and

$$d(y_{m_k}, y_{n_k}) \geq \varepsilon \qquad \text{and} \qquad d(y_{m_k-1}, y_{n_k}) < \varepsilon. \qquad (3.22)$$

Using the triangular inequality and (3.22), we have

$$\begin{aligned}
\varepsilon \leq d(y_{n_k}, y_{m_k}) &\leq d(y_{n_k}, y_{m_k-1}) + d(y_{m_k-1}, y_{m_k}) \\
&< \varepsilon + d(y_{m_k-1}, y_{m_k}).
\end{aligned}$$

By taking $k \to \infty$ in the above inequality and using (3.21), we obtain

$$\lim_{k\to\infty} d(y_{n_k}, y_{m_k}) = \varepsilon. \qquad (3.23)$$

By using (3.21), (3.23), and the triangular inequality, we deduce

$$\begin{aligned}
\lim_{k\to\infty} d(y_{n_k-1}, y_{m_k}) &= \varepsilon, \\
\lim_{k\to\infty} d(y_{m_k-1}, y_{n_k}) &= \varepsilon, \\
\lim_{k\to\infty} d(y_{m_k-1}, y_{n_k-1}) &= \varepsilon. \qquad (3.24)
\end{aligned}$$

From (3.1), we get

$$\begin{aligned}
\varphi(d(y_{n_k}, y_{m_k})) &= \varphi(d(fx_{n_k}, fx_{m_k})) \\
&\leq \eta(M(x_{n_k}, x_{m_k})), \qquad (3.25)
\end{aligned}$$

where

$$\begin{aligned}
M(x_{n_k}, x_{m_k}) &= \psi(d(Tx_{n_k}, Tx_{m_k}), d(Tx_{n_k}, fx_{n_k}), d(Tx_{m_k}, fx_{m_k}), \\
&\qquad \frac{1}{2}[d(Tx_{n_k}, fx_{m_k}) + d(Tx_{m_k}, fx_{n_k})]) \\
&\leq \psi(\max\{\varepsilon, d(y_{n_k-1}, y_{m_k-1})\}, d(y_{n_k-1}, y_{n_k}), d(y_{m_k-1}, y_{m_k}), \\
&\qquad \max\{\varepsilon, \frac{1}{2}[d(y_{n_k-1}, y_{m_k}) + d(y_{m_k-1}, y_{n_k})]\}).
\end{aligned}$$

Now, from the properties of φ, ψ and η and using (3.21), (3.23), (3.24), and the above inequality, as $k \to \infty$ in (3.25), we have

$$\varphi(\varepsilon) \leq \eta(\psi(\varepsilon, 0, 0, \varepsilon)) \leq \eta(\varepsilon) < \varphi(\varepsilon),$$

which implies that $\varepsilon = 0$, a contradiction with $\varepsilon > 0$. Thus $\{y_n\}$ is a Cauchy sequence in X. From the completeness of (X, d), there exists $z \in X$ such that

$$\lim_{n\to\infty} y_n = z. \tag{3.26}$$

From (3.16) and (3.26), we obtain

$$fx_n \to z \qquad \text{and} \qquad Tx_{n+1} \to z. \tag{3.27}$$

Since TX is closed, by (3.27), $z \in TX$. Therefore, there exists $u \in X$ such that $Tu = z$. Then $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Tx_n = Tu$. Thus f and T satisfy the (CLR_T) property. □

Theorem 3.15. *Let (X, d) be a complete metric space and f and T be self-mappings on X such that $f(x) \subset T(x)$. Suppose that f is a (T, φ, ψ, η)-contractive mapping and $T(x)$ is a closed subset of X. Then f and T have a unique point of coincidence in X. Moreover, if f and T are weakly compatible, then f and T have a unique common fixed point.*

Proof. By using Lemma 3.14 and Theorem 3.2, we get this result. □

Lemma 3.16. *Let (X, d) be a complete metric space and let f and T be self-mappings on X such that $f(x) \subset T(x)$. Suppose that f is a (T, φ, η, ϕ)-contractive mapping and $T(x)$ is a closed subset of X. Then f and T satisfy the (CLR_T) property.*

Proof. Similar to the proof of Lemma 3.14, we define sequences $\{x_n\}$ and $\{y_n\}$ in X by $y_n = fx_n = Tx_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Also we assume that $y_n \neq y_{n-1}$ for all $n \in \mathbb{N}$. Then by (3.9), we have

$$\varphi(d(y_n, yn+1)) = \varphi(d(fx_n, fx_{n+1})) \leq \eta(N(x_n, x_{n+1})) < \varphi(N(x_n, x_{n+1})), \tag{3.28}$$

Since φ is nondecreasing, we get

$$d(y_n, y_{n+1}) < N(x_n, x_{n+1}), \tag{3.29}$$

where

$$\begin{aligned} N(x_n, x_{n+1}) &= \phi(d(Tx_n, Tx_{n+1}), \frac{1}{2}d(Tx_n, fx_{n+1}), d(Tx_{n+1}, fx_n), \\ &\qquad \frac{[1 + d(Tx_n, fx_n)]d(Tx_{n+1}, fx_{n+1})}{1 + d(Tx_n, Tx_{n+1})}) \\ &= \phi(d(y_{n-1}, y_n), \frac{1}{2}d(y_{n-1}, y_{n+1},), d(y_n, y_n), \frac{[1 + d(y_{n-1}, y_n)]d(y_n, y_{n+1})}{1 + d(y_{n-1}, y_n)}) \\ &\leq \phi(d(y_{n-1}, y_n), \frac{1}{2}[d(y_{n-1}, y_n) + d(y_n, y_{n+1})], 0, d(y_n, y_{n+1})). \end{aligned} \tag{3.30}$$

Thus, from (3.29), we deduce

$$\begin{aligned} d(y_n, y_{n+1}) &< N(x_n, x_{n+1}) \\ &\leq \phi(d(y_{n-1}, y_n), \frac{1}{2}[d(y_{n-1}, y_n) + d(y_n, y_{n+1})], 0, d(y_n, y_{n+1})). \end{aligned}$$

If $d(y_{n-1}, y_n) \leq d(y_n, y_{n+1})$ for some $n \in N$, then

$$\begin{aligned} d(y_n, y_{n+1}) &< \phi(d(y_{n-1}, y_n), \frac{1}{2}[d(y_{n-1}, y_n) + d(y_n, y_{n+1})], 0, d(y_n, y_{n+1})) \\ &\leq \phi(d(y_n, y_{n+1}), d(y_n, y_{n+1}), d(y_n, y_{n+1}), d(y_n, y_{n+1})) \\ &\leq d(y_n, y_{n+1}), \end{aligned}$$

which is a contradiction, and hence $d(y_n, y_{n+1}) < d(y_{n-1}, y_n)$ for all $n \in N$. Therefore, the sequence $\{d(y_n, y_{n+1})\}$ is decreasing and bounded from below. Thus, there exists $\delta \geq 0$ such that $\lim_{n\to\infty} d(y_n, y_{n+1}) = \delta$. Also, from (3.28), (3.30) and using the properties of φ and η, we obtain

$$\begin{aligned} \varphi(d(y_n, y_{n+1})) &\leq \eta(N(x_n, x_{n+1})) \\ &\leq \eta(\phi(d(y_{n-1}, y_n), \frac{1}{2}[d(y_{n-1}, y_n) + d(y_n, y_{n+1})], 0, d(y_n, y_{n+1}))) \\ &\leq \eta(\phi(d(y_{n-1}, y_n), d(y_{n-1}, y_n), d(y_{n-1}, y_n), d(y_{n-1}, y_n))) \\ &\leq \eta(d(y_{n-1}, y_n)) < \varphi(d(y_{n-1}, y_n)). \end{aligned} \tag{3.31}$$

Consider the properties of φ and η, letting $n \to \infty$ in (3.31), we get

$$\begin{aligned} \varphi(\delta) &= \lim_{n\to\infty} \varphi(d(y_n, y_{n+1})) \\ &\leq \lim_{n\to\infty} \eta(d(y_{n-1}, y_n)) = \eta(\delta) < \varphi(\delta), \end{aligned}$$

which implies $\delta = 0$ and so

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0. \tag{3.32}$$

Now, we want to show that $\{y_n\}$ is a Cauchy sequence. Suppose, to the contrary, that $\{y_n\}$ is not a Cauchy sequence. Then there exists an $\varepsilon > 0$ for which we can find two subsequences $\{y_{m_k}\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ such that m_k is the smallest index for which $m_k > n_k > k$ and

$$d(y_{n_k}, y_{m_k}) \geq \varepsilon \quad \text{and} \quad d(y_{m_k-1}, y_{m_k}) < \varepsilon. \tag{3.33}$$

Using the triangular inequality and (3.33), we have

$$\begin{aligned} \varepsilon \leq d(y_{m_k}, y_{n_k}) &\leq d(y_{m_k}, y_{n_k-1}) + d(y_{n_k-1}, y_{n_k}) \\ &< \varepsilon + d(y_{n_k-1}, y_{n_k}). \end{aligned}$$

By taking $k \to \infty$ in the above inequality and using (3.32), we obtain

$$\lim_{k\to\infty} d(y_{m_k}, y_{n_k}) = \varepsilon. \tag{3.34}$$

By using (3.32), (3.34), and the triangular inequality, we deduce

$$\begin{aligned} \lim_{k\to\infty} d(y_{n_k-1}, y_{m_k}) &= \varepsilon, \\ \lim_{k\to\infty} d(y_{m_k-1}, y_{n_k}) &= \varepsilon, \\ \lim_{k\to\infty} d(y_{m_k-1}, y_{n_k-1}) &= \varepsilon. \end{aligned} \tag{3.35}$$

From (3.9), we get

$$\begin{aligned}\varphi(d(y_{n_k}, y_{m_k})) &= \varphi(d(fx_{n_k}, fx_{m_k})) \\ &\leq \eta(N(x_{n_k}, x_{m_k})), \end{aligned} \tag{3.36}$$

where

$$\begin{aligned} N(x_{n_k}, x_{m_k}) &= \phi(d(Tx_{n_k}, Tx_{m_k}), \frac{1}{2}d(Tx_{n_k}, fx_{m_k}), d(Tx_{m_k}, fx_{n_k}), \\ &\qquad \frac{[1 + d(Tx_{n_k}, fx_{n_k})]d(Tx_{m_k}, fx_{m_k})}{1 + d(Tx_{n_k}, Tx_{m_k})}) \\ &= \phi(d(y_{n_k-1}, y_{m_k-1}), \frac{1}{2}d(y_{n_k-1}, y_{m_k},), d(y_{m_k-1}, y_{n_k}), \\ &\qquad \frac{[1 + d(y_{n_k-1}, y_{n_k})]d(y_{m_k-1}, y_{m_k})}{1 + d(y_{n_k-1}, y_{m_k-1})}) \\ &\leq \max\{\varepsilon, N(x_{n_k}, x_{m_k})\} \\ &= \phi(\max\{\varepsilon, d(y_{n_k-1}, y_{m_k-1})\}, \frac{1}{2}\max\{\varepsilon, d(y_{n_k-1}, y_{m_k})\}, \\ &\qquad \max\{\varepsilon, d(y_{m_k-1}, y_{n_k}), \frac{[1 + d(y_{n_k-1}, y_{n_k})]d(y_{m_k-1}, y_{m_k})}{1 + d(y_{n_k-1}, y_{m_k-1})}\}). \end{aligned}$$

Therefore $\lim_{k\to\infty} \max\{\varepsilon, N(x_{n_k}, x_{m_k})\} = \phi(\varepsilon, \frac{\varepsilon}{2}, \varepsilon, 0) \leq \varepsilon$. Now, from the properties of φ and η and using (3.32), (3.34), (3.35), and the previous inequality, as $k \to \infty$ in (3.36), we have

$$\begin{aligned}\varphi(\varepsilon) &\leq \lim_{n\to\infty} \varphi(d(y_n, y_{n+1})) \\ &\leq \lim_{n\to\infty} \eta(\max\{\varepsilon, N(x_{n_k}, x_{m_k})\}) \\ &\leq \eta(\varepsilon) < \varphi(\varepsilon),\end{aligned}$$

which implies that $\varepsilon = 0$, a contradiction with $\varepsilon > 0$. Thus $\{y_n\}$ is a Cauchy sequence in X. From the completeness of (X, d), there exists $w \in X$ such that

$$\lim_{n\to\infty} y_n = w \tag{3.37}$$

and so by (3.37), we obtain

$$fx_n \to w \qquad \text{and} \qquad Tx_{n+1} \to w. \tag{3.38}$$

Since TX is closed, by (3.38), $w \in TX$. Therefore, there exists $v \in X$ such that $Tv = w$. Then $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Tx_n = Tv$. Thus f and T satisfy the (CLR_T) property. □

Theorem 3.17. *Let (X, d) be a metric space and let f and T be self-mappings on X such that $f(x) \subset T(x)$. Suppose that f is a (T, φ, η, ϕ)-contractive mapping and $T(x)$ is a closed subset of X. Then f and T have a unique point of coincidence in X. Moreover, if f and T are weakly compatible, then f and T have a unique common fixed point.*

Proof. By using Lemma 3.16 and Theorem 3.9, we get this result. □

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