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SYMMETRY IDENTITIES FOR THE 2-VARIABLE UNIFIED APOSTOL-TYPE POLYNOMIALS

BURAK KURT*, SECIL BILGIC

Abstract. In this paper, we introduce and investigate 2-variable unified Apostol-type polynomials. We obtain some symmetry identities between these polynomials and the generalized sum of integer powers. We give explicit relation for this unified family. Also, we prove some recurrence relation for these polynomials.

1. Introduction

Throughout this paper, we always make use of the following notations: $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N}_0$ denotes the set of nonnegative integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers.

The generalized Apostol-Bernoulli polynomials $B_n^{(\alpha)}(x,\lambda)$ of order $\lambda \in \mathbb{N}_0$, the generalized Apostol-Euler polynomials $E_n^{(\alpha)}(x,\lambda)$ of order $\alpha \in \mathbb{N}_0$ and the generalized Apostol-Genocchi polynomials $G_n^{(\alpha)}(x,\lambda)$ of order $\alpha \in \mathbb{N}_0$ are defined by the following generating functions ([5], [15]-[18], [20]) respectively,

$$\sum_{n=0}^{\infty} B_n^{(\alpha)}(x,\lambda)\frac{t^n}{n!} = \left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} e^{xt},\ |t| < 2\pi,\ (\text{when } \lambda = 1,\ \ |t| < |\log\lambda| \ \ \text{when } \lambda \neq 1), \tag{1}$$

$$\sum_{n=0}^{\infty} E_n^{(\alpha)}(x,\lambda)\frac{t^n}{n!} = \left(\frac{2}{\lambda e^t + 1}\right)^{\alpha} e^{xt},\ (|t| < \pi, \text{ when } \lambda = 1,\ |t| < |\log(-\lambda)|, \text{ when } \lambda \neq 1) \tag{2}$$

and

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$$\sum_{n=0}^{\infty} G_n^{(\alpha)}(x,\lambda)\frac{t^n}{n!} = \left(\frac{2t}{\lambda e^t+1}\right)^{\alpha} e^{xt},\ (|t|<\pi \text{ when } \lambda=1,\quad |t|<|\log(-\lambda)| \quad \text{when } \lambda\neq 1)\,. \tag{3}$$

The multiplier power sums are defined by Luo [16] as follows

$$S_k^{(l)}(m;\lambda) = \sum_{\substack{0\leq v_1\leq\ldots\leq v_m=l\\ v_1+\ldots+v_m}} \binom{l}{v_1,v_2,\ldots,v_m} \lambda^{v_1+2v_2+\ldots+mv_m}\left(v_1+2v_2+\ldots+mv_m\right)^{(k)}. \tag{4}$$

From 4 , we have

$$\left(\frac{1-\lambda^m e^{mt}}{1-\lambda e^t}\right)^l = \frac{1}{\lambda^l}\sum_{n=0}^{\infty}\left\{\sum_{p=0}^{n}\binom{n}{p}(-l)^{n-p} S_k^{(l)}(m;\lambda)\frac{t^n}{n!}\right\} \tag{5}$$

(see detail in [16]).

Ozden et al in [19] defined the generalized Stirling numbers of the second kind as

$$\sum_{n=0}^{\infty} S(n,v,a,b,\beta)\frac{t^n}{n!} = \frac{\left(\beta^b e^t - a^b\right)^v}{v!}. \tag{6}$$

Ozarslan in [2] defined the unified Apostol-Bernoulli, Euler and Genocchi polynomials as

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} P_{n,\beta}^{(\alpha)}(x,k,a,b)\frac{t^n}{n!}, \tag{7}$$

where $k\in\mathbb{N}_0$, a, $b\in\mathbb{R}\setminus\{0\}$, α, $\beta\in\mathbb{C}$.

Khan et al in([9]-[11]) defined the 2-variable general polynomial family as

$$e^{xt}\phi(y,t) = \sum_{n=0}^{\infty} P_n(x,y)\frac{t^n}{n!}\ ,\ \ P_0(x,y)=1 \tag{8}$$

where

$$\phi(y,t) = \sum_{n=0}^{\infty}\phi_n(y)\frac{t^n}{n!},\ \phi_0(y)\neq 0.$$

Khan et al in ([9]-[11]) gave some basic relations and explicit relations for this polynomial.

We define the 2-variable unified Apostol-Bernoulli, Euler and Genocchi polynomials ${}_pP_{n,\beta}^{(\alpha)}(x,y;k,a,b)$ of order α as

$$\sum_{n=0}^{\infty} {}_pP_{n,\beta}^{(\alpha)}(x,y;k,a,b)\frac{t^n}{n!} = \left(\frac{2^{1-k}\ t^k}{\beta^b e^t - a^b}\right)^{(\alpha)} e^{xt}\phi(y,t) \tag{9}$$

where

$$\sum_{n=0}^{\infty} P_n(x,y)\frac{t^n}{n!} = e^{xt}\phi(y,t)\quad ,\quad \phi(y,t) = \sum_{n=0}^{\infty}\phi_n(y)\frac{t^n}{n!},\ \phi_0(y)\neq 0.$$

Luo in ([15]-[18]) introduced and investigated for the Apostol-Bernoulli polynomials, the Apostol-Euler polynomials and Apostol-Genocchi polynomials. He

proved the multiplication formulas for Apostol-type polynomials. He proved some recurrence relation and explicit relationships for these polynomials. Srivastava et al in ([20]-[23]) gave some generalizations,proved the some theorems, recurrence relations for the Apostol-Bernoulli, Euler, Genocchi polynomials.

Kurt in ([13]-[14]) proved some identities and symmetric relations for these polynomials.

Ozden et al in [19] gave unified representation for these polynomials. Ozarslan ([1], [2]) proved some relation and symmetric relations for the unified Apostol-type polynomials and Hermite-based Apostol-Bernoulli, Euler and Genocchi polynomials.

Khan et al ([9]-[11]) introduced and proved some relations for the 2-variable Apostol-type polynomials.

In this work, we define the 2-variable unified Apostol-type polynomials ${}_pP_{n,\beta}^{(\alpha)}(x,y;k,a,b)$ of order $\alpha \in \mathbb{N}_0$. We give some basic relationships for these polynomials. Also, we prove some symmetric relations between for these polynomials.

2. Explicit Relation For The 2-Variable Unified Apostol-Type Polynomials

In this section, we aim to obtain the explicit relation of the polynomials ${}_pP_{n,\beta}^{(\alpha)}(x,y;k,a,b)$. We prove some relations for these polynomials and give the relations between the 2-variable unified family of generalized Apostol-type polynomials and the Stirling numbers of second kind $S(n,v,a,b,\beta)$ of order v.

For $\alpha = 1$, we write again the equation (9) as

$$F(x,y;k,a,b,\beta,t) = \sum_{n=0}^{\infty} {}_pP_{n,\beta}(x,y;k,a,b)\frac{t^n}{n!} = \left(\frac{2^{1-k}\ t^k}{\beta^b e^t - a^b}\right) e^{xt}\phi(y,t). \tag{10}$$

We can obtain the following equations easily from (9)

$$F^{(\alpha)}(x+1,y;k,a,b,\beta,t) = e^t F^{(\alpha)}(x,y;k,a,b,\beta,t), \tag{11}$$

$$F\left(x,k,a,b,\beta,t\right) F\left(w,k,a,b,\beta,t\right) = F^{(2)}(0,k,a,b,\beta,t)e^{(x+w)t}, \tag{12}$$

$$(\beta^b e^t + a^b) F\left(x,y;k,a^2,b,\beta^2,2t\right) = 2^k F\left(2x,y,k,a,b,\beta,2t\right) \tag{13}$$

and

$$F\left(x,y;k,a,b,\beta,t\right) F\left(u,0;k,a,b,\beta,t\right) = F\left(k,a,b,\beta,t\right) F\left(x+u,y;k,a,b,\beta,t\right). \tag{14}$$

Theorem 1. *The 2-variable unified Apostol-type polynomials satisfy the following equations*

$${}_pP_{n,\ \beta}^{(\alpha)}(x+1,y;k,a,b) = \sum_{l=0}^{n} \binom{n}{l}\ {}_pP_{n-l,\ \beta}^{(\alpha)}(x,y;k,a,b), \tag{15}$$

$$\begin{aligned}&\sum_{m=0}^{n}\binom{n}{m}\ {}_pP_{m,\ \beta}(x,y;k,a,b)\ {}_pP_{n-m,\ \beta}(w,y;k,a,b)\\ =&\sum_{r=0}^{n}\binom{n}{r}\ {}_pP^{(2)}_{r,\ \beta}(0,y;k,a,b)\,(x+w)^{n-r}\,,\end{aligned}\tag{16}$$

$$\begin{aligned}&\beta^b\sum_{m=0}^{n}\binom{n}{m}\ {}_pP_{m,\ \beta^2}(x,y,a^2,b)2^m+a^b\ {}_pP_{n,\ \beta^2}(x,y;k,a^2,b)2^n\\ =&\ 2^k\sum_{m=0}^{n}\binom{n}{m}\ {}_pP_{m,\ \beta}(x,2y;k,a,b)x^{n-m}\end{aligned}\tag{17}$$

and

$$\begin{aligned}&\sum_{m=0}^{n}\binom{n}{m}\ {}_pP_{m,\ \beta}(x,y;k,a,b)\ {}_pP_{n-m,\ \beta}(u,0;k,a,b)\\ =&\sum_{m=0}^{n}\binom{n}{m}\ {}_pP_{m,\ \beta}(0,0;k,a,b)\ {}_pP_{n-m,\ \beta}(x+u,y;k,a,b).\end{aligned}\tag{18}$$

Proof. (15) can be find easily. By using (9), (15), (17), (18) equation can be prove easily from (12), (13) and (14) respectively. ∎

Theorem 2. *There is the following relation between the Stirling numbers of the Second kind $S(n,v,a,b,\beta)$ and the 2-variable unified Apostol-type polynomials*

$${}_pP^{(\alpha-1)}_{n-k,\ \beta}(x,y;k,a,b)=\frac{(n-k)!}{2^k n!}\sum_{l=0}^{n}\binom{n}{l}\ {}_pP^{(\alpha)}_{n-l,\ \beta}(x,y;k,a,b)S\left(l,1,a,b,\beta\right)\tag{19}$$

Proof. From (6) and (9), we write as

$$\begin{aligned}\sum_{n=0}^{\infty}{}_pP^{(\alpha-1)}_{n,\ \beta}(x,y;k,a,b)\frac{t^n}{n!}&=\frac{1}{2^{1-k}}\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)^{(\alpha)}e^{xt}\phi\left(y,t\right)\frac{\beta^b e^t-a^b}{t^k}\\ \sum_{n=0}^{\infty}{}_pP^{(\alpha-1)}_{n,\ \beta}(x,y;k,a,b)\frac{t^{n+k}}{n!}&=\sum_{m=0}^{\infty}{}_pP^{(\alpha)}_{m,\ \beta}(x,y;k,a,b)\frac{t^m}{m!}\sum_{l=0}^{\infty}S(l,1,a,b,\beta)\frac{t^l}{l!}.\end{aligned}$$

By using Cauchy product and comprising the coefficient of $\frac{t^n}{n!}$, we have (19). ∎

Theorem 3. *The following statements of the 2-variable unified Apostol-type polynomials hold:*

(1) $\frac{d}{dx}\ {}_pP^{(\alpha)}_{n,\ \beta}(x,y;k,a,b)=n\ {}_pP^{(\alpha)}_{n-1,\ \beta}(x,y;k,a,b)$
(2) ${}_pP^{(\alpha)}_{n,\ \beta}(x,y;k,a,b)=\sum_{l=0}^{n}\binom{n}{l}\ {}_pP^{(\alpha)}_{n-1,\ \beta}(k,a,b)P_l\left(x,y\right)$
(3) ${}_pP^{(\alpha+\gamma)}_{n,\ \beta}(x+u,y;k,a,b)=\sum_{m=0}^{n}\binom{n}{m}{}_pP^{(\alpha)}_{m,\ \beta}(x,y;k,a,b)\ {}_pP^{(\gamma)}_{n-m,\ \beta}(u,y,k,a,b)$
(4) $\beta^b\ {}_pP^{(\alpha)}_{n,\ \beta}(x+1,y;k,a,b)=2^{1-k}\frac{n!}{(n-k)!}\ {}_pP^{(\alpha-1)}_{n-k,\ \beta}(x,y;k,a,b)$
(5) $\int_{x_0}^{x_1}{}_pP^{(\alpha)}_{n-1,\ \beta}(x,y;k,a,b)dx=\frac{1}{n}\left\{\ {}_pP^{(\alpha)}_{n,\ \beta}(x_1,y;k,a,b)-\ {}_pP^{(\alpha)}_{n,\ \beta}(x_0,y;k,a,b)\right\}$

3. Some Symmetry Identities for the 2-Variable Unified Apostol-type Polynomials

Kurt in ([13], [14]) proved some symmetry identities for the unified Apostol-type polynomials. Ozarslan [2] proved some relation for the Unified Apostol-Bernoulli, Euler, Genocchi polynomials.

In this section, we give new symmetry identities for the 2-variable unified Apostol-type polynomials.

Theorem 4. *The following symmetry relations for the* 2*-variable unified Apostol-type polynomials hold true:*

$$\begin{aligned}
&\sum_{m=0}^{n}\binom{n}{m}\left\{\sum_{i=0}^{c-1}\left(\frac{\beta}{a}\right)^{bi}{}_{p}P_{m,\beta}^{(\alpha)}\left(x+\frac{di}{c};dy;k,a,b\right)c^{m}\right.\\
&\left.\sum_{j=0}^{d-1}\left(\frac{\beta}{a}\right)^{jb}{}_{p}P_{n-m,\beta}^{(\alpha)}\left(X+\frac{cj}{d};cY;k,a,b\right)d^{n-m}\right\}\\
=&\sum_{m=0}^{n}\binom{n}{m}\left\{\sum_{i=0}^{d-1}\left(\frac{\beta}{a}\right)^{bi}{}_{p}P_{m,\beta}^{(\alpha)}\left(x+\frac{ci}{d};cy;k,a,b\right)d^{m}\right.\\
&\left.\sum_{j=0}^{c-1}\left(\frac{\beta}{a}\right)^{jb}{}_{p}P_{n-m,\beta}^{(\alpha)}\left(X+\frac{dj}{d};dY;k,a,b\right)c^{n-m}\right\}. \qquad (20)
\end{aligned}$$

Proof. Let $f(t)$

$$\begin{aligned}
f(t) &= \frac{2^{2(1-k)\alpha}t^{2k\alpha}e^{cdt}\phi(y,cdt)\left(\beta^{bc}e^{cdt}-a^{bc}\right)\left(\beta^{bd}e^{cdt}-a^{bd}\right)e^{dxt}\phi\left(Y,cdt\right)}{\left(\beta^{b}e^{ct}-a^{b}\right)^{\alpha+1}\left(\beta^{b}e^{ct}-a^{b}\right)^{\alpha+1}}\\
&= \frac{1}{(cd)^{k\alpha}}\left\{\left(\frac{2^{1-k}\left(ct\right)^{k}}{\beta^{b}e^{ct}-a^{b}}\right)^{(\alpha)}e^{cdx}\phi\left(y,cdt\right)a^{bc-b}\left(\frac{\left(\frac{\beta}{a}\right)^{bc}e^{cdt}-1}{\left(\frac{\beta}{a}\right)^{b}e^{dt}-1}\right)\right.\\
&\left.\left(\frac{2^{1-k}\left(dt\right)^{k}}{\beta^{b}e^{dt}-a^{b}}\right)^{(\alpha)}e^{dXt}\phi\left(Y,cdt\right).a^{bd-b}\left(\frac{\left(\frac{\beta}{a}\right)^{bd}e^{cdt}-1}{\left(\frac{\beta}{a}\right)^{b}e^{ct}-1}\right)\right\}
\end{aligned}$$

since the expression for $f\left(t\right)$ is symmetric in c and d . We can expand $f\left(t\right)$ into series in

$$\begin{aligned}
&= \frac{1}{(cd)^{k\alpha}}\left\{\left(\frac{2^{1-k}\left(ct\right)^{k}}{\beta^{b}e^{ct}-a^{b}}\right)^{(\alpha)}e^{ctx}\phi\left(dy,ct\right)a^{bc-b}\sum_{i=0}^{c-1}\left(\frac{\beta}{a}\right)^{bi}e^{dti}\left(\frac{2^{1-k}\left(dt\right)}{\beta^{b}e^{dt}-a^{b}}\right)^{(\alpha)}e^{dXt}\right.\\
&\left.\phi\left(cY,dt\right)a^{bd-b}\sum_{j=0}^{d-1}\left(\frac{\beta}{a}\right)^{jb}e^{ctj}\right\}
\end{aligned}$$

By using the Cauchy product,We have

$$= \frac{1}{(cd)^{k\alpha}} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} \left\{ \sum_{i=0}^{c-1} \left(\frac{\beta}{a}\right)^{bi} \; {}_pP_{m,\beta}^{(\alpha)} \left(x + \frac{di}{c}, dy; k, a, b\right) c^m a^{bc+bd-2b} \right.$$
$$\left. \sum_{j=0}^{d-1} \left(\frac{\beta}{a}\right)^{jb} \; {}_pP_{n-m,\beta}^{(\alpha)} \left(X + \frac{c}{d} j, cY : k, a, b\right) d^{n-m} \frac{t^n}{n!} \right. \tag{21}$$

in a similar manner,

$$f(t) = \frac{1}{(cd)^{k\alpha}} \left(\frac{2^{1-k} dt}{\beta^b e^{dt} - a^b}\right)^{(\alpha)} e^{dxt} \phi(cy, dt) a^{bd-b} \left\{ \sum_{i=0}^{d-1} \left(\frac{\beta}{a}\right)^{bi} e^{ict} \left(\frac{2^{1-k} ct}{\beta^b e^{ct} - a^b}\right)^{(\alpha)} \right.$$
$$\left. e^{Xct} \phi(dY, ct) a^{bc-b} \sum_{j=0}^{c-1} \left(\frac{\beta}{a}\right)^{bj} e^{djt} \right\}$$
$$= \frac{1}{(cd)^{k\alpha}} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} \left\{ \sum_{i=0}^{d-1} \left(\frac{\beta}{a}\right)^{bi} \; {}_pP_{m,\beta}^{(\alpha)} \left(x + \frac{ic}{d}, cy; k, a, b\right) d^m a^{bd+bc-2b} \right.$$
$$\left. \sum_{j=0}^{d-1} \left(\frac{\beta}{a}\right)^{bj} \; {}_pP_{n-m,\beta}^{(\alpha)} \left(X + \frac{dj}{c}, dY; k; a, b\right) c\hat{}n - m \right\} \frac{t^n}{n!} \tag{22}$$

∎

From, (21) and (22) ,we have (20).

Theorem 5. *This is the following relations holds true.*

$$\sum_{m=0}^{n} \binom{n}{m} \; {}_pP_{m,\beta}^{(\alpha)} (x, dy; k, a, b) c^m \; {}_pP_{n-m,\beta}^{(\alpha)} (X, cY; k, a, b) d^{n-m}$$
$$= \sum_{n=0}^{m} \binom{n}{m} \; {}_pP_{m,\beta}^{(\alpha)} (X, cy; k, a, b) d^m \; {}_pP_{n-m,\beta}^{(\alpha)} (x, dY; k, a, b) c^{n-m} \tag{23}$$

Theorem 6. *There is the following relation between the multiplier power sums and the 2-variable unified Apostol-type polynomials*

$$d^{kl} \sum_{l=0}^{n} \binom{n}{l} \left\{ {}_pP_{n-l,\beta}^{(\alpha+1)} (dx, dy; k, a, b) c^{n-l} \sum_{s=0}^{l} \binom{l}{s} \sum_{p=0}^{s} \binom{s}{p} (-\alpha)^{s-p} \right.$$
$$\left. S_k^{(\alpha)} \left(c, \left(\frac{\beta}{a}\right)^b\right) \; {}_pP_{l-s,\beta} (cX, cY; k, a, b) d^l \right\}$$
$$= c^{kl} \sum_{l=0}^{n} \binom{n}{l} \left\{ {}_pP_{n-l,\beta}^{(\alpha+1)} (cX, cY; k, a, b) d^{n-l} \sum_{s=0}^{l} \binom{l}{s} \sum_{p=0}^{s} \binom{s}{p} (-\alpha)^{s-p} \right.$$
$$\left. S_k^{(\alpha)} \left(d, \left(\frac{\beta}{a}\right)^b\right) \; {}_pP_{l-s,\beta} (dx, dy; k, a, b) c^l \right\}. \tag{24}$$

Proof. Let

$$g(t)=\frac{t^{k(\alpha+2)}2^{(1-k)(\alpha+2)}e^{cdxt}\phi(y,cdt)\left(\beta^{bd}e^{cdt}-a^{bd}\right)^{\alpha}e^{cdXt}\phi(Y,cdt)}{\left(\beta^{b}e^{ct}-a^{b}\right)^{\alpha+1}\left(\beta^{b}e^{dt}-a^{b}\right)^{\alpha+1}}$$

From the $g(t)$ is symmetric in c and d, we write as

$$=\left\{\frac{1}{c^{k(\alpha+1)}d^{k}}\left(\frac{(ct)^{k}2^{1-k}}{\beta^{b}e^{ct}-a^{b}}\right)^{(\alpha+1)}e^{dxct}\phi(dy,ct)\left(\frac{\beta^{bd}e^{cdt}-a^{bd}}{\beta^{b}e^{ct}-a^{b}}\right)^{\alpha}\right.$$
$$\left.\left(\frac{(dt)^{k}2^{1-k}}{\beta^{b}e^{dt}-a^{b}}\right)e^{dt(Xc)}\phi(cY;dt)\right\}.$$

By using the equation (5) in above equality, we write as

$$\frac{1}{c^{k(\alpha+1)}d^{k}}\sum_{n=0}^{\infty}\left\{{}_{p}P_{n,\beta}^{(\alpha+1)}(dx,dy;k,a,b)\,c^{n}\frac{t^{n}}{n!}a^{bd\alpha}\beta^{-b\alpha}\sum_{s=0}^{\infty}\sum_{p=0}^{s}\binom{s}{p}(-\alpha)^{s-p}\right.$$
$$\left.S_{k}^{(\alpha)}\left(c,\left(\frac{\beta}{a}\right)^{b}\right)d^{s}\frac{t^{s}}{s!}\sum_{n=0}^{\infty}{}_{p}P_{n,\beta}(cX,cY;k,a,b)\,d^{n}\right\}\frac{t^{n}}{n!}.$$

Using cauchy product, we get

$$=\frac{1}{c^{k(\alpha+1)}d^{k}}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}\left\{{}_{p}P_{n-l,\beta}^{(\alpha+1)}(dx,dy;k,a,b)\,c^{n-l}a^{bd\alpha}\beta^{-b\alpha}\sum_{s=0}^{l}\binom{l}{s}\right.$$
$$\left.\sum_{p=0}^{s}\binom{s}{p}(-\alpha)^{s-p}S_{k}^{(\alpha)}\left(c,\left(\frac{\beta}{a}\right)^{b}\right)d^{s}\;{}_{p}P_{n-l,\beta}^{(\alpha+1)}(cX,cY;k,a,b)\,d^{l}\right\}\frac{t^{n}}{n!}.\quad(25)$$

∎

In a similar manner

$$g(t)=\frac{1}{d^{k(\alpha+1)}c^{k}}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}\left\{{}_{p}P_{n-l,\beta}^{(\alpha+1)}(cX,cY;k,a,b)\,d^{n-l}a^{bd\alpha}\beta^{-b\alpha}\sum_{s=0}^{l}\binom{l}{s}\right.$$
$$\left.\sum_{p=0}^{s}\binom{s}{p}(-\alpha)^{s-p}S_{k}^{(\alpha)}\left(d,\left(\frac{\beta}{a}\right)^{b}\right)d^{s}\;{}_{p}P_{n-s,\beta}^{(\alpha+1)}(dx,dy;k,a,b)\,c^{l}\right\}\frac{t^{n}}{n!}.\quad(26)$$

Comparing the coefficients $\frac{t^{n}}{n!}$ both sides above equations. We have (24) .

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Burak Kurt, Department of Mathematics, Faculty of Educations University of Akdeniz, TR-07058 Antalya, Turkey,

E-mail address: burakkurt@akdeniz.edu.tr

Secil Bilgic, Department of Computer Programming (Distance Education), Anadolu Bil Vocational School, Istanbul Aydin University, TR-34295 Istanbul-Turkey

E-mail address: secilbilgic@aydin.edu.tr