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SOME HERMITE-HADAMARD TYPE INTEGRAL INEQUALITIES FOR HARMONICALLY $(p,(s,m))$-CONVEX FUNCTIONS

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Abstract. In this article, we introduce the class of harmonically p-convex functions which is generalization of convex and harmonically convex functions. Next, we define Harmonically p-quasiconvex functions and Harmonically Logarithmic p-convex functions. We also introduce and consider the class of harmonically $(p,(s,m))$-convex functions which is generalization of harmonically (s,m)-convex and harmonically p-convex functions. Finally, we establish Hermite-Hadamard type inequalities for these classes of functions. Results proved in this paper may stimulate further research in this area.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$. Then following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

holds. This double inequality is known in the literature as the Hermite-Hadamard integral inequality for convex functions. Note that some of the classical inequalities for mean can be derived from (1.1) for the appropriate particular selection of function f. Both inequalities in (1.1) hold in the reverse direction if f is concave. For useful literature details on Hemite-Hadamard type integral inequalities, (see [4,6-16,19-23]) and references therein.

In recent past, convexity has been generalised and extended in various aspects using new and different concepts, (see [4,8-11,14,15,19,21,22]) and references therein. Iscan [9], investigated and studied a new generalised class of convex functions which are called harmonically convex functions. He further generalised the concept of Harmonically convex functions and introduced the classes of harmonically s-convex functions [10] and harmonically (α, m)-convex functions [11]. In [12], authors have

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introduced the concept of harmonically (s, m)-convex functions which unify harmonic convexity and harmonic s-convexity in more general way.

Motivated by ongoing research, in this paper we introduce a new generalised class of convex and harmonically convex functions which is called harmonically p-convex functions. Next, we define harmonically p-quasiconvex, harmonically Logarithmic p-convex and harmonically $(p,(s,m))$-convex functions. We derive several Hermite-Hadamard type inequalities for these classes of functions which are much more general than already existing in literature.

2. Preliminaries

In this section, we recall basic concepts which will be helpful to understand our main contribution.

Definition 2.1. [9] *Let $I \subset \mathbb{R}/\{0\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \le tf(y) + (1-t)f(x) \tag{2.1}$$

for all $x, y \in I$ and $t \in [0,1]$. If inequality in (2.1) is reversed, then f is said to be harmonically concave.

Definition 2.2. [10] *A function $f : I \subset \mathbb{R}/\{0\} \to \mathbb{R}$ is said to be harmonically s-convex in second sense, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \le t^s f(y) + (1-t)^s f(x) \tag{2.2}$$

for all $x, y \in I$ and $t \in [0,1]$. If inequality in (2.2) is reversed, then f is said to be harmonically s-concave.

Remark. *Note that for $s = 1$, harmonically s-convexity reduces to ordinary harmonically convexity.*

Definition 2.3. [18] *For some fixed $s \in (0,1]$ and $m \in [0,1]$ a mapping $f : I \subset [0,\infty) \to \mathbb{R}$ is said to be (s,m)-convex in the second sense on I if*

$$f(tx + m(1-t)y) \le t^s f(x) + m(1-t)^s f(y)$$

holds, for all $x, y \in I$ and $t \in [0,1]$.

Definition 2.4. [11] *A function $f : (0,\infty) \to \mathbb{R}$ is said to be harmonically (α, m)-convex, where $\alpha \in [0,1]$ and $m \in (0,1]$, if*

$$f\big(\frac{mxy}{mty+(1-t)x}\big) = f\big((\frac{t}{x} + \frac{1-t}{my})^{-1}\big) \le t^\alpha f(x) + m(1-t^\alpha)f(y) \tag{2.3}$$

for all $x, y \in (0,\infty)$ and $t \in [0,1]$. If the inequality in (2.3) is reversed, then f is said to be harmonically (α, m)-concave.

Definition 2.5. [12] *A function $f : I \subset (0,\infty) \to \mathbb{R}$ is said to be harmonically (s,m)-convex in second sense, where $s \in (0,1]$ and $m \in (0,1]$ if*

$$f\big(\frac{mxy}{mty+(1-t)x}\big) = f\big((\frac{t}{x} + \frac{1-t}{my})^{-1}\big) \le t^s f(x) + m(1-t)^s f(y)$$

$\forall x, y \in I$ and $t \in [0,1]$.

Remark. *Note that for $s = 1$, (s, m)-convexity reduces to harmonically m-convexity and for $m = 1$, harmonically (s, m)-convexity reduces to harmonically s-convexity in second sense (see* [10]*) and for $s, m = 1$, harmonically (s, m)-convexity reduces to ordinary harmonically convexity (see* [9]*).*

In [25], Zhang and Wan gave definition of p-convex function as follow:

Definition 2.6. *Let I be a p-convex set. A function $f : I \to \mathbb{R}$ is said to be a p-convex function if*

$$f([tx^p + (1-t)y^p]^{\frac{1}{p}}) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in I$.

Remark. [25] *An interval I is said to be a p-convex set if $[tx^p + (1-t)y^p]^{\frac{1}{p}} \in I$ for all $x, y \in I$ and $t \in [0; 1]$, where $p = 2k + 1$ or $p = n = m$, $n = 2r + 1, m = 2t + 1$ and $k, r, t \in N$.*

Iscan [12] gave a different version of the definition of p-convex function as follow:

Definition 2.7. *Let $I \subset (0, \infty)$ be a real interval and $p \in \mathbb{R}/\{0\}$. A function $f : I \to \mathbb{R}$ is said to be a p-convex function if*

$$f([tx^p + (1-t)y^p]^{\frac{1}{p}}) \leq tf(x) + (1-t)f(y) \tag{2.4}$$

for all $x, y \in I$. If the iequality in (2.4) is reversed, then f is called a p-concave function.

3. Main Results

In this section, we define the class of the harmonically p-convex functions which is a generalization of convex functions and harmonically convex functions:

Definition 3.1. *A function $f : I \subset (0, \infty) \to \mathbb{R}$ is said to be harmonically p-convex function, where $p \in \mathbb{R}/\{0\}$, if*

$$f\left(\frac{xy}{[ty^p + (1-t)x^p]^{\frac{1}{p}}}\right) = f\left(\left(\frac{t}{x^p} + \frac{1-t}{y^p}\right)^{\frac{-1}{p}}\right) \leq tf(x) + (1-t)f(y), \tag{3.1}$$

for all $x, y \in I$ and $t \in [0, 1]$. If the inequality in (3.1) is reversed, then f is said to be harmonically p-concave.

Remark. *Note that for $p = -1$, harmonic p-convexity reduces to convexity and for $p = 1$ harmonic p-convexity reduces to harmonic convexity.*

Theorem 3.1. *Let $f : I \subset \mathbb{R}/\{0\} \to \mathbb{R}$ be harmonically p-convex and $a, b \in I$ with $a < b$. If $f \in L[a, b]$, then following inequalities hold*

$$f\left(\frac{2^{\frac{1}{p}}ab}{[a^p + b^p]^{\frac{1}{p}}}\right) \leq \frac{p(ab)^p}{b^p - a^p}\int_a^b \frac{f(x)}{x^{p+1}}dx \leq \frac{f(a) + f(b)}{2}. \tag{3.2}$$

Remark. *If, we take $p = 1$ in Theorem 3.1, then we get Theorem 1.2 of [9].*

Next, we prove the weighted version Theorem 3.1 in the following theorem.

Theorem 3.2. *Let $f : I \subset \mathbb{R}/\{0\} \to \mathbb{R}$ be harmonically p-convex and $a, b \in I$ with $a < b$. If $f \in L[a, b]$, then following inequalities hold*

$$f\left(\frac{2^{\frac{1}{p}}ab}{[a^p+b^p]^{\frac{1}{p}}}\right)\int_a^b \frac{w(x)}{x^{p+1}}dx \leq \int_a^b \frac{f(x)w(x)}{x^{p+1}}dx \leq \frac{f(a)+f(b)}{2}\int_a^b \frac{w(x)}{x^{p+1}}dx, \quad (3.3)$$

where $w(x) : [a, b] \to \mathbb{R}$ is nonnegative, integrable and satisfies

$$w(\frac{ab}{x}) = w\left(\frac{ab}{[a^p+b^p-x^p]^{\frac{1}{p}}}\right). \quad (3.4)$$

Proof. Since, f is harmonically p-convex function on $[a, b]$, we have, for all $x, y \in [a, b]$,

$$f\left(\frac{2^{\frac{1}{p}}xy}{[x^p+y^p]^{\frac{1}{p}}}\right) \leq \frac{f(x)+f(y)}{2}.$$

Now, by choosing $x = \frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}$, $y = \frac{ab}{[ta^p+(1-t)b^p]^{\frac{1}{p}}}$, we get

$$f\left(\frac{2^{\frac{1}{p}}ab}{[a^p+b^p]^{\frac{1}{p}}}\right) \leq \frac{1}{2}\left[f\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)+f\left(\frac{ab}{[ta^p+(1-t)b^p]^{\frac{1}{p}}}\right)\right] \leq \frac{f(a)+f(b)}{2}.$$

Since, $w(x)$ is nonnegative and satisfies the condition (3.4), we obtain

$$\begin{aligned} & f\left(\frac{2^{\frac{1}{p}}ab}{[a^p+b^p]^{\frac{1}{p}}}\right)w\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right) \\ \leq\ & \frac{1}{2}\left[f\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)w\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)\right. \\ +\ & \left. f\left(\frac{ab}{[ta^p+(1-t)b^p]^{\frac{1}{p}}}\right)w\left(\frac{ab}{[ta^p+(1-t)b^p]^{\frac{1}{p}}}\right)\right] \\ \leq\ & \frac{f(a)+f(b)}{2}w\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right). \end{aligned}$$

Integrating both sides of inequality with respect to t over $[0, 1]$, we have

$$\begin{aligned} & f\left(\frac{2^{\frac{1}{p}}ab}{[a^p+b^p]^{\frac{1}{p}}}\right)\int_0^1 w\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)dt \\ \leq\ & \frac{1}{2}\left[\int_0^1 f\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)w\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)dt\right. \\ +\ & \left.\int_0^1 f\left(\frac{ab}{[ta^p+(1-t)b^p]^{\frac{1}{p}}}\right)w\left(\frac{ab}{[ta^p+(1-t)b^p]^{\frac{1}{p}}}\right)dt\right] \\ \leq\ & \frac{f(a)+f(b)}{2}\int_0^1 w\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)dt. \end{aligned}$$

The proof completes. ∎

Remark. *For $w(x)=1$ in Theorem 3.2, we get Theorem 3.1.*

Remark. *If we choose $w(\frac{ab}{x})=g(x)$ in Theorem 3.2, we get following result*

$$f\left(\frac{2^{\frac{1}{p}}ab}{[a^p+b^p]^{\frac{1}{p}}}\right)\int_a^b g(x)dx \le \int_a^b f(\frac{ab}{x})g(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)dx,$$

where $0<a<b$, $g(x)$ is nonnegative, integrable and symmetric with respect to $x=[\frac{a^p+b^p}{2}]^{\frac{1}{p}}$.

Lemma 3.3. *Let $f: I\subseteq \mathbb{R}/\{0\}\to\mathbb{R}$ be a harmonically p-convex function and $a,b\in I$ with $a<b$ and let*

$$h(t)=\frac{1}{2}f\left(\frac{2^{\frac{1}{p}}ab}{[a^p+b^p-t^p]^{\frac{1}{p}}}\right)+\frac{1}{2}f\left(\frac{2^{\frac{1}{p}}ab}{[a^p+b^p+t^p]^{\frac{1}{p}}}\right),$$

for $t\in[0,b-a]$. Then, $h(t)$ is p-convex on $[0,b-a]$ and for all $t\in[0,b-a]$

$$f\left(\frac{2^{\frac{1}{p}}ab}{[a^p+b^p]^{\frac{1}{p}}}\right)\le h(t)\le \frac{f(a)+f(b)}{2}. \tag{3.5}$$

Proof. Firstly, for $x,y\in[0,b-a]$, we have

$$\begin{aligned}
h([tx^p+(1-t)y^p]^{\frac{1}{p}}) &= \frac{1}{2}f\left(\frac{2^{\frac{1}{p}}ab}{[a^p+b^p-(tx^p+(1-t)y^p)]^{\frac{1}{p}}}\right)\\
&+ \frac{1}{2}f\left(\frac{2^{\frac{1}{p}}ab}{[a^p+b^p+(tx^p+(1-t)y^p)]^{\frac{1}{p}}}\right)\\
&= \frac{1}{2}f\left(\frac{2^{\frac{1}{p}}ab}{[t((a^p+b^p-x^p)^{\frac{1}{p}})^p+(1-t)((a^p+b^p-y^p)^{\frac{1}{p}})^p]^{\frac{1}{p}}}\right)\\
&+ \frac{1}{2}f\left(\frac{2^{\frac{1}{p}}ab}{[t((a^p+b^p+x^p)^{\frac{1}{p}})^p+(1-t)((a^p+b^p+y^p)^{\frac{1}{p}})^p]^{\frac{1}{p}}}\right)\\
&\le \frac{t}{2}f\left(\frac{2^{\frac{1}{p}}ab}{(a^p+b^p-x^p)^{\frac{1}{p}}}\right)+\frac{1-t}{2}f\left(\frac{2^{\frac{1}{p}}ab}{(a^p+b^p-y^p)^{\frac{1}{p}}}\right)\\
&+ \frac{t}{2}f\left(\frac{2^{\frac{1}{p}}ab}{(a^p+b^p+x^p)^{\frac{1}{p}}}\right)+\frac{1-t}{2}f\left(\frac{2^{\frac{1}{p}}ab}{(a^p+b^p+y^p)^{\frac{1}{p}}}\right)\\
&= th(x)+(1-t)h(y),
\end{aligned}$$

and hence, h(t) is p-convex on $[0,b-a]$.
Next, if $t\in[0,b-a]$, it follows from the harmonic p-convexity of f that

$$\begin{aligned}
h(t) &= \frac{1}{2}f\left(\frac{2^{\frac{1}{p}}ab}{[a^p+b^p-t^p]^{\frac{1}{p}}}\right)+\frac{1}{2}f\left(\frac{2^{\frac{1}{p}}ab}{[a^p+b^p+t^p]^{\frac{1}{p}}}\right)\\
&\ge f\left(\frac{\frac{2^{\frac{1}{p}}ab}{[a^p+b^p-t^p]^{\frac{1}{p}}}\frac{2^{\frac{1}{p}}ab}{[a^p+b^p+t^p]^{\frac{1}{p}}}}{\left[\frac{1}{2}\frac{2(ab)^p}{[a^p+b^p-t^p]}+\frac{1}{2}\frac{2(ab)^p}{[a^p+b^p+t^p]}\right]^{\frac{1}{p}}}\right)\\
&= f\left(\frac{2^{\frac{1}{p}}ab}{[a^p+b^p]^{\frac{1}{p}}}\right).
\end{aligned}$$

It is easy to observe that

$$\begin{aligned}
h(t) &= \frac{1}{2}f\left(\frac{2^{\frac{1}{p}}ab}{[a^p+b^p-t^p]^{\frac{1}{p}}}\right)+\frac{1}{2}f\left(\frac{2^{\frac{1}{p}}ab}{[a^p+b^p+t^p]^{\frac{1}{p}}}\right)\\
&= \frac{1}{2}f\left(2^{\frac{1}{p}}ab\left(\frac{b^p-a^p+t}{b^p-a^p}a^p+\frac{b^p-a^p-t}{b^p-a^p}b^p\right)^{-\frac{1}{p}}\right)\\
&+ \frac{1}{2}f\left(2^{\frac{1}{p}}ab\left(\frac{b^p-a^p-t}{b^p-a^p}a^p+\frac{b^p-a^p+t}{b^p-a^p}b^p\right)^{-\frac{1}{p}}\right)\\
&\leq \frac{1}{2}\frac{b^p-a^p+t}{2(b^p-a^p)}f(b)+\frac{1}{2}\frac{b^p-a^p-t}{2(b^p-a^p)}f(a)\\
&+ \frac{1}{2}\frac{b^p-a^p+t}{2(b^p-a^p)}f(a)+\frac{1}{2}\frac{b^p-a^p-t}{2(b^p-a^p)}f(b)\\
&= \frac{f(a)+f(b)}{2}.
\end{aligned}$$

Thus the inequality (3.5) holds. □

Proposition 3.4. *Let $f,g : I \subseteq \mathbb{R}/\{0\} \to \mathbb{R}$ be two harmonically p-convex functions. If f,g are similarly ordered then their product fg is also harmonically p-convex function.*

Proof. Since $f,g : I \subseteq \mathbb{R}/\{0\} \to \mathbb{R}$ are two harmonically p-convex functions, we have

$$f\left(\frac{xy}{[ty^p+(1-t)x^p]^{\frac{1}{p}}}\right)g\left(\frac{xy}{[ty^p+(1-t)x^p]^{\frac{1}{p}}}\right)$$

$$\begin{aligned}
&\leq [tf(x)+(1-t)f(y)][tf(x)+(1-t)g(y)]\\
&= t^2f(x)g(x)+t(1-t)[f(x)g(y)+f(y)g(x)]+(1-t)^2f(y)g(y)\\
&\leq t^2f(x)g(x)+t(1-t)[f(x)g(x)+f(y)g(y)]+(1-t)^2f(y)g(y)\\
&= tf(x)g(x)+(1-t)f(y)g(y).
\end{aligned}$$

This shows that product of two similarly ordered harmonically p-convex functions is again harmonically p-convex function. □

Theorem 3.5. *Let $f,g : I \subseteq \mathbb{R}/\{0\} \to \mathbb{R}$ be two harmonically p-convex functions. Then following inequality*

$$\frac{p(ab)^p}{b^p-a^p}\int_a^b \frac{f(x)g(x)}{x^{p+1}}dx \leq \frac{1}{3}M(a,b)+\frac{1}{6}N(a,b) \tag{3.6}$$

holds. Where $M(a,b) = f(a)g(a)+f(b)g(b)$ and $N(a,b) = f(a)g(b)+f(b)g(a)$.

Proof. Since, f, g are harmonically p-convex functions, then we have

$$\begin{aligned}\frac{p(ab)^p}{b^p-a^p}\int_a^b \frac{f(x)g(x)}{x^{p+1}}dx &= \int_0^1 f\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)g\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)dt \\ &\leq \int_0^1 [tf(a)+(1-t)f(b)][tg(a)+(1-t)g(b)]dt \\ &= \frac{1}{3}[f(a)g(a)+f(b)g(b)]+\frac{1}{6}[f(a)g(b)+f(b)g(a)].\end{aligned}$$

□

Theorem 3.6. *Let $f, g : I \subseteq \mathbb{R}/\{0\} \to \mathbb{R}$ be two similarly ordered harmonically p-convex functions. Then following inequality*

$$\frac{p(ab)^p}{b^p-a^p}\int_a^b \frac{f(x)g(x)}{x^{p+1}}dx \leq \frac{1}{2}M(a,b) \tag{3.7}$$

holds.

Proof. Since, f, g are similarly ordered harmonically p-convex functions on I. Then using Proposition 3.4, for $a, b \in I$, we have

$$f\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)g\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)$$
$$\leq tf(a)g(a)+(1-t)f(b)g(b),$$

integrating with respect to t over $[0,1]$, we get required result. □

Definition 3.2. *A function $f : I \subset (0,\infty) \to \mathbb{R}$ is said to be harmonically p-quasiconvex function, where $p \in \mathbb{R}/\{0\}$, if*

$$f\left(\frac{xy}{[ty^p+(1-t)x^p]^{\frac{1}{p}}}\right) = f\left(\left(\frac{t}{x^p}+\frac{1-t}{y^p}\right)^{\frac{-1}{p}}\right) \leq \max\{f(x), f(y)\}, \tag{3.8}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (3.8) is reversed, then f is said to be harmonically p-quasiconcave.

Definition 3.3. *A function $f : I \subset (0,\infty) \to \mathbb{R}$ is said to be harmonically Logarithmic p-convex function, where $p \in \mathbb{R}/\{0\}$, if*

$$f\left(\frac{xy}{[ty^p+(1-t)x^p]^{\frac{1}{p}}}\right) = f\left(\left(\frac{t}{x^p}+\frac{1-t}{y^p}\right)^{\frac{-1}{p}}\right) \leq [f(x)]^t[f(y)]^{1-t}, \tag{3.9}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (3.9) is reversed, then f is said to be harmonically Logarithmic p-concave.

Theorem 3.7. *Let $f : I \subset \mathbb{R}/\{0\} \to \mathbb{R}$ be harmonically Logarithmic p-convex and $a, b \in I$ with $a < b$. If $f \in L[a,b]$, then following inequalities hold*

$$\frac{p(ab)^p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{p+1}}dx \leq \frac{f(a)-f(b)}{\ln f(a)-\ln f(b)}. \tag{3.10}$$

Proof. Since, f is harmonically Logarithmic p-convex, then for $a, b \in I$, we have

$$f\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right) \le [f(a)]^t[f(b)]^{1-t}.$$

By integrating above inequality with respect to t over $[0,1]$, we get

$$\begin{aligned} \frac{p(ab)^p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{p+1}}dx &= \int_0^1 f\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)dt \\ &\le \int_0^1 [f(a)]^t[f(b)]^{1-t}dt \\ &= \frac{f(a)-f(b)}{\ln f(a)-\ln f(b)}, \end{aligned}$$

the proof completed. □

Theorem 3.8. *Let $f : I \subset \mathbb{R}/\{0\} \to \mathbb{R}$ be harmonically Logarithmic p-convex and $a, b \in I$ with $a < b$. If $f \in L[a, b]$, then following inequalities hold*

$$f\left(\frac{2^{\frac{1}{p}}ab}{[a^p+b^p]^{\frac{1}{p}}}\right) \le \exp\left[\frac{p(ab)^p}{b^p-a^p}\int_a^b \ln\left(\frac{f(x)}{x^{p+1}}\right)dx\right] \le \sqrt{[f(a)f(b)]}. \tag{3.11}$$

Proof. Since, f is harmonically Logarithmic p-convex, then for $t = \frac{1}{2}$, we have

$$f\left(\frac{2^{\frac{1}{p}}xy}{[x^p+y^p]^{\frac{1}{p}}}\right) \le [(f(x))(f(y))]^{\frac{1}{2}}.$$

By choosing $x = \frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}$ and $y = \frac{ab}{[(1-t)b^p+ta^p]^{\frac{1}{p}}}$, we get

$$\ln f\left(\frac{2^{\frac{1}{p}}ab}{[a^p+b^p]^{\frac{1}{p}}}\right) \le \frac{1}{2}\left[\ln f\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right) + \ln f\left(\frac{ab}{[(1-t)b^p+ta^p]^{\frac{1}{p}}}\right)\right].$$

Integrating above inequality with respect to t over $[0,1]$, we have

$$\begin{aligned} &\ln f\left(\frac{2^{\frac{1}{p}}ab}{[a^p+b^p]^{\frac{1}{p}}}\right) \\ &\le \frac{1}{2}\left[\int_0^1 \ln f\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)dt + \int_0^1 \ln f\left(\frac{ab}{[(1-t)b^p+ta^p]^{\frac{1}{p}}}\right)dt\right]. \end{aligned}$$

This implies that

$$\ln f\left(\frac{2^{\frac{1}{p}}ab}{[a^p+b^p]^{\frac{1}{p}}}\right) \le \frac{p(ab)^p}{b^p-a^p}\int_a^b \ln\left(\frac{f(x)}{x^{p+1}}\right)dx. \tag{3.12}$$

Also, since f is harmonically Logarithmic p-convex, we have

$$\begin{aligned} &\frac{1}{2}\left[\int_0^1 \ln f\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)dt + \int_0^1 \ln f\left(\frac{ab}{[(1-t)b^p+ta^p]^{\frac{1}{p}}}\right)dt\right] \\ &\le \frac{1}{2}\left[\int_0^1 (t\ln f(a)+(1-t)\ln f(b))dt + \int_0^1 (t\ln f(b)+(1-t)\ln f(a))dt\right], \end{aligned}$$

and this implies that

$$\frac{p(ab)^p}{b^p-a^p}\int_a^b \ln\left(\frac{f(x)}{x^{p+1}}\right)dx \le \frac{\ln f(a)+\ln f(b)}{2} = \ln[f(a)f(b)]^{\frac{1}{2}}. \tag{3.13}$$

From (3.12) and (3.13), we have

$$\ln f\left(\frac{2^{\frac{1}{p}}ab}{[a^p+b^p]^{\frac{1}{p}}}\right) \leq \frac{p(ab)^p}{b^p-a^p}\int_a^b \ln\left(\frac{f(x)}{x^{p+1}}\right)dx \leq \ln[f(a)f(b)]^{\frac{1}{2}},$$

taking antilog of above inequality, we get

$$f\left(\frac{2^{\frac{1}{p}}ab}{[a^p+b^p]^{\frac{1}{p}}}\right) \leq \exp\left[\frac{p(ab)^p}{b^p-a^p}\int_a^b \ln\left(\frac{f(x)}{x^{p+1}}\right)dx\right] \leq \sqrt{[f(a)f(b)]}.$$

This completes the proof. □

Theorem 3.9. *Let $f,g : I \subset \mathbb{R}/\{0\} \to \mathbb{R}$ be harmonically Logarithmic p-convex and $a,b \in I$ with $a < b$. If $f,g \in L[a,b]$, then following inequality*

$$\frac{p(ab)^p}{b^p-a^p}\int_a^b \frac{f(x)g(x)}{x^{p+1}}dx \leq \frac{1}{4}\Theta(a,b) \tag{3.14}$$

holds. Where $\Theta(a,b) = (f(a))^2 + (f(b))^2 + (g(a))^2 + (g(b))^2$.

Proof. Since, f,g are harmonically Logarithmic p-convex functions, then we have

$$\frac{p(ab)^p}{b^p-a^p}\int_a^b \frac{f(x)g(x)}{x^{p+1}}dx$$

$$\begin{aligned}
&= \int_0^1 f\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)g\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)dt \\
&\leq \frac{1}{2}\int_0^1 \left[\left(f\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)\right)^2 + \left(g\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)\right)^2\right]dt \\
&\leq \frac{1}{2}\int_0^1 \left[\left([f(a)]^t[f(b)]^{1-t}\right)^2 + \left([g(a)]^t[g(b)]^{1-t}\right)^2\right]dt \\
&= \frac{[f(a)]^2-[f(b)]^2}{4(\ln f(a)-\ln f(b))} + \frac{[g(a)]^2-[g(b)]^2}{4(\ln g(a)-\ln g(b))} \\
&= \frac{1}{2}\frac{f(a)+f(b)}{2}\frac{f(a)-f(b)}{\ln f(a)-\ln f(b)} + \frac{1}{2}\frac{g(a)+g(b)}{2}\frac{g(a)-g(b)}{\ln g(a)-\ln g(b)} \\
&\leq \frac{1}{4}[(f(a))^2+(f(b))^2+(g(a))^2+(g(b))^2].
\end{aligned}$$

This completes the proof. □

Theorem 3.10. *Let $f,g : I \subset \mathbb{R}/\{0\} \to \mathbb{R}$ be harmonically Logarithmic p-convex and $a,b \in I$ with $a < b$. If $f,g \in L[a,b]$, then following inequality*

$$\frac{p(ab)^p}{b^p-a^p}\int_a^b \frac{f(x)g(x)}{x^{p+1}}dx \leq \frac{1}{8}\Theta(a,b) + \frac{1}{4}M(a,b) \tag{3.15}$$

holds.

Proof. Since, f,g are harmonically Logarithmic p-convex functions and using inequality

$$ab \leq \frac{1}{4}(a+b)^2,$$

for all $a, b \in \mathbb{R}$, we have

$$\begin{aligned}
&\frac{p(ab)^p}{b^p - a^p}\int_a^b \frac{f(x)g(x)}{x^{p+1}}dx \\
= &\int_0^1 f\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)g\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)dt \\
\leq &\frac{1}{4}\int_0^1 \left[f\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)+g\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)\right]^2 dt \\
\leq &\frac{1}{4}\int_0^1 \left[(f(a))^t f(b))^{1-t} + (g(a))^t g(b))^{1-t}\right]^2 dt \\
= &\frac{[f(b)]^2}{4}\int_0^1 \left[\frac{f(a)}{f(b)}\right]^{2t} dt + \frac{[g(b)]^2}{4}\int_0^1 \left[\frac{g(a)}{g(b)}\right]^{2t} dt \\
+ &\frac{f(b)g(b)}{2}\int_0^1 \left[\frac{f(a)g(a)}{f(b)g(b)}\right]^t dt \\
= &\frac{1}{8}\frac{[f(a)]^2-[f(b)]^2}{\ln f(a)-\ln f(b)} + \frac{1}{8}\frac{[g(a)]^2-[g(b)]^2}{\ln g(a)-\ln g(b)} + \frac{1}{2}\frac{f(a)g(a)-f(b)]g(b)}{\ln f(a)g(a)-\ln f(b)g(b)} \\
= &\frac{1}{4}\frac{f(a)+f(b)}{2}\frac{f(a)-f(b)}{\ln f(a)-\ln f(b)} + \frac{1}{4}\frac{g(a)+g(b)}{2}\frac{g(a)-g(b)}{\ln g(a)-\ln g(b)} \\
+ &\frac{1}{2}\frac{f(a)g(a)-f(b)]g(b)}{\ln f(a)g(a)-\ln f(b)g(b)} \\
\leq &\frac{1}{8}[(f(a))^2+(f(b))^2+(g(a))^2+(g(b))^2] + \frac{1}{4}[f(a)g(a)+f(b)g(b)].
\end{aligned}$$

The proof is completed. □

Theorem 3.11. *Let $f, g : I \subset \mathbb{R}/\{0\} \to \mathbb{R}$ be harmonically Logarithmic p-convex and $a, b \in I$ with $a < b$. If $f, g \in L[a, b]$ and $\alpha + \beta = 1$, then following inequality*

$$\begin{aligned}
&\frac{p(ab)^p}{b^p - a^p}\int_a^b \frac{f(x)g(x)}{x^{p+1}}dx \\
\leq \alpha\ &\frac{f(a)+f(b)}{2}\left[L_{(\frac{1}{\alpha}-1)}(f(a), f(b))\right]^{\frac{1-\alpha}{\alpha}} + \beta\ \frac{g(a)+g(b)}{2}\left[L_{(\frac{1}{\beta}-1)}(g(a), g(b))\right]^{\frac{1-\beta}{\beta}}
\end{aligned} \tag{3.16}$$

holds. Where $L_r(.,.)$ is extended logarithmic mean.

Proof. Since, f, g are harmonically Logarithmic p-convex functions, then, using inequality

$$ab \leq \alpha a^{\frac{1}{\alpha}} + \beta b^{\frac{1}{\beta}},$$

for $\alpha, \beta > 0$, $\alpha + \beta = 1$, we have

$$\begin{aligned}
& \frac{p(ab)^p}{b^p-a^p}\int_a^b \frac{f(x)g(x)}{x^{p+1}}dx \\
= & \int_0^1 f\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)g\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)dt \\
\leq & \int_0^1 \left[\alpha\left(f\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)\right)^{\frac{1}{\alpha}}+\beta\left(g\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)\right)^{\frac{1}{\beta}}\right]dt \\
\leq & \int_0^1 \left[\alpha([f(a)]^t[f(b)]^{1-t})^{\frac{1}{\alpha}}+\beta([g(a)]^t[g(b)]^{1-t})^{\frac{1}{\beta}}\right]dt \\
= & \alpha[f(b)]^{\frac{1}{\alpha}}\int_0^1\left[\frac{f(a)}{f(b)}\right]^{\frac{t}{\alpha}}dt+\beta[g(b)]^{\frac{1}{\beta}}\int_0^1\left[\frac{g(a)}{g(b)}\right]^{\frac{t}{\beta}}dt \\
= & \alpha^2\frac{[f(a)]^{\frac{1}{\alpha}}-[f(b)]^{\frac{1}{\alpha}}}{\ln f(a)-\ln f(b)}+\beta^2\frac{[g(a)]^{\frac{1}{\beta}}-[g(b)]^{\frac{1}{\beta}}}{\ln g(a)-\ln g(b)} \\
= & \alpha^2\frac{[f(a)]^{\frac{1}{\alpha}}-[f(b)]^{\frac{1}{\alpha}}}{f(a)-f(b)}L(f(a),f(b))+\beta^2\frac{[g(a)]^{\frac{1}{\beta}}-[g(b)]^{\frac{1}{\beta}}}{g(a)-g(b)}L(g(a),g(b)) \\
= & \alpha\left[L_{(\frac{1}{\alpha}-1)}(f(a),f(b))\right]^{\frac{1-\alpha}{\alpha}}L(f(a),f(b))+\beta\left[L_{(\frac{1}{\beta}-1)}(g(a),g(b))\right]^{\frac{1-\beta}{\beta}}L(g(a),g(b)) \\
\leq & \alpha\frac{f(a)+f(b)}{2}\left[L_{(\frac{1}{\alpha}-1)}(f(a),f(b))\right]^{\frac{1-\alpha}{\alpha}}+\beta\frac{g(a)+g(b)}{2}\left[L_{(\frac{1}{\beta}-1)}(g(a),g(b))\right]^{\frac{1-\beta}{\beta}}.
\end{aligned}$$

This completes the proof. □

Theorem 3.12. *Let $f,g: I\subset\mathbb{R}/\{0\}\to\mathbb{R}$ be increasing harmonically Logarithmic p-convex and $a,b\in I$ with $a<b$. If $f,g\in L[a,b]$, then following inequality*

$$\begin{aligned}
&\frac{p(ab)^p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{p+1}}dxL(g(a),g(b))+\frac{p(ab)^p}{b^p-a^p}\int_a^b\frac{g(x)}{x^{p+1}}dxL(f(a),f(b)) \\
&\qquad\leq \frac{p(ab)^p}{b^p-a^p}\int_a^b\frac{f(x)g(x)}{x^{p+1}}dx+L(f(a)g(a),f(b)g(b)) \qquad (3.17)
\end{aligned}$$

holds.

Proof. Since, f,g are harmonically Logarithmic p-convex functions on $[a,b]$, then

$$f\left(\frac{ab}{[t(b)^p+(1-t)a^p]^{\frac{1}{p}}}\right)\leq[f(a)]^t[f(b)]^{1-t},$$

$$g\left(\frac{ab}{[t(b)^p+(1-t)a^p]^{\frac{1}{p}}}\right)\leq[g(a)]^t[g(b)]^{1-t}.$$

Now, using the fact that $(x_1-x_2)(x_3-x_4)\geq 0$ for $x_1,x_2,x_3,x_4\in\mathbb{R}$ with $x_1<x_2$ and $x_3<x_4$, we have

$$\begin{aligned}
&f\left(\frac{ab}{[t(b)^p+(1-t)a^p]^{\frac{1}{p}}}\right)[g(a)]^t[g(b)]^{1-t}+g\left(\frac{ab}{[t(b)^p+(1-t)a^p]^{\frac{1}{p}}}\right)[f(a)]^t[f(b)]^{1-t} \\
&\leq f\left(\frac{ab}{[t(b)^p+(1-t)a^p]^{\frac{1}{p}}}\right)g\left(\frac{ab}{[t(b)^p+(1-t)a^p]^{\frac{1}{p}}}\right)+[f(a)]^t[f(b)]^{1-t}[g(a)]^t[g(b)]^{1-t}.
\end{aligned}$$

Integrating above inequality with respect to t over $[0,1]$, we get

$$\int_0^1 f\left(\frac{ab}{[t(b)^p+(1-t)a^p]^{\frac{1}{p}}}\right)[g(a)]^t[g(b)]^{1-t}dt + \int_0^1 g\left(\frac{ab}{[t(b)^p+(1-t)a^p]^{\frac{1}{p}}}\right)[f(a)]^t[f(b)]^{1-t}dt$$

$$\leq \int_0^1 f\left(\frac{ab}{[t(b)^p+(1-t)a^p]^{\frac{1}{p}}}\right)g\left(\frac{ab}{[t(b)^p+(1-t)a^p]^{\frac{1}{p}}}\right)dt + \int_0^1 [f(a)]^t[f(b)]^{1-t}[g(a)]^t[g(b)]^{1-t}dt.$$

Now, since f,g are increasing, then we have

$$\int_0^1 f\left(\frac{ab}{[t(b)^p+(1-t)a^p]^{\frac{1}{p}}}\right)dt\int_0^1 [g(a)]^t[g(b)]^{1-t}dt + \int_0^1 g\left(\frac{ab}{[t(b)^p+(1-t)a^p]^{\frac{1}{p}}}\right)dt\int_0^1 [f(a)]^t[f(b)]^{1-t}dt$$

$$\leq \int_0^1 f\left(\frac{ab}{[t(b)^p+(1-t)a^p]^{\frac{1}{p}}}\right)g\left(\frac{ab}{[t(b)^p+(1-t)a^p]^{\frac{1}{p}}}\right)dt + \int_0^1 [f(a)]^t[f(b)]^{1-t}[g(a)]^t[g(b)]^{1-t}dt.$$

Now, after simple integration, we have

$$\frac{p(ab)^p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{p+1}}dxL(g(a),g(b)) + \frac{p(ab)^p}{b^p-a^p}\int_a^b \frac{g(x)}{x^{p+1}}dxL(f(a),f(b))$$

$$\leq \frac{p(ab)^p}{b^p-a^p}\int_a^b \frac{f(x)g(x)}{x^{p+1}}dx + L(f(a)g(a),f(b)g(b)).$$

□

Now, we define the class of the harmonically $(p,(s,m))$-convex functions:

Definition 3.4. *A function $f : I \subset (0,\infty) \to \mathbb{R}$ is said to be harmonically $(p,(s,m))$-convex function, where $p \in \mathbb{R}/\{0\}$, $s,m \in (0,1]$, if*

$$f\left(\frac{mxy}{[t(my)^p+(1-t)x^p]^{\frac{1}{p}}}\right) = f\left(\left(\frac{t}{x^p}+\frac{1-t}{(my)^p}\right)^{\frac{-1}{p}}\right) \leq t^s f(x) + m(1-t)^s f(y), \tag{3.18}$$

for all $x,y \in I$ with $my \in I$ and $t \in [0,1]$. If the inequality in (3.18) is reversed, then f is said to be harmonically $(p,(s,m))$-concave.

Remark. *Note that for $p=1$, harmonic $(p,(s,m))$-convexity reduce to harmonic (s,m)-convexity and for $p=-1$, harmonic $(p,(s,m))$-convexity reduce to (s,m)-convexity in second sense.*

Theorem 3.13. *Let $f : I \subset \mathbb{R}/\{0\} \to \mathbb{R}$ be harmonically $(p,(s,1))$-convex and $a, b \in I$ with $a < b$. If $f \in L[a,b]$, then following inequality*

$$f\left(\frac{2^{\frac{1}{p}}ab}{[a^p+b^p]^{\frac{1}{p}}}\right) \le \frac{1}{2^{s-1}}\frac{p(ab)^p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{p+1}}dx \le \frac{f(a)+mf(b)}{(s+1)2^{s-1}}, \tag{3.19}$$

holds.

Proof. Since, f is harmonically $(p,(s,1))$-convex function on I, for $a, b \in I$, we have

$$f\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right) \le t^s f(a) + (1-t)^s f(b),$$

integrating with respect to t over $[0,1]$, we get

$$\begin{aligned} \frac{p(ab)^p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{p+1}}dx &= \int_0^1 f\left(\frac{ab}{[t(b)^p+(1-t)a^p]^{\frac{1}{p}}}\right)dt \\ &\le f(a)\int_0^1 t^s dt + f(b)\int_0^1 (1-t)^s dt \\ &= \frac{f(a)+f(b)}{s+1}. \end{aligned}$$

Now from (3.18), for $m = 1$ and $t = \frac{1}{2}$ we have

$$f\left(\frac{2^{\frac{1}{p}}xy}{[y^p+x^p]^{\frac{1}{p}}}\right) \le \frac{f(x)+f(y)}{2^s},$$

by choosing $x = \frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}$, $y = \frac{ab}{[ta^p+(1-t)b^p]^{\frac{1}{p}}}$ and integrating with respect to t over $[0,1]$. We get

$$\begin{aligned} & f\left(\frac{2^{\frac{1}{p}}ab}{[(a)^p+b^p]^{\frac{1}{p}}}\right) \\ \le\ & \frac{1}{2^s}\left[\int_0^1 f\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)dt + \int_0^1 f\left(\frac{ab}{[ta^p+(1-t)b^p]^{\frac{1}{p}}}\right)dt\right] \\ =\ & \frac{1}{2^s}\left[\frac{p(ab)^p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{p+1}}dx + \frac{p(ab)^p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{p+1}}dx\right] \\ =\ & \frac{1}{2^{s-1}}\frac{p(ab)^p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{p+1}}dx, \end{aligned}$$

the proof is completed. □

For finding some new more general inequalities of Hermite-Hadamard type for the functions whose derivatives are harmonically $(p,(s,m)$-convex in second sense, we need the following lemma:

Lemma 3.14. *Let $f : I \subset \mathbb{R}/\{0\} \to \mathbb{R}$ be a differentiable function on I° with $a < b$. If $f \in L[a,b]$ and $p \in \mathbb{R}/\{0\}$, then*

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{p(ab)^p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{p+1}}dx \\ &\qquad = \frac{ab(b^p-a^p)}{2p}\int_0^1 \frac{1-2t}{[tb^p+(1-t)a^p]^{\frac{p+1}{p}}} f'\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)dt. \end{aligned}$$

Proof. Let

$$I = \frac{ab(b^p - a^p)}{2p} \int_0^1 \frac{1-2t}{[tb^p + (1-t)a^p]^{\frac{p+1}{p}}} f'\left(\frac{ab}{[tb^p + (1-t)a^p]^{\frac{1}{p}}}\right)dt.$$

Integrating by parts, we have

$$I = \frac{1}{2}\left|(2t-1)f\left(\frac{ab}{[tb^p + (1-t)a^p]^{\frac{1}{p}}}\right)\right|_0^1 - \int_0^1 f\left(\frac{ab}{[tb^p + (1-t)a^p]^{\frac{1}{p}}}\right)dt.$$

Setting $x = \frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}$, we have $dt = \frac{p(ab)^p}{a^p-b^p}\frac{dx}{x^{p+1}}$, hence we obtain

$$I = \frac{f(a)+f(b)}{2} - \frac{p(ab)^p}{b^p - a^p}\int_a^b \frac{f(x)}{x^{p+1}}dx,$$

this completes the proof. □

Now, we give some more generalised Hermite-Hadamard type inequalities from which one can obtain the results presented in [2,4,5,6] for the particular values of p, s, m.

Theorem 3.15. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a, \frac{b}{m} \in I^\circ$ with $a < b$, $m \in (0,1]$, $f' \in L[a,b]$ and $p \in \mathbb{R}/\{0\}$. If $|f'|^q$ is harmonically $(p,(s,m))$-convex in second sense on $[a, \frac{b}{m}]$ for $q \geq 1$ with $s \in [0,1]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{p(ab)^p}{b^p - a^p}\int_a^b \frac{f(x)}{x^{p+1}}dx\right|$$

$$\leq \frac{ab(b^p-a^p)}{2p}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\sigma_1(s,p,q;a,b)|f'(a)|^q + m\sigma_2(s,p,q;a,b)|f'(\frac{b}{m})|^q\right]^{\frac{1}{q}}$$

holds, where

$$\sigma_1(s,p,q;a,b) = \begin{cases} \frac{1}{b^{(p+1)q}}\beta(1,s+2).{}_2F_1(\frac{(p+1)q}{p},1;s+3;1-(\frac{a}{b})^p) \\ -\frac{1}{b^{(p+1)q}}\beta(2,s+1).{}_2F_1(\frac{(p+1)q}{p},2;s+3;1-(\frac{a}{b})^p) \\ +\frac{1}{2^s}\left(\frac{2}{a^p+b^p}\right)^{\frac{(p+1)q}{p}}\beta(2,s+1).{}_2F_1(\frac{(p+1)q}{p},2;s+3;\frac{b^p-a^p}{b^p+a^p}), & p>0 \\ \\ \frac{1}{a^{(p+1)q}}\beta(s+2,1).{}_2F_1(\frac{(p+1)q}{p},s+2;s+3;1-(\frac{b}{a})^p) \\ -\frac{1}{a^{(p+1)q}}\beta(s+1,2).{}_2F_1(\frac{(p+1)q}{p},s+1;s+3;1-(\frac{b}{a})^p) \\ +\frac{1}{2^s a^{(p+1)q}}\beta(s+1,2).{}_2F_1(\frac{(p+1)q}{p},s+1;s+3;\frac{a^p-b^p}{2a^p}), & p<0 \end{cases}$$

and

$$\sigma_2(s,p,q;a,b) = \begin{cases} \frac{1}{b^{(p+1)q}}\beta(s+2,1).{}_2F_1(\frac{(p+1)q}{p},s+2;s+3;1-(\frac{a}{b})^p) \\ -\frac{1}{b^{(p+1)q}}\beta(s+1,2).{}_2F_1(\frac{(p+1)q}{p},s+1;s+3;1-(\frac{a}{b})^p) \\ +\frac{1}{2^s b^{(p+1)q}}\beta(s+1,2).{}_2F_1(\frac{(p+1)q}{p},s+1;s+3;\frac{b^p-a^p}{2b^p}), & p>0 \\ \\ \frac{1}{a^{(p+1)q}}\beta(1,s+2).{}_2F_1(\frac{(p+1)q}{p},1;s+3;1-(\frac{b}{a})^p) \\ -\frac{1}{a^{(p+1)q}}\beta(2,s+1).{}_2F_1(\frac{(p+1)q}{p},2;s+3;1-(\frac{b}{a})^p) \\ +\frac{1}{2^s}\left(\frac{2}{a^p+b^p}\right)^{\frac{(p+1)q}{p}}\beta(2,s+1).{}_2F_1(\frac{(p+1)q}{p},2;s+3;\frac{a^p-b^p}{a^p+b^p}), & p<0 \end{cases}.$$

Proof. From above Lemma and using power mean inequality, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{p(ab)^p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{p+1}}dx\right|$$

$$\begin{aligned}
&\leq \frac{ab(b^p-a^p)}{2p}\int_0^1\frac{|1-2t|}{[tb^p+(1-t)a^p]^{\frac{p+1}{p}}}\left|f'\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)\right|dt\\
&\leq \frac{ab(b^p-a^p)}{2p}\left(\int_0^1|1-2t|dt\right)^{1-\frac{1}{q}}\\
&\times \left(\int_0^1\frac{|1-2t|}{[tb^p+(1-t)a^p]^{\frac{(p+1)q}{p}}}\left|f'\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Since, $|f'|^q$ is harmonically $(p,(s,m))$-convex function in second sense, we have

$$\begin{aligned}
&\leq \frac{ab(b^p-a^p)}{2p}\left(\int_0^1|1-2t|dt\right)^{1-\frac{1}{q}}\\
&\times \left(\int_0^1\frac{|1-2t|[t^s|f'(a)|^q+m(1-t)^s|f'(\frac{b}{m})|^q]}{[tb^p+(1-t)a^p]^{\frac{(p+1)q}{p}}}dt\right)^{\frac{1}{q}}\\
&= \frac{ab(b^p-a^p)}{2p}(\frac{1}{2})^{1-\frac{1}{q}}\left[\sigma_1(s,p,q;a,b)|f'(a)|^q+m\sigma_2(s,p,q;a,b)|f'(\frac{b}{m})|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

□

Corollary 3.16. *Let $f: I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I°, $a,b\in I^\circ$ with $a<b$, $f'\in L[a,b]$ and $p\in\mathbb{R}/\{0\}$. If $|f'|^q$ is harmonically $(p,(1,1))$-convex in second sense on $[a,b]$ for $q\geq 1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{p(ab)^p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{p+1}}dx\right|$$

$$\leq \frac{ab(b^p-a^p)}{2p}(\frac{1}{2})^{1-\frac{1}{q}}\left[\sigma_1(1,p,q;a,b)|f'(a)|^q+\sigma_2(1,p,q;a,b)|f'(b)|^q\right]^{\frac{1}{q}}.$$

Corollary 3.17. *Let $f: I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I°, $a,\frac{b}{m}\in I^\circ$ with $a<b$, $m\in(0,1]$, $f'\in L[a,b]$ and $p\in\mathbb{R}/\{0\}$. If $|f'|^q$ is harmonically $(1,(s,m))$-convex in second sense on $[a,\frac{b}{m}]$ for $q\geq 1$ with $s\in[0,1]$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{(ab)}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|$$

$$\leq \frac{ab(b-a)}{2p}(\frac{1}{2})^{1-\frac{1}{q}}\left[\sigma_1(s,1,q;a,b)|f'(a)|^q+m\sigma_2(s,1,q;a,b)|f'(\frac{b}{m})|^q\right]^{\frac{1}{q}}.$$

Corollary 3.18. *If we take $m=1$ in Theorem 3.15, then we get*

$$\left|\frac{f(a)+f(b)}{2}-\frac{p(ab)^p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{p+1}}dx\right|$$

$$\leq \frac{ab(b^p-a^p)}{2p}(\frac{1}{2})^{1-\frac{1}{q}}\left[\sigma_1(s,p,q;a,b)|f'(a)|^q+\sigma_2(s,p,q;a,b)|f'(b)|^q\right]^{\frac{1}{q}}.$$

Corollary 3.19. *If we take $s = 1$ in Theorem 3.15, then we get*

$$\left|\frac{f(a)+f(b)}{2} - \frac{p(ab)^p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{p+1}}dx\right|$$

$$\leq \frac{ab(b^p-a^p)}{2p}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\sigma_1(1,p,q;a,b)|f'(a)|^q + m\sigma_2(1,p,q;a,b)|f'(\frac{b}{m})|^q\right]^{\frac{1}{q}}.$$

Theorem 3.20. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I, $a, \frac{b}{m} \in I^\circ$ with $a < b$, $m \in (0,1]$, $f' \in L[a,b]$ and $p \in \mathbb{R}/\{0\}$. If $|f'|^q$ is harmonically $(p,(s,m))$-convex in second sense on $[a,\frac{b}{m}]$ for $q \geq 1$ with $s \in [0,1]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{p(ab)^p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{p+1}}dx\right|$$

$$\leq \frac{ab(b^p-a^p)}{2p}\sigma_1^{1-\frac{1}{q}}(1,p,1;a,b)\left[\sigma_1(s,p,1;a,b)|f'(a)|^q + m\sigma_2(s,p,1;a,b)|f'(\frac{b}{m})|^q\right]^{\frac{1}{q}}.$$

Proof. From Lemma, Power mean inequality and harmonically $(p,(s,m))$-convexity in second sense of$|f'|^q$ on $[a,\frac{b}{m}]$,we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{p(ab)^p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{p+1}}dx\right|$$

$$\begin{aligned}
&\leq \frac{ab(b^p-a^p)}{2p}\int_0^1 \frac{|1-2t|}{[tb^p+(1-t)a^p]^{\frac{p+1}{p}}}\left|f'\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)\right|dt \\
&\leq \frac{ab(b^p-a^p)}{2p}\left(\int_0^1 \frac{|1-2t|}{[tb^p+(1-t)a^p]^{\frac{p+1}{p}}}dt\right)^{1-\frac{1}{q}} \\
&\times \left(\int_0^1 \frac{|1-2t|}{[tb^p+(1-t)a^p]^{\frac{p+1}{p}}}\left|f'\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{ab(b^p-a^p)}{2p}\left(\int_0^1 \frac{|1-2t|}{[tb^p+(1-t)a^p]^{\frac{p+1}{p}}}dt\right)^{1-\frac{1}{q}} \\
&\times \left(\int_0^1 \frac{|1-2t|[t^s|f'(a)|^q + m(1-t)^s|f'(\frac{b}{m})|^q]}{[tb^p+(1-t)a^p]^{\frac{p+1}{p}}}dt\right)^{\frac{1}{q}} \\
&= \frac{ab(b^p-a^p)}{2p}\sigma_1^{1-\frac{1}{q}}(1,p,1;a,b)\left[\sigma_1(s,p,1;a,b)|f'(a)|^q + m\sigma_2(s,p,1;a,b)|f'(\frac{b}{m})|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

□

Corollary 3.21. *If we take $s = m = 1$ in Theorem 3.20, we get*

$$\left|\frac{f(a)+f(b)}{2} - \frac{p(ab)^p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{p+1}}dx\right|$$

$$\leq \frac{ab(b^p-a^p)}{2p}\sigma_1^{1-\frac{1}{q}}(1,p,1;a,b)\left[\sigma_1(1,p,1;a,b)|f'(a)|^q + \sigma_2(1,p,1;a,b)|f'(b)|^q\right]^{\frac{1}{q}}.$$

Corollary 3.22. *If we take $s=1$ in Theorem 3.20, we get*

$$\left|\frac{f(a)+f(b)}{2}-\frac{p(ab)^p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{p+1}}dx\right|$$

$$\leq\frac{ab(b^p-a^p)}{2p}\sigma_1^{1-\frac{1}{q}}(1,p,1;a,b)\left[\sigma_1(1,p,1;a,b)|f'(a)|^q+m\sigma_2(1,p,1;a,b)|f'(\frac{b}{m})|^q\right]^{\frac{1}{q}}.$$

Corollary 3.23. *If we take $m=1$ in Theorem 3.20, we get*

$$\left|\frac{f(a)+f(b)}{2}-\frac{p(ab)^p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{p+1}}dx\right|$$

$$\leq\frac{ab(b^p-a^p)}{2p}\sigma_1^{1-\frac{1}{q}}(1,p,1;a,b)\left[\sigma_1(s,p,1;a,b)|f'(a)|^q+\sigma_2(s,p,1;a,b)|f'(b)|^q\right]^{\frac{1}{q}}.$$

Corollary 3.24. *If we take $p=1$ in Theorem 3.20, we get*

$$\left|\frac{f(a)+f(b)}{2}-\frac{p(ab)}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|$$

$$\leq\frac{ab(b-a)}{2}\sigma_1^{1-\frac{1}{q}}(1,1,1;a,b)\left[\sigma_1(s,1,1;a,b)|f'(a)|^q+m\sigma_2(s,1,1;a,b)|f'(\frac{b}{m})|^q\right]^{\frac{1}{q}}.$$

Theorem 3.25. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I°, $a,\frac{b}{m}\in I^\circ$ with $a<b$, $m\in(0,1]$, $f'\in L[a,b]$ and $p\in\mathbb{R}/\{0\}$. If $|f'|^q$ is harmonically $(p,(s,m))$-convex in second sense on $[a,\frac{b}{m}]$ for $q>1$, $\frac{1}{r}+\frac{1}{q}$ with $s\in[0,1]$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{p(ab)^p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{p+1}}dx\right|$$

$$\leq\frac{ab(b^p-a^p)}{2p}\left(\frac{1}{r+1}\right)^{\frac{1}{r}}\left[\varpi_1(s,p,q;a,b)|f'(a)|^q+m\varpi_2(s,p,q;a,b)|f'(\frac{b}{m})|^q\right]^{\frac{1}{q}},$$

where

$$\varpi_1(s,p,q;a,b)=\begin{cases}\frac{1}{b^{(p+1)q}}\beta(1,s+1).{}_2F_1(\frac{(p+1)q}{p},1;s+2;1-(\frac{a}{b})^p), & p>0\\ \frac{1}{a^{(p+1)q}}\beta(s+1,1).{}_2F_1(\frac{(p+1)q}{p},s+1;s+2;1-(\frac{b}{a})^p), & p<0\end{cases}$$

and

$$\varpi_2(s,p,q;a,b)=\begin{cases}\frac{1}{b^{(p+1)q}}\beta(s+1,1).{}_2F_1(\frac{(p+1)q}{p},s+1;s+2;1-(\frac{a}{b})^p), & p>0\\ \frac{1}{a^{(p+1)q}}\beta(1,s+1).{}_2F_1(\frac{(p+1)q}{p},1;s+2;1-(\frac{b}{a})^p), & p<0\end{cases}.$$

Proof. From Lemma, Power mean inequality and harmonically $(p,(s,m))$-convexity in second sense of$|f'|^q$ on $[a,\frac{b}{m}]$,we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{p(ab)^p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{p+1}}dx\right|$$

$$
\begin{aligned}
&\leq \frac{ab(b^p-a^p)}{2p}\int_0^1 \frac{|1-2t|}{[tb^p+(1-t)a^p]^{\frac{p+1}{p}}}\left|f'\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)\right|dt \\
&\leq \frac{ab(b^p-a^p)}{2p}\left(\int_0^1 |1-2t|^r dt\right)^{\frac{1}{r}} \\
&\times \left(\int_0^1 \frac{1}{[tb^p+(1-t)a^p]^{\frac{(p+1)q}{p}}}\left|f'\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{ab(b^p-a^p)}{2p}\left(\int_0^1 |1-2t|^r dt\right)^{\frac{1}{r}} \\
&\times \left(\int_0^1 \frac{[t^s|f'(a)|^q+m(1-t)^s|f'(\frac{b}{m})|^q]}{[tb^p+(1-t)a^p]^{\frac{(p+1)q}{p}}}dt\right)^{\frac{1}{q}} \\
&= \frac{ab(b^p-a^p)}{2p}\left(\frac{1}{r+1}\right)^{\frac{1}{r}}\left[\varpi_1(s,p,q;a,b)|f'(a)|^q+m\varpi_2(s,p,q;a,b)|f'(\frac{b}{m})|^q\right]^{\frac{1}{q}}.
\end{aligned}
$$

□

Corollary 3.26. *If we take $s=m=1$ in Theorem 3.25, we get*

$$\left|\frac{f(a)+f(b)}{2}-\frac{p(ab)^p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{p+1}}dx\right|$$

$$\leq \frac{ab(b^p-a^p)}{2p}\left(\frac{1}{r+1}\right)^{\frac{1}{r}}\left[\varpi_1(1,p,q;a,b)|f'(a)|^q+\varpi_2(1,p,q;a,b)|f'(b)|^q\right]^{\frac{1}{q}}.$$

Corollary 3.27. *If we take $s=1$ in Theorem 3.25, we get*

$$\left|\frac{f(a)+f(b)}{2}-\frac{p(ab)^p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{p+1}}dx\right|$$

$$\leq \frac{ab(b^p-a^p)}{2p}\left(\frac{1}{r+1}\right)^{\frac{1}{r}}\left[\varpi_1(1,p,q;a,b)|f'(a)|^q+m\varpi_2(1,p,q;a,b)|f'(\frac{b}{m})|^q\right]^{\frac{1}{q}}.$$

Corollary 3.28. *If we take $m=1$ in Theorem 3.25, we get*

$$\left|\frac{f(a)+f(b)}{2}-\frac{p(ab)^p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{p+1}}dx\right|$$

$$\leq \frac{ab(b^p-a^p)}{2p}\left(\frac{1}{r+1}\right)^{\frac{1}{r}}\left[\varpi_1(s,p,q;a,b)|f'(a)|^q+\varpi_2(s,p,q;a,b)|f'(b)|^q\right]^{\frac{1}{q}}.$$

Corollary 3.29. *If we take $p=1$ in Theorem 3.25, we get*

$$\left|\frac{f(a)+f(b)}{2}-\frac{(ab)}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right|$$

$$\leq \frac{ab(b-a)}{2}\left(\frac{1}{r+1}\right)^{\frac{1}{r}}\left[\varpi_1(s,1,q;a,b)|f'(a)|^q+m\varpi_2(s,1,q;a,b)|f'(\frac{b}{m})|^q\right]^{\frac{1}{q}}.$$

4. Conclusion

In this paper, we have studied the class of Harmonically p-convex (Harmonically p-concave) functions which is larger then that convex (concave) and Harmonically convex (Harmonically concave) functions and have developed some interesting results for this new class. Also, we have introduced the concept of Harmonically p-quasiconvex (Harmonically p-quasiconcave) and Harmonically Logarithmic p-convex (Harmonically Logarithmic p-concave) functions. Further, we gave the results for the product of Harmonically p-convex (Harmonically p-concave) and Harmonically Logarithmic p-convex (Harmonically Logarithmic p-concave) functions.
The interested researchers are encouraged to find the particular examples of the results presented in this paper.

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