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ESSENTIAL NORM AND ESSENTIAL NUMERICAL RADIUS INEQUALITIES FOR HILBERT SPACE OPERATORS

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Abstract. We obtain some sharp inequalities for essential numerical radius $w_e(\cdot)$ of operators. Moreover, we give some applications of our results in estimation of essential norm, and compare our results with some usual norm known results. In particular, we obtain under suitable conditions if $T \in \mathcal{B}(\mathcal{H})$, then

$$w(T) \leq \|T\|_e \leq \|T\|.$$

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$. For $T \in \mathcal{B}(\mathcal{H})$

$$\begin{aligned} w(T) &= \sup\{|\langle Tx, x\rangle| : \ \|x\| = 1\}, \\ \|T\| &= \sup\{\|Tx\| : \ \|x\| = 1\}, \\ |T| &= (T^*T)^{\frac{1}{2}}, \end{aligned}$$

denote the numerical radius, the usual operator norm and the absolute value of T, respectively. It is well known that, $w(\cdot)$ defines a norm on $\mathcal{B}(\mathcal{H})$. This norm is equivalent to the operator norm. In fact, the following more precise result holds:

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\|. \tag{1.1}$$

for any $T \in \mathcal{B}(\mathcal{H})$. For basic properties and applications of the numerical radius, we refer to [3,6,7,8,10].

The inequalities in (1.1) have been improved considerably by in [9,11,14,15], it has been shown that if $T \in \mathcal{B}(\mathcal{H})$, then

$$w(T) \leq \frac{1}{2}\|(|T| + |T^*|)\| \leq \frac{1}{2}(\|T\| + \|T^2\|^{\frac{1}{2}}) \leq \|T\|, \tag{1.2}$$

$$\frac{1}{4}\|T^*T + TT^*\| \leq w^2(T) \leq \frac{1}{2}\|T^*T + TT^*\|. \tag{1.3}$$

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Moreover if T is invertible, then

$$\|T\| \leq \sqrt{2} w(T).$$

If $T = A + iB$ is the cartesian decomposition of T, then A and B are self-adjoint and $T^*T - TT^* = 2(A^2 + B^2)$. Thus, the inequality in (1.3) can be written as

$$\frac{1}{2}\|A^2 + B^2\| \leq w^2(T) \leq \|A^2 + B^2\|.$$

Let $T \in \mathcal{B}(\mathcal{H})$, the essential norm of T is defined by

$$\|T\|_e = \|T + \mathcal{K}(\mathcal{H})\| = \|\pi(T)\| = \inf\{\|T + K\|; K \in \mathcal{K}(\mathcal{H})\} \tag{1.4}$$

where $\mathcal{K}(\mathcal{H})$ is the set of compact operators on complex Hilbert space $\mathcal{H}$.

The essential numerical range $W_e(T)$ of an operator T on $\mathcal{H}$ is given by

$$W_e(T) = \bigcap_{K \in \mathcal{K}(\mathcal{H})} \overline{W(T + K)},$$

and note that the most important properties of the essential numerical range are that it is convex and its closure contains the spectrum of the operator (see [4]). The following properties of $W_e(T)$ are also well known:

(I) $W_e(T + K) = W_e(T)$ for all $K \in \mathcal{K}(\mathcal{H})$.
(II) $W_e(T^*) = \overline{W_e(T)}$.
(III) If $a, b \in \mathbb{C}, W_e(aT + bI_H) = aW_e(T) + b$.
(IV) If $U \in \mathcal{B}(\mathcal{H})$ is a unitary, then $W_e(UTU^*) = W_e(T)$.

The essential numerical radius of T is defined by

$$w_e(T) = \sup\{|\lambda|; \lambda \in W_e(T)\}$$

hence $w_e(\cdot)$ defines a norm on the Calkin algebra.

The purpose of this paper is to establish some essential norm inequalities. We also define a new concept for essential numerical radius. Based on this, we obtain some related inequalities. Essential norm inequalities and a related essential numerical radius inequality for the sum of two operators are also given.

2. Essential norm

Recently, Kriete and Moorhouse in [13] showed the following definition of the essential norm. We state it for the sake of convenience.

Definition 2.1. *Let $T \in \mathcal{B}(\mathcal{H})$, then*

$$\|T\|_e = \sup_{\{f_n\} \in E} (\limsup_{n \to \infty} \|Tf_n\|) \tag{2.1}$$

where E is the collection of all sequences $\{f_n\}$ of unit vector in $\mathcal{H}$ which tend to zero weakly.

Remark. *Since $\mathcal{H}$ is reflexive, so for every $T \in \mathcal{B}(\mathcal{H})$, $\|T\|_e = \|T^*\|_e$ (see [2]).*

To prove our generalized inequality, we need the following basic lemma.

Lemma 2.1. *Let $T, S \in \mathcal{B}(\mathcal{H})$, then*

(i) $\|TS\|_e \leq \|T\|_e \|S\|_e$.
(ii) If $0 \leq S \leq T$, *then* $\|S\|_e \leq \|T\|_e$.
(iii) $\|T\|_e^2 = \|T^*T\|_e$.

(iv) $\|T\|_e^3 = \|TT^*T\|_e$.
(v) If T is a normal operator, then $\|T\|_e^2 = \|T^2\|_e$.

Proof. (i) For $K_1, K_2 \in \mathcal{K}(\mathcal{H})$, we have

$$\begin{aligned}\|TS\|_e &\leq \|TS + TK_2 + K_1S + K_1K_2\| \quad \text{(by (1.4))}\\ &\leq \|T + K_1\| \, \|S + K_2\| .\end{aligned}$$

Take infimum on K_1, K_2, we get

$$\|TS\|_e \leq \|T\|_e \|S\|_e.$$

(ii) Let $T - S \geq 0$, then $T - S = C^*C$ for some $C \in \mathcal{B}(\mathcal{H})$, therefore

$$\begin{aligned}T - S + \mathcal{K}(\mathcal{H}) &= C^*C + \mathcal{K}(\mathcal{H})\\ &= (C^* + \mathcal{K}(\mathcal{H}))(C + \mathcal{K}(\mathcal{H}))\\ &= (C + \mathcal{K}(\mathcal{H}))^*(C + \mathcal{K}(\mathcal{H})) \geq 0.\end{aligned}$$

Which gives the inequality

$$\|S\|_e \leq \|T\|_e.$$

(iii) Let $\{f_n\}$ be unit vector in $\mathcal{H}$ which tend to zero weakly, then

$$\|Tf_n\|^2 = \langle Tf_n, Tf_n\rangle = \langle T^*Tf_n, f_n\rangle \leq \|(T^*T) f_n\| .$$

By taking $\sup\limits_{\{f_n\}\in E} \left(\lim\limits_{n\to\infty} \sup\right)$ on both sides of the inequality we have

$$\|T\|_e^2 \leq \|T^*T\|_e.$$

We can also state that

$$\|T^*T\|_e \leq \|T\|_e^2 \qquad \text{(by Lemma 2.1}\textit{(i)}\text{ and Remark 2.1)}$$

The proof is complete. *(iv)* It is not hard to see that

$$\|TT^*T\|_e \leq \|T\|_e^3 \qquad \text{(by Lemma 2.1}\textit{(i)}\text{ and Remark 2.1)}$$

On the other hand

$$\begin{aligned}\|T\|_e^4 = \left(\|T\|_e^2\right)^2 &= \|T^*T\|_e^2\\ &= \|T^*TT^*T\|_e \qquad \text{(by Lemma 2.1}\textit{(iii)}\text{)}\\ &\leq \|T^*\|_e \|TT^*T\|_e \qquad \text{(by Lemma 2.1}\textit{(i)}\text{)}\\ &= \|T\|_e \|TT^*T\|_e \qquad \text{(by Remark 2.1)}\end{aligned}$$

which is exactly the desired result.

(v) Let $\{f_n\}$ be unit vector in $\mathcal{H}$ which tend to zero weakly. Then by using normality of the operator T

$$\left\|T^2 f_n\right\| = \|T(Tf_n)\| = \|T^*(Tf_n)\| \geq \langle T^*Tf_n, f_n\rangle = \langle Tf_n, Tf_n\rangle = \|Tf_n\|^2$$

by taking $\sup\limits_{\{f_n\}\in E} \left(\lim\limits_{n\to\infty} \sup\right)$ on both sides of inequality we get

$$\left\|T^2\right\|_e \geq \|T\|_e^2 .$$

On the other hand by this Lemma *(i)*

$$\left\|T^2\right\|_e \leq \|T\|_e{}^2$$

The proof is complete. □

Remark. *When T is non normal, the inequality in Lemma 2.1(v) is not true, this can be seen from the example* $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

3. Essential numerical radius

We will often use the following well-known lemma later.

Lemma 3.1. *(see [4, corollary 5.2]). Each of the following conditions is necessary and sufficient in order that*

(i) $\lambda \in W_e(T)$;
(ii) $\langle Tf_n, f_n\rangle \to \lambda$ *for some sequence* $\{f_n\}$ *of unit vectors such that* $f_n \to 0$ *weakly;*
(iii) $\langle Tf_n, f_n\rangle \to \lambda$ *for some orthonormal sequence* $\{f_n\}$.

To achieve our main results, we state for easy reference the following definition of essential numerical radius, that will be applied below.

Definition 3.1. *Let $T \in \mathcal{B}(\mathcal{H})$ and E is the collection of all sequences $\{f_n\}$ of unit vector in $\mathcal{H}$ which tend to zero weakly then the essential numerical radius of T is defined by*

$$w_e(T) = \sup_{\{f_n\}\in E} \left\{ \limsup_{n\to\infty} |\langle Tf_n, f_n\rangle| \right\}.$$

Proposition 3.2. *Let $T \in \mathcal{B}(\mathcal{H})$, then*

(i) $w_e(T^2) \le w_e^2(T)$.
(ii) If T is a self-adjoint operator on $\mathcal{H}$, then $\|T\|_e = w_e(T)$.

Proof. (i) Let $T \in \mathcal{B}(\mathcal{H})$ and $\{f_n\}, \{g_n\}$ be unit vector in $\mathcal{H}$ which tend to zero weakly we verify (with the help of parallelogram identity) the following inequality

$$\begin{aligned} |2(\langle Tf_n, g_n\rangle + \langle Tg_n, f_n\rangle)| &= |\langle T(f_n+g_n), f_n+g_n\rangle - \langle T(f_n-g_n), f_n-g_n\rangle| \\ &\le w_e(T)\left(\|f_n+g_n\|^2 + \|f_n-g_n\|^2\right) \\ &= 2w_e(T)\left(\|f_n\|^2 + \|g_n\|^2\right) \\ &= 4w_e(T). \end{aligned}$$

Taking $Tf_n \neq 0$ and $g_n = \|Tf_n\|^{-1}Tf_n$ we conclude that

$$\|Tf_n\|^2 + |\langle T^2 f_n, f_n\rangle| \le 2w_e(T)\|Tf_n\|$$

Then

$$\begin{aligned} 0 &\le 2w_e(T)\|Tf_n\| - \|Tf_n\|^2 - |\langle T^2 f_n, f_n\rangle| \\ &= -(w_e(T) - \|Tf_n\|)^2 + w_e^2(T) - |\langle T^2 f_n, f_n\rangle| \\ &\le w_e^2(T) - |\langle T^2 f_n, f_n\rangle| \end{aligned}$$

Hence

$$|\langle T^2 f_n, f_n\rangle| \le w_e^2(T)$$

by taking $\sup\limits_{\{f_n\}\in E}\left(\lim\limits_{n\to\infty}\sup\right)$ on both sides of inequality we have

$$w_e\left(T^2\right) \le w_e^2\left(T\right).$$

(ii) Due to the fact that $|\langle Tf_n, f_n\rangle| \leq \|Tf_n\|\|f_n\|$ and $\|f_n\| = 1$, we deduce

$$w_e(T) \leq \|T\|_e.$$

On the other hand, let $M = \sup_{\{f_n\}\in E}\left\{\limsup_{n\to\infty} |\langle Tf_n, f_n\rangle|\right\}$, where $f_n, g_n \in \mathcal{H}$, then

$$\begin{aligned}
&\langle T(f_n+g_n), f_n+g_n\rangle - \langle T(f_n-g_n), f_n-g_n\rangle \\
&= 2(\langle Tf_n, g_n\rangle + \langle Tg_n, f_n\rangle) \\
&= 4\operatorname{Re}\langle Tf_n, g_n\rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\operatorname{Re}\langle Tf_n, g_n\rangle &\leq \frac{M}{4}\left(\|f_n+g_n\|^2 + \|f_n-g_n\|^2\right) \\
&= \frac{M}{2}\left(\|f_n\|^2 + \|g_n\|^2\right).
\end{aligned}$$

Now, suppose $\|f_n\| = 1$ and $Tf_n \neq 0$,if we get $g_n = \frac{Tf_n}{\|Tf_n\|}$, then

$$\operatorname{Re}\langle Tf_n, g_n\rangle = \operatorname{Re}\left\langle Tf_n, \frac{Tf_n}{\|Tf_n\|}\right\rangle = \|Tf_n\|$$

and by

$$\operatorname{Re}\langle Tf_n, g_n\rangle \leq \frac{M}{2}(\|f_n\|^2 + \|\frac{Tf_n}{\|Tf_n\|}\|^2) = M.$$

Consequently, $\|T\|_e \leq w_e(T)$.

□

As a consequence of this result we get

Corollary 3.3. *Let $T, S \in \mathcal{B}(\mathcal{H})$, then*

(i) $w_e(TS) \leq 4w_e(T)w_e(S)$.

(ii) If $T, S \in \mathcal{B}(\mathcal{H})$ and $TS = ST$. Then $w_e(TS) \leq 2w_e(T)w_e(S)$.

Proof. (i) It is not hard to see that

$$w_e(TS) \leq \|TS\|_e \leq \|T\|_e\|S\|_e \leq 4w_e(T)w_e(S).$$

(ii) First, we may assume that $w_e(T) = w_e(S) = 1$ and show that $w_e(TS) \leq 2$. Since the essential numerical radius is a norm, by triangle inequality and sub additivity of w_e we have

$$\begin{aligned}
w_e(TS) &= w_e\left(\frac{1}{4}\left[(T+S)^2 - (T-S)^2\right]\right) \\
&\leq \frac{1}{4}\left[w_e(T+S)^2 + w_e(T-S)^2\right] \\
&\leq \frac{1}{4}\left[(w_e(T+S))^2 + (w_e(T-S))^2\right] \quad \text{(by Proposition 3.1}(i)) \\
&\leq \frac{1}{4}\left[(w_e(T)+w_e(S))^2 + (w_e(T)-w_e(S))^2\right] \\
&= 2.
\end{aligned}$$

□

We will need the following observation.If S is a subset of the complex plane let $co(S)$ denote the convex hull of S.

Lemma 3.4. *If $T = A \oplus B$ on $\mathcal{H} \oplus \mathcal{H}$, then $W_e(T) = co(W_e(A) \cup W_e(B))$*

With the above lemma of essential numerical range in [1], we obtain the result of essential numerical radius.

Theorem 3.5. *If $T = A \oplus B$ on $\mathcal{H} \oplus \mathcal{H}$, then $w_e(T) = \max\{(w_e(A), w_e(B))\}$.*

Proof. Since $W_e(A) \subseteq W_e(A \oplus B)$ and $W_e(B) \subseteq W_e(A \oplus B)$, we have $w_e(A) \leq w_e(A \oplus B)$ and $w_e(B) \leq w_e(A \oplus B)$, and therefore

$$\max\{w_e(A), w_e(B)\} \leq w_e(A \oplus B).$$

On the other hand

$$\begin{aligned} w_e(A \oplus B) &= \sup\{|\lambda| : \lambda \in W_e(A \oplus B)\} \\ &= \sup\left\{|t_1x_1 + t_2x_2| : t_1 + t_2 = 1, \begin{array}{c} x_1 \in w_e(A) \\ x_2 \in w_e(B) \end{array}\right\} \\ &\leq \sup\left\{t_1|x_1| + t_2|x_2| : t_1 + t_2 = 1, \begin{array}{c} x_1 \in w_e(A) \\ x_2 \in w_e(B) \end{array}\right\} \\ &\leq \max\{w_e(A), w_e(B)\}. \end{aligned}$$

□

Utilizing the Cartesian decomposition for operators and same strategies used in [12], one can obtain the following inequality.

Theorem 3.6. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$\frac{1}{4}\|T^*T + TT^*\|_e \leq w_e^2(T) \leq \frac{1}{2}\|T^*T + TT^*\|_e.$$

As a particular case of interest, we can state that:

Corollary 3.7. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$\frac{1}{2}\|T\|_e \leq w_e(T) \leq \|T\|_e.$$

Proof. The desired result follows from the chain of the inequalities

$$\frac{1}{4}\|T\|_e^2 \leq \frac{1}{4}\|T^*T + TT^*\|_e \leq w_e^2(T) \leq \frac{1}{2}\|T^*T + TT^*\|_e \leq \|T\|^2{}_e. \tag{3.1}$$

The first inequality in (3.1) follows from the fact that $0 \leq T^*T \leq T^*T + TT^*$ and lemma(2.1(ii)), and the last inequality in (3.1) follows by the triangle inequality essential norm in Calkin algebra and lemma(2.1(iii)) □

Corollary 3.8. *Let $T \in \mathcal{B}(\mathcal{H})$ with the Cartesian decomposition $T = A + iB$, then*

$$\frac{1}{2}\|A^2 + B^2\|_e \leq w_e^2(T) \leq \|A^2 + B^2\|_e. \tag{3.2}$$

Remark

Finally in this section, we obtain the relation between essential norm and numerical radius.

In general the inequality $w(T) \leq \|T\|_e$ is not true. For example if $T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then $\|T\|_e = 0$. In the following we show that $w(T) \leq \|T\|_e$ is true under suitable conditions.

We need the next lemma, which will be very crucial for our purpose. For the reader's convenience, we state it again (see [5, lemma 2.1]]).

Lemma 3.9. *Let $T \in \mathcal{B}(\mathcal{H})$, then there exists an orthonormal sequence f_n such that $\|Tf_n\| \to \|T\|_e$. Furthermore, if P is an infinite rank projection and $TP = T$, then we can choose f_n so that the additional condition $Pf_n = f_n$ for all n is satisfied.*

Theorem 3.10. *Let $T \in \mathcal{B}(\mathcal{H})$, if there exists an infinite rank projection P such that $TP = T$, then $w(T) \leq \|T\|_e$.*

Proof. As shown in the proof of Lemma 4.4, there exists an orthonormal sequence $\{f_n\}$ such that $\|Tf_n\| \leq \|T\|_e$ for all n, then for $x = \sum\limits_{n=1}^{\infty} \langle x, f_n \rangle f_n$ we have

$$\begin{aligned}
|\langle Tx, x\rangle| &= \left| \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \langle x, f_n \rangle \overline{\langle x, f_m \rangle} \langle Tf_n, f_m \rangle \right| \\
&\leq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\langle x, f_n \rangle| \, |\langle x, f_m \rangle| \, |\langle Tf_n, f_m \rangle| \\
&\leq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\langle x, f_n \rangle| \, |\langle x, f_m \rangle| \, \|Tf_n\| \\
&\leq \|T\|_e \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\langle x, f_n \rangle| \, |\langle x, f_m \rangle| \\
&\leq \|T\|_e \left(\sum_{n=1}^{\infty} |\langle x, f_n \rangle|^2 \right)^{\frac{1}{2}} \left(\sum_{m=1}^{\infty} |\langle x, f_m \rangle|^2 \right)^{\frac{1}{2}} && \text{(by the Cauchy-Schwarz inequality)} \\
&= \|T\|_e \|x\|^2. && \text{(by the Parseval equation)}
\end{aligned}$$

Now the result follows by taking the supremum over all unit vectors in $\mathcal{H}$. □

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