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ON OSTROWSKI TYPE INEQUALITIES FOR GENERALIZED k-FRACTIONAL INTEGRALS

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Abstract. Over the years, the integral inequalities received special attention from many researchers due to their importance and applications. Consequently, this theory has been extended and generalized in various different directions by using innovative ideas and techniques. The concept of fractional calculus plays an important role in many fields of mathematics. Recently, by applying the fractional integral operators, many researchers have obtained a lot of fractional integral inequalities. In this paper, we use the left and right-sided Riemann-Liouville fractional integrals to establish some new integral inequalities of Ostrowski type in terms of a new parameter $k > 0$. From our results, the classical Ostrowski inequalities can be deduced as some special cases.

1. Introduction

Fractional calculus was introduced at the end of the nineteenth century by Liouville and Riemann. This subject has become a rapidly growing area and has found applications in many fields of sciences and engineering such as biophysics, quantum mechanics, wave theory, continuum mechanics, Lie theory, spectroscopy and in group theory. In particular, fractional integrals, fractional derivatives and inequalities involving these two operators arise in many engineering and scientific disciplines. Therefore, the researchers have a great interest in this field due to vast applications of these inequalities.

The classical Ostrowski's inequality [1] was proved by Ostrowski in 1938. It gives an upper bound for the approximation of the integral average $\frac{1}{b-a}\int_a^b f(x)dx$ by the value $f(x)$ at point $x \in [a,b]$. It is defined as follows:

Definition 1. Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Assume that $|f'(x)| \leq M$ for all $x \in (a,b)$, then

$$|f(x) - T(f;a,b)| \leq \frac{M}{b-a}\frac{(b-x)^2 + (x-a)^2}{2} \tag{1.1}$$

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for all $x \in (a,b)$, where $T(f;a,b) = \frac{1}{b-a}\int_a^b f(x)dx$.

During past several years, many researchers and mathematicians [2, 3, 4, 5, 6, 7] have worked on this inequality and found many generalizations, variants and extensions of this inequality.

In the years thereafter, numerous generalizations, extensions and variants of Grüss inequality [8] have appeared in the literature (see [9, 10, 11, 12, 13]). It states that:

Definition 2. If $f, g : [a,b] \to \mathbb{R}$ be integrable functions such that $\varphi < f(x) < \Phi$ and $\psi < g(x) < \Psi$ for all $x \in [a,b]$, where $\varphi, \Phi, \psi, \Psi \in \mathbb{R}$. Then

$$\left| \frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx \int_a^b g(x)dx \right| \leq \frac{1}{4}(\Phi-\varphi)(\Psi-\psi), \tag{1.2}$$

where the constant $\frac{1}{4}$ is sharp.

Now, we recall definitions and preliminary facts of fractional calculus which will be used in this paper.

Definition 3. Korkine's identity [14] states that if f and g be integrable functions on $[a,b]$, then

$$T(fg;a,b) - T(f;a,b)T(g;a,b) = \frac{1}{2(b-a)^2}\int_a^b \int_a^b (f(t)-f(s))(g(t)-g(s))dsdt. \tag{1.3}$$

Definition 4. A function f is said to be in $L_{p,r}[a,b]$ if

$$\left(\int_a^b |f(t)|^p t^r dt\right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty, r \in \mathbb{R}\backslash\{-1\}.$$

The Riemann-Liouville fractional integrals play a major role in fractional calculus. They are defined as follows:

Definition 5. Let $f \in L_1[a,b]$. Then left and right sided Riemann-Liouville fractional integral of order $\alpha \geq 0$ are defined by

$$I_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x \in [a,b]$$

and

$$I_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x \in [a,b],$$

where Γ is the Euler gamma function.

Definition 6. let $f \in L_{1,r}[a,b]$, then the generalized left and right sided Riemann-Liouville fractional integral of order $\alpha \geq 0$ and $\mathbb{R}\backslash\{-1\}$, introduced by Katugompola [15], are defined by

$$I_{a^+}^{\alpha,r} f(x) = \frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)}\int_a^x (x^{r+1}-t^{r+1})^{\alpha-1} t^r f(t)dt, \quad x \in [a,b]$$

and

$$I_{b^-}^{\alpha,r} f(x) = \frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)} \int_x^b (t^{r+1} - x^{r+1})^{\alpha-1} t^r f(t) dt, \quad x \in [a,b],$$

where Γ is the Euler gamma function.

Many researchers [16, 17, 18, 19, 20, 21, 22, 23] paid attention to these integrals and obtained many genaralizations and inequalities involving Riemann-Liouville fractional integrals like Grüss, Chebyshev and Hermite-Hadamard type inequalities for integrable functions as well as for convex functions.

Recently, Diaz and Pariguan [24] have defined the generalization of the classical gamma and beta functions in terms of a new parameter $k > 0$, called gamma and beta k-functions respectively.

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n! k^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}}, \; k > 0, x \in \mathbb{C} \setminus kZ^-,$$

where

$$(x)_{n,k} = x(x+k)(x+2k)\cdots(x+(n-1)k), \quad n \geq 1,$$

is called Pochhammer k-symbol. The integral representation of gamma k-function is

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}} dt, \; Re(x) > 0.$$

In 2010, Krasniqi [25] proved some inequalities and monotonicity for the ratio of gamma k-function. Later, in 2012, by using the above definitions, Mubeen and Habibullah [26] have introduced the left sided k-fractional integral of the Riemann Liouville type as

Definition 7. Let $f \in L_1[a,b]$, then

$$I_{a^+,k}^{\alpha} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt, \quad x \in [a,b], \tag{1.4}$$

where Γ_k is the Euler gamma k-function.

In 2013, Kokologiannaki and Krasniqi [27] gave completely monotonicity properties and inequalities for functions involving the gamma and psi k-functions. They also introduced the k-analogue of the Riemann Zeta function and obtained some inequalities relating gamma and zeta k-functions. Romero et al. [28] introduced a new fractional operator called k-Riemann-Liouville fractional derivative by using gamma k-function. They also proved some properties of this newly defined fractional operator and found its relationship with Riemann-Liouville k-fractional integral.

In 2014, Sarikaya and Karaca [29] gave a generalization of Riemann-Liouville k-fractional integral with some properties. Also they proved some new integral inequalities involving this generalized Riemann-Liouville k-fractional integral.

In 2015, Sarikaya et al. [30] have introduced the left sided (k,r)-fractional integral of the Reimann type as

Definition 8. Let $f \in L_{1,r}[a,b]$

$$I^{\alpha,r}_{a^+,k}f(x) = \frac{(r+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^x (x^{r+1} - t^{r+1})^{\frac{\alpha}{k}-1} t^r f(t)dt, \quad x > a. \tag{1.5}$$

where Γ_k is the Euler gamma k-function.

In view of the wider applications, integral inequalities have received considerable attention. Recently, different versions of such inequalities have been developed which are useful in the study of different classes of differential and integral equations. It is well known that Ostrowski type inequalities play a crucial role in studying the qualitative behavior of differential and difference equations, as well as many other areas of mathematics.

Motivated by the definitions 7 and 8, we have introduced right sided generalized Riemann-Liouville k-fractional integral. Due to the wide application of fractional integrals, some authors extended the theory of fractional integral inequalities in many directions. The main aim of this work is to develop new integral inequalities of Ostrowski type involving the left and right sided generalized Riemann-Liouville fractional integrals in terms of a new parameter $k > 0$. From our results, the classical Ostrowski type inequality can be deduced as some special case and some further special cases are also discussed.

2. Main Results

In this section, we establish some new Ostrowski type inequalities involving the left and right sided Riemann-Liouville fractional integrals in terms of a new parameter $k > 0$, which generalize the inequalities of [31]. First of all, we define the right sided generalized Riemann-Liouville k-fractional integral.

Definition 9. Let $f \in L_{1,r}[a,b]$, then the right sided generalized Riemann-Liouville k-fractional integral, denoted by $I^{\alpha,r}_{b^-,k}$, of order $\alpha \geq 0$ and $r \in \mathbb{R}\backslash\{-1\}$ is defined as

$$I^{\alpha,r}_{b^-,k}f(x) = \frac{(r+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_x^b (t^{r+1} - x^{r+1})^{\frac{\alpha}{k}-1} t^r f(t)dt, \quad b > x, \tag{2.1}$$

where Γ_k is the Euler gamma k-function.

Now, we prove that the integral defined in (2.1) is well defined.

Theorem 2.1. *For $k > 0$, let $f \in L_{1,r}[a,b]$, $r \in \mathbb{R}\backslash\{-1\}$ and $\alpha \geq 0$, then $I^{\alpha,r}_{b^-,k}$ exists for any $x \in [a,b]$.*

Proof. Let $\Omega := [a,b] \times [a,b]$ and $F : \Omega \to \mathbb{R}$ is defined by

$$F(x,t) = (t^{r+1} - x^{r+1})^{\frac{\alpha}{k}-1} t^r.$$

It is clear to see that $F = F_+ + F_-$, where

$$F_+(x,t) := \begin{cases} (t^{r+1} - x^{r+1})^{\frac{\alpha}{k}-1} t^r & a \leq x \leq t \leq b \\ \\ 0 & a \leq t \leq x \leq b \end{cases}$$

and

$$F_-(x,t) := \begin{cases} (x^{r+1} - t^{r+1})^{\frac{\alpha}{k}-1} t^r & a \leq x \leq t \leq b \\ \\ 0 & a \leq t \leq x \leq b. \end{cases}$$

Since F is measurable on Ω, then we can write

$$\int_a^b F(x,t)dt = \int_x^b F(x,t)dt = \int_x^b (t^{r+1}-x^{r+1})^{\frac{\alpha}{k}-1} t^r dt = \frac{k}{\alpha(r+1)}(b^{r+1}-x^{r+1})^{\frac{\alpha}{k}}.$$

Now, by using repeated integrals, we have

$$\begin{aligned}\int_a^b \left(\int_a^b F(x,t)|f(x)|dt\right)dx &= \int_a^b |f(x)| \left(\int_a^b F(x,t)dt\right)dx \\ &= \frac{k}{\alpha(r+1)}\int_a^b (b^{r+1}-x^{r+1})^{\frac{\alpha}{k}}|f(x)|dx \\ &\leq \frac{k}{\alpha(r+1)}(b^{r+1}-a^{r+1})^{\frac{\alpha}{k}}\int_a^b |f(x)|dx \\ &= \frac{k}{\alpha(r+1)}(b^{r+1}-a^{r+1})^{\frac{\alpha}{k}}\|f(x)\|_{L_1[a,b]} \\ &< \infty.\end{aligned}$$

Therefore, the function $G:\Omega\to\mathbb{R}$, $G(x,t) := F(x,t)f(x)$ is integrable over Ω by Tonelli's theorem. Hence, by Fubini's theorem $\int_a^b F(x,t)f(x)dx$ is an integrable function on $[a,b]$, as a function of $t\in[a,b]$. That is, $I^{\alpha,r}_{b^-,k}f(x)$ exists. □

Theorem 2.2. *For $k>0$, if $f,g\in L_{1,r}[a,b]$, $r\in\mathbb{R}\backslash\{-1\}$, then*

$$\begin{aligned}& I^{\alpha,r}_{a^+,k}[f(b)g(b)] - \frac{\Gamma_k(\alpha+k)(r+1)^{\frac{\alpha}{k}}}{(b^{r+1}-a^{r+1})^{\frac{\alpha}{k}}} I^{\alpha,r}_{a^+,k}f(b) I^{\alpha,r}_{a^+,k}g(b) \\ &= \frac{\alpha(r+1)^{2-\frac{\alpha}{k}}}{2k^2\Gamma_k(\alpha)(b^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}\int_a^b\int_a^b (f(t)-f(s))(g(t)-g(s)) \\ &\quad \times (b^{r+1}-s^{r+1})^{\frac{\alpha}{k}-1}(b^{r+1}-t^{r+1})^{\frac{\alpha}{k}-1}t^r s^r dsdt. \end{aligned} \tag{2.2}$$

Proof. For $(f(t)-f(s))(g(t)-g(s))$, we have

$$\begin{aligned}&\int_a^b\int_a^b (f(t)-f(s))(g(t)-g(s))(b^{r+1}-s^{r+1})^{\frac{\alpha}{k}-1}(b^{r+1}-t^{r+1})^{\frac{\alpha}{k}-1}t^r s^r dsdt \\ &\qquad = 2\left[\int_a^b (b^{r+1}-s^{r+1})^{\frac{\alpha}{k}-1}s^r ds \int_a^b (b^{r+1}-t^{r+1})^{\frac{\alpha}{k}-1} f(t)g(t)t^r dt\right] \\ &\qquad -2\left[\int_a^b (b^{r+1}-s^{r+1})^{\frac{\alpha}{k}-1}g(s)s^r ds \int_a^b (b^{r+1}-s^{r+1})^{\frac{\alpha}{k}-1} f(t)t^r dt\right] \\ &= \frac{2k^2(b^{r+1}-a^{r+1})^{\frac{\alpha}{k}}\Gamma_k(\alpha)}{\alpha(r+1)^{2-\frac{\alpha}{k}}} I^{\alpha,r}_{a^+,k}f(b)g(b) - \frac{2k^2\Gamma_k(\alpha)}{(r+1)^{2-\frac{\alpha}{k}}} I^{\alpha,r}_{a^+,k}f(b) I^{\alpha,r}_{a^+,k}g(b),\end{aligned}$$

which gives the required equality. □

Remark. *If we use $\alpha=k$ in (2.2), we obtain the following identity*

$$\begin{aligned}& T_r(fg;a,b) - T_r(f;a,b)T_r(g;a,b) \\ &\quad = \frac{(r+1)^2}{2(b^{r+1}-a^{r+1})^2}\int_a^b\int_a^b (f(t)-f(s))(g(t)-g(s))t^r s^r dsdt,\end{aligned} \tag{2.3}$$

where $T_r(f;a,b)=\frac{(r+1)}{2(b^{r+1}-a^{r+1})}\int_a^b f(t)t^r dt.$

Theorem 2.3. *Let f be differentiable function on $[a,b]$ and $|f'(x)|\leq M$ for any $x\in[a,b]$. Then the following generalized fractional inequality holds for $\alpha\geq 0$ and $r\in\mathbb{R}\backslash\{-1\}$*

$$\begin{aligned}
&\frac{(r+1)^{-\frac{\alpha}{k}}}{k\Gamma_k(\alpha+k)}\left[(x^{r+1}-a^{r+1})^{\frac{\alpha}{k}}+(b^{r+1}-x^{r+1})^{\frac{\alpha}{k}}\right]x^r f(x)\\
&\quad-\frac{1}{k}\left[I^{\alpha,r}_{x^-,k}(a^r f(a))+I^{\alpha,r}_{x^+,k}(b^r f(b))\right]-r\left[I^{\alpha+k,r}_{x^-,k}\frac{f(a)}{a}-I^{\alpha+k,r}_{x^+,k}\frac{f(b)}{b}\right]\\
&\qquad\leq\frac{(r+1)^{-\frac{\alpha}{k}-1}M}{\Gamma_k(\alpha+2k)}\left[(x^{r+1}-a^{r+1})^{\frac{\alpha}{k}+1}+(b^{r+1}-x^{r+1})^{\frac{\alpha}{k}+1}\right].
\end{aligned}\tag{2.4}$$

Proof. Since

$$I^{\alpha+k,r}_{x^-,k}f'(a)=\frac{(r+1)^{-\frac{\alpha}{k}}}{k\Gamma_k(\alpha+k)}\int_a^x(t^{r+1}-a^{r+1})^{\frac{\alpha}{k}}t^r f'(t)dt.$$

Integration by parts gives us

$$\begin{aligned}
I^{\alpha+k,r}_{x^-,k}f'(a)=&\frac{(r+1)^{-\frac{\alpha}{k}}x^r(x^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha+k)}f(x)\\
&-\frac{1}{k}I^{\alpha,r}_{x^-,k}(a^r f(a))-rI^{\alpha+k,r}_{x^-,k}\left(\frac{f(a)}{a}\right).
\end{aligned}\tag{2.5}$$

Similarly

$$\begin{aligned}
I^{\alpha+k,r}_{x^+,k}f'(b)=&\frac{-(r+1)^{-\frac{\alpha}{k}}x^r(b^{r+1}-x^{r+1})^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha+k)}f(x)\\
&+\frac{1}{k}I^{\alpha,r}_{x^+,k}(b^r f(b))-rI^{\alpha+k,r}_{x^+,k}\left(\frac{f(b)}{b}\right).
\end{aligned}\tag{2.6}$$

Now as

$$\begin{aligned}
&I^{\alpha+k,r}_{x^-,k}f'(a)-I^{\alpha+k,r}_{x^+,k}f'(b)\\
&=\frac{(r+1)^{-\frac{\alpha}{k}}}{k\Gamma_k(\alpha+k)}\left[\int_a^x(t^{r+1}-a^{r+1})^{\frac{\alpha}{k}}t^r f'(t)dt-\int_x^b(b^{r+1}-t^{r+1})^{\frac{\alpha}{k}}t^r f'(t)dt\right]\\
&\leq\frac{(r+1)^{-\frac{\alpha}{k}-1}M}{\Gamma_k(\alpha+2k)}\left[(x^{r+1}-a^{r+1})^{\frac{\alpha}{k}+1}+(b^{r+1}-x^{r+1})^{\frac{\alpha}{k}+1}\right].
\end{aligned}\tag{2.7}$$

The inequality (2.7) together with relations (2.5) and (2.6) yields the required inequality. □

Remark. *If we use $r=0$ in inequality (2.4), we get*

$$\begin{aligned}
&\left|\frac{(x-a)^{\frac{\alpha}{k}}+(b-x)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}f(x)-I^{\alpha}_{x^-,k}f(a)-I^{\alpha}_{x^+,k}f(b)\right|\\
&\qquad\leq\frac{Mk\left((x-a)^{\frac{\alpha}{k}+1}+(b-x)^{\frac{\alpha}{k}+1}\right)}{\Gamma_k(\alpha+2k)}.
\end{aligned}\tag{2.8}$$

Remark. *Take $\alpha=k=1$ in (2.8), we get*

$$f(x)-T(f;a,b)\leq\frac{M}{b-a}\frac{(b-x)^2+(x-a)^2}{2},$$

which is Ostrowski's inequality.

Theorem 2.4. *Let $f:[a,b]\to\mathbb{R}$ be differentiable function and $f'\in L_{2,r}[a,b]$. If f' is bounded on $[a,b]$ with $m\le f'(x)\le M$, then the following inequality*

$$\begin{aligned}
&\frac{(r+1)^{-\frac{\alpha}{k}+1}(x^{r+1}-a^{r+1})^{\frac{\alpha}{k}-1}}{\Gamma_k(\alpha)}\left\{x^r f(x)-\frac{k}{\alpha+k}\int_a^x f'(t)t^r dt\right\}\\
&\quad-\frac{\alpha(r+1)}{x^{r+1}-a^{r+1}}\left\{I^{\alpha,r}_{x^-,k}(a^r f(a)+rkI^{\alpha+k,r}_{x^-,k}\left(\frac{f(a)}{a}\right)\right\}\\
&+\frac{(r+1)^{-\frac{\alpha}{k}+1}(b^{r+1}-x^{r+1})^{\frac{\alpha}{k}-1}}{\Gamma_k(\alpha)}\left\{x^r f(x)+\frac{k}{\alpha+k}\int_x^b f'(t)t^r dt\right\}\\
&\quad-\frac{\alpha(r+1)}{b^{r+1}-x^{r+1}}\left\{I^{\alpha,r}_{x^+,k}(b^r f(b)-rkI^{\alpha+k,r}_{x^+,k}\left(\frac{f(b)}{b}\right)\right\}\\
&\qquad\le \eta_k(x^{r+1}-a^{r+1})^{\frac{\alpha}{k}}\left\{T_r(f'^2;a,x)-T_r^2(f';a,x)\right\}^{\frac{1}{2}}\\
&\qquad\quad+\eta_k(b^{r+1}-x^{r+1})^{\frac{\alpha}{k}}\left\{T_r(f'^2;x,b)-T_r^2(f';x,b)\right\}^{\frac{1}{2}}\\
&\qquad\le \eta_k(M-m)\left[\frac{(x^{r+1}-a^{r+1})^{\frac{\alpha}{k}}+(b^{r+1}-x^{r+1})^{\frac{\alpha}{k}}}{2}\right]
\end{aligned}\tag{2.9}$$

holds, where

$$\eta_k=\frac{(r+1)^{-\frac{\alpha}{k}}}{\Gamma_k(\alpha)}\sqrt{\frac{k}{2\alpha+k}-\frac{k^2}{(\alpha+k)^2}}$$

Proof. From relations (2.5) and (2.6), we have

$$\begin{aligned}
\frac{(r+1)^{-\frac{\alpha}{k}}}{\Gamma_k(\alpha)(x^{r+1}-a^{r+1})}\int_a^x(t^{r+1}-a^{r+1})^{\frac{\alpha}{k}}t^r f'(t)dt&=\frac{(r+1)^{-\frac{\alpha}{k}+1}x^r(x^{r+1}-a^{r+1})^{\frac{\alpha}{k}-1}}{\Gamma_k(\alpha)}f(x)\\
&\quad-\frac{\alpha(r+1)}{x^{r+1}-a^{r+1}}\left\{I^{\alpha,r}_{x^-,k}(a^r f(a))+rkI^{\alpha+k,r}_{x^-,k}\left(\frac{f(a)}{a}\right)\right\}.
\end{aligned}\tag{2.10}$$

and

$$\begin{aligned}
\frac{(r+1)^{-\frac{\alpha}{k}}}{\Gamma_k(\alpha)(b^{r+1}-x^{r+1})}\int_x^b(b^{r+1}-t^{r+1})^{\frac{\alpha}{k}}t^r f'(t)dt&=\frac{(r+1)^{-\frac{\alpha}{k}+1}x^r(b^{r+1}-x^{r+1})^{\frac{\alpha}{k}-1}}{\Gamma_k(\alpha)}f(x)\\
&\quad-\frac{\alpha(r+1)}{b^{r+1}-x^{r+1}}\left\{I^{\alpha,r}_{x^+,k}(b^r f(b))-rkI^{\alpha+k,r}_{x^+,k}\left(\frac{f(b)}{b}\right)\right\}.
\end{aligned}\tag{2.11}$$

Adding

$$-\frac{k(r+1)^{-\frac{\alpha}{k}+1}}{(\alpha+k)\Gamma_k(\alpha)}\left\{(x^{r+1}-a^{r+1})^{\frac{\alpha}{k}-1}\int_a^x t^r f'(t)dt-(b^{r+1}-x^{r+1})^{\frac{\alpha}{k}-1}\int_x^b t^r f'(t)dt\right\}$$

on both sides after adding (2.10) and (2.11), we get

$$\begin{aligned}
&\frac{(r+1)^{-\frac{\alpha}{k}+1}}{\Gamma_k(\alpha)}(x^{r+1}-a^{r+1})^{\frac{\alpha}{k}-1}\left\{x^r f(x)-\frac{k}{\alpha+k}\int_a^x t^r f'(t)dt\right\}\\
&\quad-\frac{\alpha(r+1)}{x^{r+1}-a^{r+1}}\left\{I^{\alpha,r}_{x^-,k}(a^r f(a))+rkI^{\alpha+k,r}_{x^-,k}\left(\frac{f(a)}{a}\right)\right\}\\
&+\frac{(r+1)^{-\frac{\alpha}{k}+1}}{\Gamma_k(\alpha)}(b^{r+1}-x^{r+1})^{\frac{\alpha}{k}-1}\left\{x^r f(x)+\frac{k}{\alpha+k}\int_x^b t^r f'(t)dt\right\}\\
&\quad-\frac{\alpha(r+1)}{b^{r+1}-x^{r+1}}\left\{I^{\alpha,r}_{x^+,k}(b^r f(b))-rkI^{\alpha+k,r}_{x^+,k}\left(\frac{f(b)}{b}\right)\right\}\\
&=\frac{(r+1)^{\frac{\alpha}{k}+1}}{\Gamma_k(\alpha)(x^{r+1}-a^{r+1})}\int_a^x (t^{r+1}-a^{r+1})^{\frac{\alpha}{k}}t^r f'(t)dt\\
&\quad-\frac{k(r+1)^{-\frac{\alpha}{k}+1}}{(\alpha+k)\Gamma_k(\alpha)}(x^{r+1}-a^{r+1})^{\frac{\alpha}{k}-1}\int_a^x t^r f'(t)dt\\
&\quad-\frac{(r+1)^{\frac{\alpha}{k}+1}}{\Gamma_k(\alpha)(b^{r+1}-x^{r+1})}\int_x^b (b^{r+1}-t^{r+1})^{\frac{\alpha}{k}}t^r f'(t)dt\\
&\quad+\frac{k(r+1)^{-\frac{\alpha}{k}+1}}{(\alpha+k)\Gamma_k(\alpha)}(b^{r+1}-x^{r+1})^{\frac{\alpha}{k}-1}\int_x^b t^r f'(t)dt. \qquad (2.12)
\end{aligned}$$

Now using Korkine's identity for the generalized Riemann-Liouville k-fractional integral on the right hand side of (2.12), we get

$$\begin{aligned}
&\frac{(r+1)^{-\frac{\alpha}{k}+1}}{\Gamma_k(\alpha)}(x^{r+1}-a^{r+1})^{\frac{\alpha}{k}-1}\left\{x^r f(x)-\frac{k}{\alpha+k}\int_a^x t^r f'(t)dt\right\}\\
&\quad-\frac{\alpha(r+1)}{x^{r+1}-a^{r+1}}\left\{I^{\alpha,r}_{x^-,k}(a^r f(a))+rkI^{\alpha+k,r}_{x^-,k}\left(\frac{f(a)}{a}\right)\right\}\\
&\quad+\frac{(r+1)^{-\frac{\alpha}{k}+1}}{\Gamma_k(\alpha)}(b^{r+1}-x^{r+1})^{\frac{\alpha}{k}-1}\left\{x^r f(x)+\frac{k}{\alpha+k}\int_x^b t^r f'(t)dt\right\}\\
&\quad-\frac{\alpha(r+1)}{b^{r+1}-x^{r+1}}\left\{I^{\alpha,r}_{x^+,k}(b^r f(b))-rkI^{\alpha+k,r}_{x^+,k}\left(\frac{f(b)}{b}\right)\right\}\\
&=\frac{(r+1)^{-\frac{\alpha}{k}+2}}{2\Gamma_k(\alpha)(x^{r+1}-a^{r+1})^2}\int_a^x\int_a^x\left[(t^{r+1}-a^{r+1})^{\frac{\alpha}{k}}-(s^{r+1}-a^{r+1})^{\frac{\alpha}{k}}\right](f'(t)-f'(s))t^r s^r dsdt\\
&\quad+\frac{(r+1)^{-\frac{\alpha}{k}+2}}{2\Gamma_k(\alpha)(b^{r+1}-x^{r+1})^2}\int_x^b\int_x^b\left[(b^{r+1}-s^{r+1})^{\frac{\alpha}{k}}-(b^{r+1}-t^{r+1})^{\frac{\alpha}{k}}\right](f'(t)-f'(s))t^r s^r dsdt \qquad (2.13)
\end{aligned}$$

Using the Cauchy-Schwarz inequality on the double integrals in 2.13, we get

$$\begin{aligned}
&\int_a^x\int_a^x\left[(t^{r+1}-a^{r+1})^{\frac{\alpha}{k}}-(s^{r+1}-a^{r+1})^{\frac{\alpha}{k}}\right](f'(t)-f'(s))t^{r/2}t^{r/2}s^{r/2}s^{r/2}dsdt\\
&\quad\le\left(\int_a^x\int_a^x\left[(t^{r+1}-a^{r+1})^{\frac{\alpha}{k}}-(s^{r+1}-a^{r+1})^{\frac{\alpha}{k}}\right]^2 t^r s^r dsdt\right)^{\frac{1}{2}}\\
&\qquad\times\left(\int_a^x\int_a^x (f'(t)-f'(s))^2 t^r s^r dsdt\right)^{\frac{1}{2}}. \qquad (2.14)
\end{aligned}$$

However

$$\int_a^x\int_a^x\left[(t^{r+1}-a^{r+1})^{\frac{\alpha}{k}}-(s^{r+1}-a^{r+1})^{\frac{\alpha}{k}}\right]^2 t^r s^r dsdt$$
$$=\frac{2k(x^{r+1}-a^{r+1})^{\frac{2\alpha}{k}+2}}{(r+1)^2}\left(\frac{1}{2\alpha+k}-\frac{k}{(\alpha+k)^2}\right) \tag{2.15}$$

and

$$\int_a^x\int_a^x(f'(t)-f'(s))^2 t^r s^r dsdt$$
$$=\frac{2(x^{r+1}-a^{r+1})^2}{(r+1)^2}\left[T_r(f'^2;a,x)-T_r^2(f';a,x)\right] \tag{2.16}$$

So by (2.14)-(2.16), we have the following inequality

$$\frac{(r+1)^{-\frac{\alpha}{k}+2}}{2\Gamma_k(\alpha)(x^{r+1}-a^{r+1})^2}\int_a^x\int_a^x\left[(t^{r+1}-a^{r+1})^{\frac{\alpha}{k}}-(s^{r+1}-a^{r+1})^{\frac{\alpha}{k}}\right](f'(t)-f'(s))t^r s^r dsdt$$
$$\leq (x^{r+1}-a^{r+1})^{\frac{\alpha}{k}}\eta_k\left[T_r(f'^2;a,x)-T_r^2(f';a,x)\right]^{\frac{1}{2}}. \tag{2.17}$$

Similarly

$$\frac{(r+1)^{-\frac{\alpha}{k}+2}}{2\Gamma_k(\alpha)(b^{r+1}-x^{r+1})^2}\int_x^b\int_x^b\left[(b^{r+1}-s^{r+1})^{\frac{\alpha}{k}}-(b^{r+1}-t^{r+1})^{\frac{\alpha}{k}}\right](f'(t)-f'(s))t^r s^r dsdt$$
$$\leq (b^{r+1}-x^{r+1})^{\frac{\alpha}{k}}\eta_k\left[T_r(f'^2;x,b)-T_r^2(f';x,b)\right]^{\frac{1}{2}}. \tag{2.18}$$

By using (2.13), (2.17) and (2.18), we obtain first inequality of 2.9. To prove second inequality, use the condition $m\leq f'(x)\leq M$ on $[a,b]$. Then by Grüss inequality, we have

$$T_r(f'^2;a,x)-T_r^2(f';a,x)\leq\frac{1}{4}(M-m) \tag{2.19}$$

and

$$T_r(f'^2;x,b)-T_r^2(f';x,b)\leq\frac{1}{4}(M-m). \tag{2.20}$$

□

Remark. *Letting $r=0$ in (2.9), we obtain*

$$\frac{\alpha f(x)+kf(a)}{(\alpha+k)\Gamma_k(\alpha)}(x-a)^{\frac{\alpha}{k}-1}-\frac{\alpha}{x-a}I^{\alpha}_{x^-,k}f(a)$$
$$+\frac{\alpha f(x)+kf(b)}{(\alpha+k)\Gamma_k(\alpha)}(b-x)^{\frac{\alpha}{k}-1}-\frac{\alpha}{b-x}I^{\alpha}_{x^+,k}f(b)$$
$$\leq\sqrt{\frac{k}{2\alpha+k}-\frac{k^2}{(\alpha+k)^2}}\frac{1}{\Gamma_k(\alpha)}(x-a)^{\frac{\alpha}{k}}\left\{T(f'^2;a,x)-T^2(f';a,x)\right\}^{\frac{1}{2}}$$
$$+\sqrt{\frac{k}{2\alpha+k}-\frac{k^2}{(\alpha+k)^2}}\frac{1}{\Gamma_k(\alpha)}(b-x)^{\frac{\alpha}{k}}\left\{T(f'^2;x,b)-T^2(f';x,b)\right\}^{\frac{1}{2}}$$
$$\leq\sqrt{\frac{k}{2\alpha+k}-\frac{k^2}{(\alpha+k)^2}}\frac{1}{\Gamma_k(\alpha)}\frac{(x-a)^{\frac{\alpha}{k}}+(b-x)^{\frac{\alpha}{k}}}{2}(M-m). \tag{2.21}$$

Corollary 2.5. *Under the assumptions of theorem 2.4 with $\alpha = k$, then inequality (2.9) becomes*

$$\begin{aligned}
&2x^r f(x) + \frac{1}{2}\left\{\int_x^b f'(t)t^r dt - \int_a^x f'(t)t^r dt\right\} \\
&\quad -\frac{r+1}{x^{r+1}-a^{r+1}}\left\{I^{k,r}_{x^-,k}(a^r f(a)) + rkI^{2k,r}_{x^-,k}\left(\frac{f(a)}{a}\right)\right\} \\
&\quad -\frac{r+1}{b^{r+1}-x^{r+1}}\left\{I^{k,r}_{x^+,k}(b^r f(b)) - rkI^{2k,r}_{x^+,k}\left(\frac{f(b)}{b}\right)\right\} \\
&\quad\quad \leq \frac{1}{4\sqrt{3}}\frac{1}{r+1}(x^{r+1}-a^{r+1})\left\{M_r(f'^2,a,x) - M_r^2(f',a,x)\right\}^{\frac{1}{2}} \\
&\quad\quad\quad +\frac{1}{4\sqrt{3}}\frac{1}{r+1}(b^{r+1}-x^{r+1})\left\{M_r(f'^2,x,b) - M_r^2(f',x,b)\right\}^{\frac{1}{2}} \\
&\quad\quad\quad\quad \leq \frac{1}{4\sqrt{3}}\frac{1}{r+1}(b^{r+1}-a^{r+1})(M-m). \qquad (2.22)
\end{aligned}$$

Remark. *Use $r = 0$ in (2.22), we get*

$$\begin{aligned}
&\left|2f(x) + \frac{f(b)+f(a)}{2} - \frac{1}{x-a}\int_a^x f(t)dt - \frac{1}{b-x}\int_x^b f(t)dt\right| \\
&\qquad\qquad \leq \frac{1}{4\sqrt{3}}(b-a)(M-m). \qquad (2.23)
\end{aligned}$$

Remark. *Put $x = \frac{a+b}{2}$ in (2.23), we have*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right) + \frac{f(b)+f(a)}{4} - \frac{1}{b-a}\int_a^b f(t)dt\right| \\
&\qquad\qquad \leq \frac{1}{8\sqrt{3}}(b-a)(M-m). \qquad (2.24)
\end{aligned}$$

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References

[1] A. Ostrowski, *Über die Absolutabweichung einer differentiierbaren Funktion von ihrem Integralmittelwert*, Comment. Math. Helv., **10(1)**, (1937), 226227.

[2] S. S. Dramomir, *On the Ostrowski's integral inequality for mappings with bounded variation and applications*, RGMIA Research report collection, **2(1)**, (1999).

[3] S. S. Dramomir, *The Ostrowski's Integral Inequality for Lipschitzian Mappings and Applications*, Computers and Mathematics with Applications, **38**, (1999), 33-37.

[4] P. Kumar, *The Ostrowski Type Moment Integral Inequalities and Moment-Bounds for Continuous Random Variables*, Computers and Mathematics with Applications, **0**, (2005), 1-0.

[5] M. Z. Sarikaya, *On the Ostrowski type integral inequality*, Acta Math. Univ. Comenianae, **LXXIX(1)**, (2010), 129-134.

[6] M. M. Jamei and S. S. Dragomir, *An analogue of the Ostrowski inequality and applications*, Filomat, **28(2)**, (2014), 373381.

[7] M. Z. Sarikaya, H.Yaldiz and S. Erden, *On the weighted Ostrowski type inequalities for double integrals*, Kragujevac Journal of Mathematics, **38(2)**, (2014), 303-314.

[8] Grüss, G: Über das maximum des absoluten Betrages $\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx \frac{1}{b-a}\int_a^b g(x)dx$. Math. Z. **39**, 215-226(1935).

[9] S. S. Dragomir, *A generalization of Grüss inequality in inner product spaces and applications*, J. Math. Anal. Appl. **237**, (1999), 74-82.

[10] N. Elezovic, L. J. Marangunic and J. Pecaric, *Some improvements of Grüss type inequality*, J. Math. Ineq. **1**, (2007), 425-436.

[11] X. Li, R. N. Mohapatra, and R. S. Rodriguez, *Grüss-type inequalities*, J. Math. Anal. Appl, **267**, (2002), 434-443.

[12] B. G. Pachpatte, *On multidimensional Grüss type inequalities*, J. Inequal. Pure Appl. Math. **3**, (2002), 1-15.

[13] G. Wang, P. Agarwal and M. Chand, *Certain Grüss type inequalities involving the generalized fractional integral operator*. Journal of Inequalities and Applications. **2014**, (2014), 147.

[14] A. N. Korkine, *Sur une theoreme de M. Tchebychef*, C.R. Acad. Sci. Paris., **96**, (1883), 216-27.

[15] U. N. Katugompola, *New approach generalized fractional integral*, Appl. Math. Comput. **218**, (2011), 860-865J.

[16] J. Tariboon, S.K. Ntouyas and W. Sudsutad, *Some New Riemann-Liouville Fractional Integral Inequalities*, International Journal of Mathematics and Mathematical Sciences, **2014**, (2014), 6 pages.

[17] M. Z. Sarikaya, H. Filiz and M. E. Kiris, *On Some Generalized Integral Inequalities for Riemann-Liouville Fractional Integrals*, Filomat **29(6)**, (2015), 1307-1314.

[18] Z. Dahmani, *About some integral inequalities using Riemann-Liouville integrals*. General Mathematics, **20(4)**, (2012), 63-69.

[19] Y. Liao, J. Deng and J. Wang, *Riemann-Liouville fractional Hermite-Hadamard inequalities. Part I: for once differentiable geometric-arithmetically s-convex functions*, Journal of Inequalities and Applications, **2013**, (2013), 443.

[20] F. Chen, *A note on Hermite-Hadamard inequalities for products of convex functions via Riemann-Liouville fractional inegrals*, Italian journal of pure and applied mathematics, **2014(33)**, (2014), 299-306.

[21] J. Park, *Some Integral Inequalities for Convex Functions via Riemann-Liouville Integrals*, Applied Mathematical Sciences, **9(27)**, (2015), 1341-1353.

[22] M. Tunc, *On new inequalities for h-convex functions via Riemann-Liouville fractional integration*, Filomat, **27(4)**, (2013) 559-565.

[23] E. Kacar, H. Yildirim, *Grüss type integral inequalities for generalized Riemann-Liouville fractional integrals*, International Journal of Pure and Applied Mathematics, **101(1)**, (2015), 55-70.

[24] R. Diaz and E. Pariguan, *On hypergeometric functions and Pochhammer k-symbol*, Divulgaciones Matemticas, **15**, (2007), 179-192.

[25] V. Krasniqi, *Inequalities and monotonicity for the ration of k-gamma functions*, Scientia Magna, **6(1)**, (2010), 40-45.

[26] S. Mubeen and G. M. Habibullah, *k-fractional integrals and application*, Int. J. Contemp. Math. Sci. **7**, (2012), 89-94.

[27] C. G. Kokologiannaki, and V. Krasniqi, *Some properties of the k-gamma function*, Le Matematiche, **68(1)**, (2013), 13-22.

[28] L. G. Romero, L. L. Luque, G. A. Dorrego and R. A. Cerutti, *On the k-Riemann-Liouville Fractional Derivative*, Int. J. Contemp. Math. Sciences, **8(1)**, (2013), 41-51.

[29] M. Z. Sarikaya and A. Karaca, *On the k-Riemann-Liouville fractional integral and applications*, International Journal of Statistics and Mathematics, **1(3)**, (2014), 033-043.

[30] M. Z. Sarikaya, Z. Dahmani, M. E. Kiris and F. Ahmad, *(k,s)-Riemann-Liouville fractional integral and applications*, Hacettepe Journal of Mathematics and Statistics, (2015). In press.

[31] H. Yilidrim and Z. Kirtay, *Owstrowski inequality for generalized fractional integral and related inequalities*, Malaya journal of matematik, **2(3)**(2014), 322-329.

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