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ON PARAMETRIZATION OF THE q-BERNSTEIN BASIS FUNCTIONS AND THEIR APPLICATIONS

YILMAZ SIMSEK

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Abstract. In order to investigate the fundamental properties of q-Bernstein basis functions, we give generating functions for these basis functions and their functional and differential equations. In [16], [15] and [17], we construct a novel collection of generating functions to derive many known and some new identities for the classical Bernstein basis functions. The main purpose of this paper is to construct analogous generating functions for the q-Bernstein basis functions. By using an approach similar to that of our methods in [16] as well as some properties of interpolation functions, we can derive some known and some new identities, relations and formulas for the q-Bernstein basis functions, including the partition of unity property, formulas for representing the monomials, recurrence relations, formulas for derivatives, subdivision identities and integral representations. Furthermore, we give plots of not only our new basis functions, but also their generating functions. Also, we simulate q-Bezier type curves for some selected q values and control points.

1. Introduction

The Bernstein polynomials are used in many branches of mathematics and other sciences, for instance, in approximation theory, probability theory, statistic theory, number theory, the solution of the differential equations, numerical analysis, constructing Bezier curves, q-calculus, operator theory and applications in computer graphics. The Bernstein polynomials are used to construct Bezier curves.

In this paper, we study on generating functions and functional equations for the Bernstein basis functions. In order to present our results, we need to introduce some notations and definitions:

$$[x:q]=\frac{1-q^x}{1-q}. \tag{1.1}$$

Observe that

$$\lim_{q\to 1}[x:q]=x.$$

For $q\in\mathbb{C}$, we assume that $\mid q \mid<1$. For $q\in\mathbb{R}$, we assume that $0<q<1$.

We summarize our paper's sections as follows:

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In Section 2, we construct generating function of the q-Bernstein basis functions. Using these generating, some identities and properties of the q-Bernstein basis functions can be driven. We give some properties of these generating functions and the basis functions including partition of unity, alternating sum, subdivision property and recurrence retaliations and derivative formulas.

In Section 3, by applying the Laplace transform to the modified generating function for the q-Bernstein basis functions, we derive some new identity and infinite series representation of the q-Bernstein basis functions.

In Section 4, by applying the Cauchy residue theorem to our generating functions, we construct interpolation function of the q-Bernstein polynomials.

In Section 5, we study q-Bezier type curves and their construction. We give further remarks and observations on these curves.

In Section 6, we give integral representation for the q-Bernstein basis functions with combinatorial sums.

In Section 7, simulations of the q-Bernstein basis functions and their generating functions are given.

2. Modified generating function for the q-Bernstein basis type functions

In this paper, we modify the generating functions for the q-Bernstein basis functions. We investigate and study many properties of these functions. By using these functions and their functional equations, we derive fundamental properties and some identities of the q-Bernstein basis functions, which are defined as follows.

Definition 2.1. *Let $x \in [0,1]$. Let k and n be nonnegative integers with $n \geq k$. Then we define*

$$\mathfrak{b}_k^n(x;q) = \binom{n}{k} [x:q]^k q^{(n-k)x} \left[(1-x):q\right]^{n-k}, \tag{2.1}$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

and $k = 0,1,2,...,n.$

The generating functions for the q-Bernstein basis functions $\mathfrak{b}_k^n(x;q)$ can be defined as follows.

Definition 2.2. *Let $x \in [0,1]$ and $t \in \mathbb{C}$. Let k be nonnegative integer. Then we define*

$$\mathcal{F}_{k,q}(t,x) = \sum_{n=0}^{\infty} \mathfrak{b}_k^n(x;q) \frac{t^n}{n!}. \tag{2.2}$$

It is clear to say that there is one generating function for each value of k.

We modify the generating functions for the q-Bernstein type basis functions as follows:

Theorem 2.1. *Let $x \in [0,1]$ and $t \in \mathbb{C}$. Then we have*

$$\mathcal{F}_{k,q}(t,x) = \frac{1}{k!} t^k [x:q]^k \exp\left(q^x \left[(1-x):q\right] t\right). \tag{2.3}$$

Proof. By substituting (2.1) into the right hand side of (2.2), we obtain

$$\begin{aligned}\mathcal{F}_{k,q}(t,x) &= \sum_{n=0}^{\infty}\left(\left(\begin{array}{c} n \\ k \end{array}\right)[x:q]^k q^{(n-k)x}\left[(1-x):q\right]^{n-k}\right)\frac{t^n}{n!} \\ &= \frac{t^k[x:q]^k}{k!}\sum_{n=k}^{\infty}\frac{\left(q^x\left[(1-x):q\right]t\right)^{n-k}}{(n-k)!}.\end{aligned}$$

The right hand side of the above equation is a Taylor series for

$$\exp\left(q^x\left[(1-x):q\right]t\right),$$

thus we arrive at the desired result. □

2.1. **Some properties for the q-Bernstein basis functions.** In [15] and [16], Simsek presented a lot of background material on computations functional equation of the generating function for the Bernstein basis functions. We give some functional equations which are used to find some new identities related to the q-Bernstein basis functions. Our method is similar to that of Simsek's [15].

2.2. **Partition of unity.** The polynomials $\mathfrak{b}_k^n(x;q)$ have *partition of unity*, which is given by the following theorem.

Theorem 2.2. (*Sum of the polynomials* $\mathfrak{b}_k^n(x;q)$)

$$\sum_{k=0}^{n}\mathfrak{b}_k^n(x;q)=1.$$

Proof. By using (2.3), we give the following functional equation:

$$\sum_{k=0}^{\infty}\mathcal{F}_{k,q}(t,x)=\exp\left(q^x\left[1-x:q\right]t\right)\sum_{k=0}^{\infty}\frac{1}{k!}t^k[x:q]^k.$$

The right hand side of the above equation is a Taylor series for

$$\exp\left(\left[x:q\right]t\right),$$

thus we obtain

$$\sum_{k=0}^{\infty}\mathcal{F}_{k,q}(t,x)=\exp\left(\left(q^x\left[1-x:q\right]+\left[x:q\right]\right)t\right). \tag{2.4}$$

If we substitute the following identity

$$\left[a+b:q\right]=\left[a:q\right]+q^a\left[b:q\right],$$

into the right-hand side of (2.4), we obtain

$$\sum_{k=0}^{\infty}\mathcal{F}_{k,q}(t,x)=\exp(t).$$

By using (2.2) and Taylor expansion of $\exp(t)$ in the above equation, we get

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\mathfrak{b}_k^n(x;q)\right)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of the above equation, we arrive at the desired result. □

Remark. Simsek and Acikgoz [17] defined the q-Bernstein type basis functions as follows:

$$Y_n(k,x;q) = \begin{pmatrix} n \\ k \end{pmatrix} [x:q]^k \left[1-x:q\right]^{n-k}. \tag{2.5}$$

The polynomials $Y_n(k,x;q)$ have not *partition of unity.* That is,

$$\sum_{k=0}^{n} Y_n(k,x;q) = \left([x:q] + [1-x:q]\right)^n \neq 1. \tag{2.6}$$

By using (2.1) and (2.5), one can easily see that

$$\mathfrak{b}_k^n(x;q) = q^{x(n-k)} Y_n(k,x;q).$$

Thus generating functions of the polynomials $\mathfrak{b}_k^n(x;q)$ give us modification that of the polynomials $Y_n(k,x;q)$.

Remark. In the special case when $q \to 1$, Definition 2.1 immediately yields the corresponding well known results concerning the classical Bernstein basis functions $B_k^n(x)$:

$$B_k^n(x) = \begin{pmatrix} n \\ k \end{pmatrix} x^k (1-x)^{n-k}, \tag{2.7}$$

where $k = 0, 1, \cdots, n$ and $x \in [0,1]$ cf. ([1]-[17]).

Since

$$q^x \left[(1-x):q\right] = 1 - [x:q],$$

we modify Definition 2.1 as follows:

$$\mathfrak{b}_k^n(x;q) = \begin{pmatrix} n \\ k \end{pmatrix} [x:q]^k \left(1 - [x:q]\right)^{n-k}. \tag{2.8}$$

Observe that substituting $x = [x:q]$ into (2.7), we also get (2.8). That is, we get the following corollary.

Corollary 2.3.

$$\mathfrak{b}_k^n(x;q) = B_k^n\left([x:q]\right).$$

2.3. **Alternating sum.** By using (2.3), we obtain the following functional equation:

$$\sum_{k=0}^{\infty} (-1)^k \mathcal{F}_{k,q}(t,x) = \exp\left(\left(q^x\left[1-x:q\right] - [x:q]\right)t\right). \tag{2.9}$$

By using the same method in our paper [16] and (2.9), we derive a formula for the alternating sum, which is given the following theorem.

Theorem 2.4. (*Alternating sum*)

$$\sum_{k=0}^{n} (-1)^k \mathfrak{b}_k^n(x;q) = \left(1 - 2\left[x:q\right]\right)^n. \tag{2.10}$$

Remark. If we let $q \to 1$ in (2.10), then we arrive at the well-known Goldman's results [4]-[3, Chapter 5, pages 299-306] and see also [16]:

$$\sum_{k=0}^{n} (-1)^k B_k^n(x) = (1-2x)^n.$$

2.4. **Subdivision property.** By using similar method in [16] and [15], we derive the following functional equation:

$$\mathcal{F}_{k,q}(t,xy)=\mathcal{F}_{k,q}(t\left[y:q^{x}\right],x)\exp\left(q^{xy}\left[1-y:q^{x}\right]t\right). \tag{2.11}$$

By using the above functional equation, we derive subdivision property for the q-Bernstein basis functions by the following theorem.

Theorem 2.5. *The following identity holds:*

$$\mathfrak{b}_{j}^{n}(xy;q)=\sum_{k=j}^{n}\mathfrak{b}_{j}^{k}(x;q)\mathfrak{b}_{k}^{n}(y;q^{x}).$$

Remark. If we let $q\to 1$ in Theorem 2.5, we have

$$B_{j}^{n}(xy)=\sum_{k=j}^{n}B_{j}^{k}(x)B_{k}^{n}(y). \tag{2.12}$$

The above identity is fundamental in the subdivision property for the Bernstein basis functions (*cf.* [4]-[3, Chapter 5, pages 299-306], [16], [15]).

2.5. **Recurrence relations and derivative of the q-Bernstein basis functions.** In this section, we give higher order derivatives of the Bernstein basis functions. We define

$$\mathcal{F}_{k,q}(t,x)=g_{k,q}(t,x)h_{q}(t,x), \tag{2.13}$$

where

$$g_{k,q}(t,x)=\frac{t^{k}\left[x:q\right]^{k}}{k!}$$

and

$$h_{q}(t,x)=\exp\left(q^{x}\left[1-x\right]t\right).$$

In this section we shall differentiate (2.13) with respect to t to derive a recurrence relation for the Bernstein basis functions. Using Leibnitz's formula for the vth derivative, with respect to t, we obtain the following higher order partial differential equation:

$$\frac{\partial^{v}\mathcal{F}_{k,q}(t,x)}{\partial t^{v}}=\sum_{j=0}^{v}\left(\begin{array}{c}v\\ j\end{array}\right)\left(\frac{\partial^{j}g_{k,q}(t,x)}{\partial t^{j}}\right)\left(\frac{\partial^{v-j}h_{q}(t,x)}{\partial t^{v-j}}\right). \tag{2.14}$$

From the above equation, we get the following partial derivative for the generating functions as follows.

Theorem 2.6.

$$\frac{\partial^{v}\mathcal{F}_{k,q}(t,x)}{\partial t^{v}}=\sum_{j=0}^{v}\mathfrak{b}_{j}^{v}(x;q)\mathcal{F}_{k-j,q}(t,x). \tag{2.15}$$

By the same method in [16] and [15], Theorem 2.6 is proved by induction on v using (2.14).

Using (2.2) and (2.7) in Theorem 2.6, we obtain a recurrence relation for the Bernstein basis functions by the following theorem.

Theorem 2.7.

$$\mathfrak{b}_{k}^{n}(x;q)=\sum_{j=0}^{v}\mathfrak{b}_{j}^{v}(x;q)\mathfrak{b}_{k-j}^{n-v}(x;q). \tag{2.16}$$

Proof. By substituting right hand side of (2.2) into (2.15), we get

$$\frac{\partial^v}{\partial t^v}\left(\sum_{n=0}^{\infty}\mathfrak{b}_k^n(x;q)\frac{t^n}{n!}\right)=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{v}\mathfrak{b}_j^v(x;q)\mathfrak{b}_{k-j}^{n-v}(x;q)\right)\frac{t^n}{n!}.$$

Therefore,

$$\sum_{n=v}^{\infty}\mathfrak{b}_k^n(x;q)\frac{t^{n-v}}{(n-v)!}=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{v}\mathfrak{b}_j^v(x;q)\mathfrak{b}_{k-j}^{n-v}(x;q)\right)\frac{t^n}{n!}.$$

From the above equation, we get

$$\sum_{n=v}^{\infty}\mathfrak{b}_k^n(x;q)\frac{t^{n-v}}{(n-v)!}=\sum_{n=v}^{\infty}\left(\sum_{j=0}^{v}\mathfrak{b}_j^v(x;q)\mathfrak{b}_{k-j}^{n-v}(x;q)\right)\frac{t^{n-v}}{(n-v)!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of the above equation, we arrive at the desired result. □

Remark. If we let $q\to 1$ in (2.17), then we arrive at Theorem 9 in [16].

By using (2.3), we obtain derivative of the q-Bernstein basis functions for in the next theorem.

Theorem 2.8. *Let $x\in[0,1]$. Let k and n be nonnegative integers with $n\geq k$. Then we have*

$$\frac{d}{dx}\mathfrak{b}_k^n(x;q)=\frac{q^x\log(q^n)}{q-1}\left(\mathfrak{b}_{k-1}^{n-1}(x;q)-\mathfrak{b}_k^{n-1}(x;q)\right). \tag{2.17}$$

Remark. If we let $q\to 1$ in (2.17), then we arrive at Corollary 1 in [16].

3. Unified generating functions with their applications

In this section we apply the Laplace transform to the generating function for the q-Bernstein basis functions. We derive a new identity.

From (2.3), we get the following generating functions:

$$e^{[x]t}\sum_{n=0}^{\infty}\mathfrak{b}_k^n(x;q)\frac{t^n}{n!}=\frac{[x:q]^k}{k!}t^ke^t, \tag{3.1}$$

$$e^{-t}\sum_{n=0}^{\infty}\mathfrak{b}_k^n(x;q)\frac{t^n}{n!}=\frac{[x:q]^k}{k!}t^ke^{-[x]t}, \tag{3.2}$$

$$e^{-q^x[1-x]t}\sum_{n=0}^{\infty}\mathfrak{b}_k^n(x;q)\frac{t^n}{n!}=\frac{[x:q]^k}{k!}t^k. \tag{3.3}$$

Theorem 3.1.

$$\sum_{n=0}^{\infty}[x]\,\mathfrak{b}_k^n(x;q)=1. \tag{3.4}$$

Proof. Integrating equation (3.2) (by parts) with respect to t from zero to infinity, we have

$$\sum_{n=0}^{\infty}\frac{\mathfrak{b}_k^n(x;q)}{n!}\int_0^{\infty}e^{-t}t^ndt=\frac{[x:q]^k}{k!}\int_0^{\infty}t^ke^{-[x:q]t}dt. \tag{3.5}$$

We here assume that

$$x>0$$

and for the Laplace transform of the function $f(t)=t^k$, we have

$$\mathcal{L}(t^k)=\frac{k!}{[x:q]^{k+1}}.$$

Therefore the both sides of equation (3.5) reduce to the following formula:

$$\sum_{n=0}^{\infty}\mathfrak{b}_k^n(x;q)=\frac{1}{[x:q]}.$$

Thus, the proof of the theorem is completed. □

Remark. If we let $q\to 1$ in (3.4), then we arrive at Theorem 15 in [16].

4. Interpolation Function

In this section, we construct interpolation function for the q-Bernstein basis functions. This function interpolates the q-Bernstein basis functions at negative integers.

Let $s\in\mathbb{C}$, and $x\in R$ with $x\neq 1$. By applying the Mellin transformation to (2.3), we give integral representation of the interpolation function $I_q(s,k;x)$ as follows:

$$I_q(s,k;x)=\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}\mathcal{F}_{k,q}(-t,x)dt,$$

where $\Gamma(s)$ denotes the Euler gamma function. By using the above integral representation, we are now ready to define interpolation function of the q-Bernstein polynomials.

Definition 4.1. *Let k be a nonnegative integer. Let $s\in\mathbb{C}$, and $x\in R$ with $x\neq 1$. The interpolation function $I_q(s,k;x)$ is defined by*

$$I_q(s,k;x)=(-1)^k\frac{\Gamma(s+k)}{\Gamma(s)\Gamma(k+1)}\frac{[x:q]^k}{q^{x(k+s)}[1-x:q]^{k+s}}.$$

Theorem 4.1. *Let n be a positive integer. Then we have*

$$I_q(-n,k;x)=\mathfrak{b}_k^n(x;q).$$

The proof of this theorem is same as that of Theorem 12 in [14]. So we omit it.

5. Bezier type curve

The Bezier curves are constructed by the Bernstein polynomials and control points. The Bezier curves are widely used in computer graphics to model smooth curves. The history of the Bezier curves can be traced back to Pierre Bezier, who was an engineer with the Renault car company and set out in the early 1960's to develop a curve formulation which would lend itself to shape design. Engineers may find it most understandable to think of the Bezier curves in terms of the center of mass of a set of point masses.

The q-Bezier type curves $B_n(x:q)$ with control points $P_0,\cdots,P_n$ are defined by

$$B_n(x:q)=\sum_{k=0}^{n}P_k\mathfrak{b}_k^n(x;q).$$

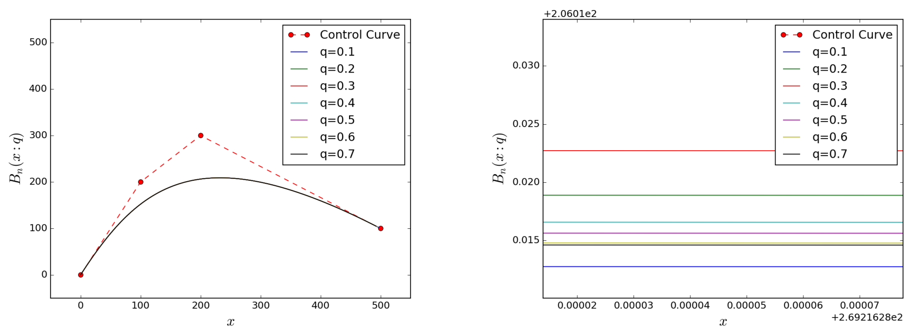
Observe that if $q\to 1$, we have the standard Bezier curves as follows:

$$B_n(x:1)=B_n(x)=\sum_{k=0}^{n}P_kB_k^n(x)$$

(cf. [2]).

If we substitute $\mathfrak{b}_k^n(x;q) = B_k^n([x : q])$ in the above equation, then the q-Bezier type curves have same properties as standard Bezier curves. Because the q-Bernstein basis functions are parametrization of the standard Bernstein basis functions. The q-Bernstein basis functions might be the affected by selecting different q values on the shape of the curves.

5.1. **Graphs of the q-Bezier type curves.** Here, we plot q-Bezier type curves by selecting q from 0.1 to 0.7 with control points as $(0,0)$, $(100,200)$, $(200,300)$ and $(500,100)$. Also, we give some observations and comments about these graphs.



(a) q-Bezier type curves obtained varying q from 0.1 to 0.7 and control points $(0,0)$, $(100,200)$, $(200,300)$ and $(500,100)$.

(b) Looking closer to Figure 1a.

Figure 1 An example for the q-Bezier type curves

In Figure 1a, there are 7 different q-Bezier type cubic curves having same control points. Although all curves are plotted in Figure 1a, only a single q-Bezier type curve seems due to overlapping of different colors. Because of the drawing sensitivity, we give Figure 1b for observing the shape of the graphs for various q values.

6. Integral representation for the q-Bernstein basis functions

In this section, we give not only the Cuachy integral representation, but also the Riemann integral representation for the q-Bernstein basis functions.

6.1. **Cauchy integral representation for the q-Bernstein basis functions.** Here, by applying the Cauchy Residue Theorem to the generating function for the q-Bernstein basis functions, we derive a Cauchy integral representation for these basis functions by the following theorem.

Theorem 6.1.

$$\mathfrak{b}_k^n(x;q) = \frac{n!}{2\pi i}\int_{\mathcal{C}} \mathcal{F}_{k,q}(z,x)\frac{dz}{z^{n+1}}, \tag{6.1}$$

where $\mathcal{C}$ is a circle around the origin and the integration is in positive direction, $z \in \mathbb{C}$, $n \in \mathbb{Z}^+ = \{1,2,3,\cdots\}$ and $x \in [0,1]$.

Proof. In [14], we give integral representation for the q-Bernstein basis functions. By combining (2.2) and (2.3) with the Cauchy Residue Theorem, we obtain

$$\frac{n!}{2\pi i}\int_{\mathcal{C}}\mathcal{F}_{k,q}(z,x)\frac{dz}{z^{n+1}}=\frac{n!}{2\pi i}\left(2\pi i Res\left(\frac{\mathcal{F}_{k,q}(t,x)}{z^{n+1}},0\right)\right),$$

where $Res(f(z),a)$ denotes the residue of $f(z)$ function at $z=a$. We compute residue of the function $\frac{1}{z^{n+1}}\mathcal{F}_{k,q}(t,x)$ at $z=0$ by the Laurent series expansion as follows:

$$\mathfrak{b}_0^n(x;q)\frac{1}{z^{n+1}}+\mathfrak{b}_k^n(x;q)\frac{1}{z^n}+\cdots+\frac{\mathfrak{b}_k^n(x;q)}{n!}\frac{1}{z}+\mathfrak{b}_k^{n+1}(x;q)+\cdots.$$

By using the above equation, we obtain

$$Res\left(\frac{\mathcal{F}_{k,q}(t,x)}{z^{n+1}},0\right)=\frac{\mathfrak{b}_k^n(x;q)}{n!}.$$

Consequently, the proof of the theorem is completed. □

We note that our proof method is similar to that of Lopez et al. [11] and Kim et al. [7].

6.2. **Riemann integral representation for the q-Bernstein basis functions.** Here, by applying the Riemann integral to the function $[x:q]$ and $\mathfrak{b}_k^n(x;q)$, we derive two integral formulas.

The Riemann integral representation for the function $[x:q]$ is given by the following lemma.

Lemma 6.2.

$$\int_0^1 [x:q]dx=\frac{\ln q+1-q}{(1-q)\ln q}. \tag{6.2}$$

Proof. By integrating both side of equation (1.1) from 0 to 1, we get desired result. □

Lemma 6.3.

$$\int_0^1 [x:q]^m dx=\frac{1}{(1-q)^m\ln q}\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\frac{q^{m-j}-1}{m-j}. \tag{6.3}$$

The proof of the above lemma is same as that of (6.2).

The Riemann integral representation for the q-Bernstein basis functions is given by the following theorem.

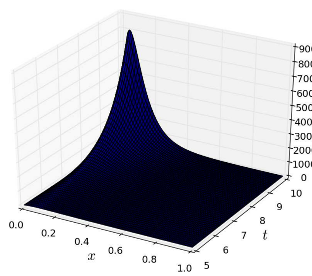
Theorem 6.4.

$$\int_0^1 \mathfrak{b}_k^n(x;q)dx = \binom{n}{k}\sum_{j=0}^{n-k}\sum_{l=0}^{n-j}\binom{n-k}{j}\binom{n-j}{l}\frac{(-1)^{k+l}}{n-j-l}\times\frac{q^{n-j-l}-1}{(1-q)^{n-j}\ln q}.$$

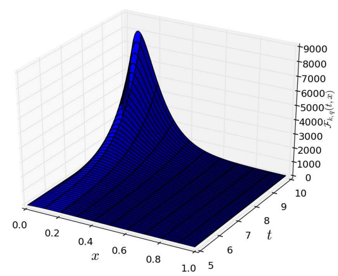
Proof. Combining (6.2) with (6.3), we get the desired result. □

7. Simulations of the q-Bernstein basis functions and their generating functions

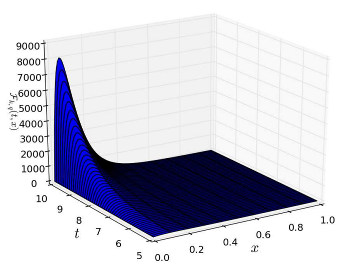
In this section, we first give graphs of the generating functions of the q-Bernstein basis functions for some special q and k values. Secondly, we draw the q-Bernstein basis functions for some selected numerical values of q.



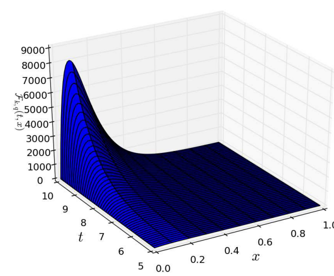
(a) Generating function of the q-Bernstein basis functions for $q = 0.2$ and $k = 1$.



(b) Generating function of the q-Bernstein basis functions for $q = 0.6$ and $k = 1$



(c) Different perspective of Figure 2a



(d) Different perspective of Figure 2b

Figure 2 Generating functions for $x \in [0, 1]$ and $t \in [5, 10]$.

From Figure 2 and Figure 3, we observe that the graphs of the q-Bernstein basis functions and their generating functions are affected by q values.

We also note that when q goes to 1, the maximum point of the curves of the q-Bernstein basis functions are shifted to the right in Figure 3.

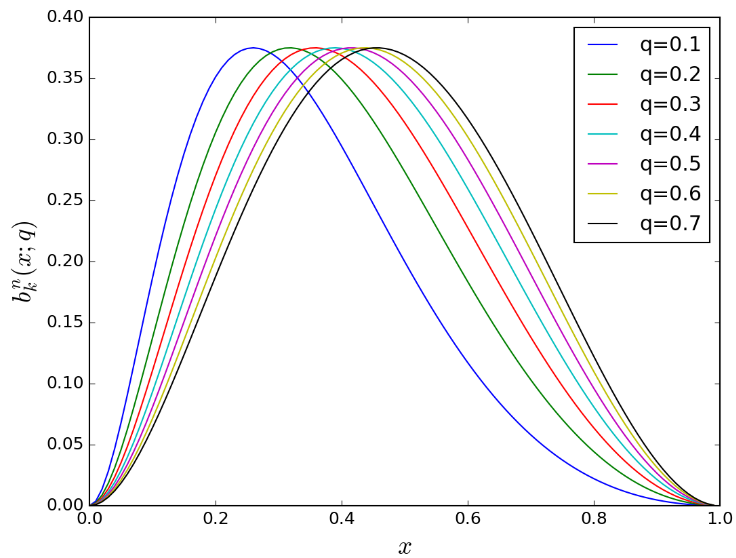


Figure 3 The q-Bernstein basis functions obtained by varying q values.

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Yilmaz Simsek, Department of Mathematics, Faculty of Science University of Akdeniz, TR–07058 Antalya, Turkey
E-mail address: ysimsek@akdeniz.edu.tr