Journal of Inequalities and Special Functions
ISSN: 2217-4303, URL: http://ilirias.com/jiasf
Volume 7 Issue 4(2016), Pages 137- 142.

A TAUBERIAN THEOREM FOR THE GENERALIZED NÖRLUND-EULER SUMMABILITY METHOD

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Abstract. Let (p_n) and (q_n) be any two non-negative real sequences with

$$R_n := \sum_{k=0}^{n} p_k q_{n-k} \neq 0 \ \ (n \in \mathbb{N}).$$

And E_n^1- Euler summability method. Let (x_n) be a sequence of real or complex numbers and set

$$N_{p,q}^n E_n^1 := \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} x_v$$

for $n \in \mathbb{N}$. In this paper, we present necessary and sufficient conditions under which the existence of the limit $\lim_{n\to\infty} x_n = L$ follows from that of $\lim_{n\to\infty} N_{p,q}^n E_n^1 = L$. These conditions are one-sided or two-sided if (x_n) is a sequence of real or complex numbers, respectively.

1. Introduction

In what follows we give the concept of the summability method known as the generalized Nörlund summability method (N, p, q) (see [1]). Given two non-negative sequences (p_n) and (q_n), the convolution $(p \star q)$ is defined by

$$R_n := (p \star q)_n = \sum_{k=0}^{n} p_k q_{n-k} = \sum_{k=0}^{n} p_{n-k} q_k.$$

With E_n^1- we will denote the Euler summability method. Let (x_n) be a sequence. When $(p \star q)_n \neq 0$ for all $n \in \mathbb{N}$, the generalized Nörlund-Euler transform of the sequence (x_n) is the sequence $N_{p,q}^n E_n^1$ obtained by putting

$$N_{p,q}^n E_n^1 = \frac{1}{(p \star q)_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} x_v. \tag{1.1}$$

We say that the sequence (x_n) is generalized Nörlund-Euler summable to L determined by the sequences (p_n) and (q_n) or briefly summable $N_{p,q}^n E_n^1$ to L if

$$\lim_{n\to\infty} N_{p,q}^n E_n^1 = L. \tag{1.2}$$

2010 *Mathematics Subject Classification.* 40G15, 41A36.
Key words and phrases. generalized Nörlund-Euler summability; one-sided and two-sided Tauberian conditions.

Submitted July 15, 2016. Published September 12, 2016.

Suppose throughout the paper we assume that the sequence $q = (q_n)$ satisfies the following conditions:

$$q_{\lambda_n - k} \leq 2q_{n-k}, k = 0, 1, 2, 3, \cdots, n; \lambda > 1, \tag{1.3}$$

$$q_{n-k} \leq 2q_{\lambda_n - k}, k = 0, 1, 2, 3, \cdots, \lambda_n; 0 < \lambda < 1, \tag{1.4}$$

where $\lambda_n = [\lambda \cdot n]$.

If

$$\lim_{n\to\infty} x_n = L \tag{1.5}$$

implies (1.2), then the method $N^n_{p,q}E^1_n$ is called to be regular.

Notice that (1.2) may imply (1.5) under a certain condition, which is called a Tauberian condition. Any theorem which states that convergence of a sequence follows from its $N^n_{p,q}E^1_n$ summability and some Tauberian condition is said to be a Tauberian theorem for the $N^n_{p,q}E^1_n$ summability method. The inclusion and Tauberian type theorems are proved in the papers [4, 5, 2, 3], and some theorems of inclusion, Tauberian and convexity type for certain families of generalized Nörlund methods are obtained in [6].

In this paper, we present necessary and sufficient conditions under which the existence of the limit $\lim_{n\to\infty} x_n = L$ follows from that of $\lim_{n\to\infty} N^n_{p,q}E^1_n = L$. These conditions are one-sided or two-sided if (x_n) is a sequence of real or complex numbers, respectively.

2. MAIN RESULTS

In the following theorem we characterize the converse implication when the ordinary convergence follows from its $N^n_{p,q}E^1_n$ summability.

Theorem 1. *Let (p_n) and (q_n) be any two non-negative real sequences such that*

$$\liminf_{n\to\infty} \frac{R_{\lambda_n}}{R_n} > 1, \quad \textit{for every} \quad \lambda > 1, \tag{2.1}$$

where $\lambda_n := [\lambda n]$ denotes the integral part of λn for every $n \in \mathbb{N}$, and let (x_n) be a sequence of real numbers which is $N^n_{p,q}E^1_n$ summable to a finite number L. Then (x_n) is convergent to the same number L if and only if the following two conditions hold:

$$\lim_{\lambda\to 1^+} \liminf_{n\to\infty} \frac{1}{R_{\lambda_n} - R_n} \sum_{k=n+1}^{\lambda_n} p_k q_{\lambda_n - k} \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} (x_v - x_n) \geq 0, \tag{2.2}$$

and

$$\lim_{\lambda\to 1^-} \liminf_{n\to\infty} \frac{1}{R_n - R_{\lambda_n}} \sum_{k=\lambda_n+1}^{n} p_k q_{n-k} \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} (x_n - x_v) \geq 0. \tag{2.3}$$

In the next result we will consider the case where $x = (x_n)$ is a sequence of complex numbers.

Theorem 2. *Let condition (2.1) be satisfied and let (x_n) be a sequence of complex numbers which is $N^n_{p,q}E^1_n$ summable to a finite number L. Then (x_n) is convergent* *to the same number L if and only if one of the following two conditions holds:*

$$\lim_{\lambda\to 1^+}\limsup_{n\to\infty}\left|\frac{1}{R_{\lambda_n}-R_n}\sum_{k=n+1}^{\lambda_n}p_kq_{\lambda_n-k}\frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v}(x_v-x_n)\right|=0, \tag{2.4}$$

or

$$\lim_{\lambda\to 1^-}\limsup_{n\to\infty}\left|\frac{1}{R_n-R_{\lambda_n}}\sum_{k=\lambda_n+1}^{n}p_kq_{n-k}\frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v}(x_n-x_v)\right|=0. \tag{2.5}$$

3. AUXILIARY RESULTS

In what follows we list some auxiliary lemmas which are needful in the sequel.

Lemma 3. *The condition given by relation (2.1) is equivalent to the condition*

$$\liminf_{n\to\infty}\frac{R_n}{R_{\lambda_n}}>1,\quad 0<\lambda<1. \tag{3.1}$$

Proof. Suppose that relation (2.1) is valid, $0<\lambda<1$ and $m=\lambda_n=[\lambda n]$, $n\in\mathbb{N}$. Then it follows that

$$\frac{1}{\lambda}>1\Rightarrow\frac{m}{\lambda}=\frac{[\lambda n]}{t}\leq n.$$

From above relation and definition of the positive real sequences (p_n) and (q_n), we obtain:

$$\frac{R_n}{R_{\lambda_n}}\geq\frac{R_{[\frac{m}{\lambda}]}}{R_{\lambda_n}}\Rightarrow\liminf_{n\to\infty}\frac{R_n}{R_{\lambda_n}}\geq\liminf_{n\to\infty}\frac{R_{[\frac{m}{\lambda}]}}{R_{\lambda_n}}>1.$$

Conversely, suppose that relation (3.1) is valid. Let $\lambda>1$ be given number and let λ_1 be chosen such that $1<\lambda_1<\lambda$. Set $m=\lambda_n=[\lambda n]$. From $0<\frac{1}{\lambda}<\frac{1}{\lambda_1}<1$, it follows that:

$$n\leq\frac{\lambda n-1}{\lambda_1}<\frac{[\lambda n]}{\lambda_1}=\frac{m}{\lambda_1},$$

provided $\lambda_1\leq\lambda-\frac{1}{n}$, which is a case where if n is large enough. Under this conditions we have:

$$\frac{R_{\lambda_n}}{R_n}\geq\frac{R_{\lambda_n}}{R_{\left[\frac{m}{\lambda_1}\right]}}\Rightarrow\liminf_{n\to\infty}\frac{R_{\lambda_n}}{R_n}\geq\liminf_{n\to\infty}\frac{R_{\lambda_n}}{R_{\left[\frac{m}{\lambda_1}\right]}}>1.$$

□

Proposition 4. *Let us suppose that relation (2.1) is satisfied and let $x=(x_k)$ be a sequence of complex numbers which is generalized Nörlund-Cesáro summable to L. Then*

$$\lim_n\frac{1}{R_{\lambda_n}-R_n}\sum_{j=n+1}^{\lambda_n}p_jq_{\lambda_n-j}\frac{1}{2^j}\sum_{v=0}^{j}\binom{j}{v}x_v=L,\quad \textit{for}\quad \lambda>1 \tag{3.2}$$

and

$$\lim_n\frac{1}{R_n-R_{\lambda_n}}\sum_{j=\lambda_n+1}^{n}p_jq_{n-j}\frac{1}{2^j}\sum_{v=0}^{j}\binom{j}{v}x_v=L,\quad \textit{for}\quad 0<\lambda<1. \tag{3.3}$$

Proof. (I) Let us consider the case where $\lambda > 1$. Then we obtain

$$\frac{1}{R_{\lambda_n}-R_n}\sum_{k=n+1}^{\lambda_n} p_k q_{\lambda_n-k}\frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v}(x_v-L)=\frac{R_{\lambda_n}}{R_{\lambda_n}-R_n}\frac{1}{R_{\lambda_n}}\sum_{k=0}^{\lambda_n} p_k q_{\lambda_n-k}\frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v}(x_v-L)-$$

$$\frac{R_n}{R_{\lambda_n}-R_n}\frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k}\frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v}(x_v-L)=$$

$$\frac{R_{\lambda_n}}{R_{\lambda_n}-R_n}\frac{1}{R_{\lambda_n}}\sum_{k=0}^{\lambda_n} p_k q_{\lambda_n-k}\frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v}(x_v-L)-\frac{R_n}{R_{\lambda_n}-R_n}\frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k}\frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v}(x_v-L)-$$

$$\frac{R_n}{R_{\lambda_n}-R_n}\frac{1}{R_n}\sum_{k=0}^{n} p_k (q_{\lambda_n-k}-q_{n-k})\frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v}(x_v-L). \tag{3.4}$$

From relation (3.4), definition of the sequence (q_n), and relation

$$\limsup_n \frac{R_{\lambda_n}}{R_{\lambda_n}-R_n}<\infty,$$

we get relation (3.2).

(II) In this case we have that $0<\lambda<1$. Then

$$\frac{1}{R_n-R_{\lambda_n}}\sum_{k=\lambda_n+1}^{n} p_k q_{n-k}\frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v}x_v=\frac{R_n}{R_n-R_{\lambda_n}}\frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k}\frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v}x_v-$$

$$\frac{R_{\lambda_n}}{R_n-R_{\lambda_n}}\frac{1}{R_{\lambda_n}}\sum_{k=0}^{\lambda_n} p_k q_{n-k}\frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v}x_v=\frac{R_n}{R_n-R_{\lambda_n}}\frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k}\frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v}x_v-$$

$$\frac{R_{\lambda_n}}{R_n-R_{\lambda_n}}\frac{1}{R_{\lambda_n}}\sum_{k=0}^{\lambda_n} p_k q_{\lambda_n-k}\frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v}x_v-\frac{1}{R_n-R_{\lambda_n}}\sum_{k=0}^{\lambda_n} p_k (q_{n-k}-q_{\lambda_n-k})\frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v}x_v.$$

Now proof of the proposition is similar to the first part. □

4. PROOFS OF THE THEOREMS

Proof of Theorem 1. Necessity. Suppose that $\lim_{n\to\infty} x_n = L$, and (2.1) holds. Following Proposition 4, we have

$$\lim_{n\to\infty}\frac{1}{R_{\lambda_n}-R_n}\sum_{k=n+1}^{\lambda_n} p_k q_{\lambda_n-k}\frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v}(x_v-x_n)=\lim_{n\to\infty}\frac{1}{R_{\lambda_n}-R_n}\sum_{k=n+1}^{\lambda_n} p_k q_{\lambda_n-k}\frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v}x_v-x_n=0,$$

for every $\lambda > 1$. In case where $0<\lambda<1$, we find that

$$\lim_{n\to\infty}\frac{1}{R_n-R_{\lambda_n}}\sum_{k=\lambda_n+1}^{n} p_k q_{n-k}\frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v}(x_n-x_v)=x_n-\lim_{n\to\infty}\frac{1}{R_n-R_{\lambda_n}}\sum_{k=\lambda_n+1}^{n} p_k q_{n-k}\frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v}x_v=0.$$

Sufficiency. Assume that conditions (2.2) and (2.3) are satisfied. In what follows we will prove that $\lim_{n\to\infty} x_n = L$. Given any $\epsilon > 0$, by relation (2.2) we can choose $\lambda_1 > 0$ such that

$$\liminf_{n\to\infty}\frac{1}{R_{\lambda_{n_1}}-R_n}\sum_{k=n+1}^{\lambda_{n_1}} p_k q_{\lambda_{n_1}-k}\frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v}(x_v-x_n)\geq-\epsilon, \tag{4.1}$$

where $\lambda_{n_1} = [\lambda_1 n]$. By the assumed summability $N^n_{p,q}E^1_n$ of (x_n), Proposition 4 and relation (4.1), we have

$$\limsup_{n\to\infty} x_n \leq L + \epsilon, \tag{4.2}$$

for any $\lambda > 1$.

On the other hand, if $0 < \lambda < 1$, for every $\epsilon > 0$, we can choose $0 < \lambda_2 < 1$ such that

$$\liminf_{n\to\infty} \frac{1}{R_n - R_{\lambda_{n_2}}} \sum_{k=\lambda_{n_2}+1}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} (x_n - x_v) \geq -\epsilon,$$

where $\lambda_{n_2} = [\lambda_2 n]$. By the assumed summability $N^n_{p,q}C^1_n$ of (x_n), Proposition 4 and above relation, we have

$$\liminf_{n\to\infty} x_n \geq L - \epsilon, \tag{4.3}$$

for any $0 < \lambda < 1$.

Since $\epsilon > 0$ is arbitrary, combining relations (4.2) and(4.3) we obtain

$$\lim_{n\to\infty} x_n = L. \quad \square$$

Proof of Theorem 2.

Necessity. If both (1.2) and (1.5) hold, then Proposition 4 yields (2.4) for every $\lambda > 1$ and (2.5) for every $0 < \lambda < 1$.

Sufficiency. First we will suppose that (2.1), (1.2) and the condition (2.4) are satisfied. For any given $\epsilon > 0$, there exists some $\lambda_1 > 1$ such that

$$\limsup_{n\to\infty} \left| \frac{1}{R_{\lambda_{n_1}} - R_n} \sum_{k=n+1}^{\lambda_{n_1}} p_k q_{\lambda_{n_1}-k} \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} (x_v - x_n) \right| \leq \epsilon,$$

where $\lambda_{n_1} = [\lambda_1 n]$. Taking into account the fact that (x_n) is $N^n_{p,q}E^1_n$ summable to L and Proposition 4, we get the following estimation

$$\limsup_{n\to\infty} |L - x_n| \leq \lim_{n\to\infty} \sup \left| L - \frac{1}{R_{\lambda_{n_1}} - R_n} \sum_{k=n+1}^{\lambda_{n_1}} p_k q_{\lambda_{n_1}-k} \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} x_v \right| +$$

$$\limsup_{n\to\infty} \left| \frac{1}{R_{\lambda_{n_1}} - R_n} \sum_{k=n+1}^{\lambda_{n_1}} p_k q_{\lambda_{n_1}-k} \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} (x_v - x_n) \right| \leq \epsilon.$$

In this case we will suppose that (2.1), (1.2) and the condition (2.5) are satisfied. For any given $\epsilon > 0$, there exists some $0 < \lambda_2 < 1$ such that

$$\limsup_{n\to\infty} \left| \frac{1}{R_n - R_{\lambda_{n_2}}} \sum_{k=\lambda_{n_2}+1}^{n} p_k q_{n-k} \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} (x_n - x_v) \right| \leq \epsilon,$$

where $\lambda_{n_2} = [\lambda_2 n]$. Taking into account the fact that (x_n) is $N^n_{p,q}E^1_n$ summable to L and from Proposition 4, we get the following estimation

$$\limsup_{n\to\infty} |L - x_n| \leq \limsup_{n} \left| L - \frac{1}{R_n - R_{\lambda_{n_2}}} \sum_{k=\lambda_{n_2}+1}^{n} p_k q_{n-k} \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} x_v \right| +$$

$$\limsup_{n\to\infty}\left|\frac{1}{R_n-R_{\lambda_{n_2}}}\sum_{k=\lambda_{n_2}+1}^{n}p_kq_{n-k}\frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v}(x_n-x_v)\right|\leq\epsilon.$$

Since $\epsilon>0$ is arbitrary, in either case we get $\lim_{n\to\infty}x_n=L$. □

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