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ON SOME STATISTICAL AND PROBABILISTIC INEQUALITIES

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Abstract. In this paper, we show how to obtain some important integral inequalities from statistical and probabilistic point of view. We also explore the properties of the covariance leading to new classes of inequalities including the Ostrowski and Ostrowski-Grüss inequalities.

1. Introduction

Let $L^1[a,b]$ denote the space of real and measurable functions with the norm

$$\|f\|_1 = \int_a^b |f(t)|\,dt < \infty,$$

and $L^\infty[a,b]$ the space of bounded functions with

$$\|f\|_\infty = \|f\| = \sup_{a\le t\le b} |f(t)| < \infty.$$

Suppose that U is a random variable with uniform distribution $w_U(x) = 1/(b-a)$ on $[a,b]$. So $EU = (b+a)/2$, $\sigma^2(U) = (b-a)^2/12$ and

$$Ef(U) = \frac{1}{b-a}\int_a^b f(t)dt.$$

It is clear that if X has a non-uniform density function, say $w_X(x)$, then

$$Ef(X) = \int_a^b f(t)w_X(t)dt.$$

However, we can mainly focus on the uniform random variable U due to the identity

$$Ef(X) = Ef(W^{-1}(U)),$$

where $W_X(x) = P(X \le x)$, $W^{-1}(x)$ is its inverse and $U \sim U(0,1)$.

Our main aim in this paper is to find appropriate estimates for the difference between $Ef(X)$ and $f(EX)$. To reach this goal, in the next section, we recall Jensen's inequality and some of its basic extensions. To estimate $Ef(X) - f(EX)$, there exist two approaches in the literature. In the first one, authors obtaine results by the Taylor expansion of f, which is explained in Secion 3. In the second approach, bounds for $Ef(X) - f(EX)$ are obtained by the covariance of random variables

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and its general properties, which is usually neglected in the literature. In section 4, we first use properties of the covariance and correlation of random variabels to find back some older inequalities and then obtain some new improvements. Throughout this paper, we use U to denote a random variable with uniform distribution and use X to denote an arbitrary random variable defined on $[a, b]$.

2. Jensen's inequality and some of its extensions

When f is a convex function, the famous inequality of Jensen [13] states that

$$f(EX) \leq Ef(X),$$

and for concave functions we have

$$Ef(X) \leq f(EX).$$

There are various ways to estimate the difference $Ef(X) - f(EX)$. For instance, as in [37], let α, β be real numbers and f be twice differentiable such that f'' is bounded in $[a, b]$. Let $A(t) = f(t) - \alpha t^2$ and $B(t) = f(t) - \beta t^2$. Since $A''(t) = f''(t) - 2\alpha$ and $B''(t) = f''(t) - 2\beta$, if α, β are selected in such a way that $\alpha < f''(t)/2 < \beta$ for any $a \leq t \leq b$, then we have $A''(t) > 0$ for $a \leq t \leq b$ and $A(.)$ is convex. In a similar way we can show that $B(.)$ is concave. Therefore, according to Jensen inequality

$$EA(X) \geq A(EX),$$

and

$$EB(X) \leq B(EX),$$

which respectively yield

$$Ef(X) - \alpha EX^2 \geq f(EX) - \alpha E^2 X,$$

and

$$Ef(X) - \beta EX^2 \leq f(EX) - \beta E^2 X.$$

In other words

$$\alpha Var(X) \leq Ef(X) - f(EX) \leq \beta Var(X). \tag{2.1}$$

Note that even if f is convex, (2.1) can be sharper than Jensen's inequality. Inequalities of the form (2.1) have received some attention in finance, cf. [1] or [2].

Another upperbound is given in the following proposition.

Proposition 2.1. *Let f be a convex function on $[a, b]$ and X be a random variable defined on $[a, b]$. Then we have*

$$0 \leq Ef(X) - f(EX) \leq \max(A, B) S_f(a, b),$$

in which $A = (b - EX)/(b - a)$, $B = (EX - a)/(b - a)$ and

$$S_f(a, b) = f(a) + f(b) - 2f((a + b)/2).$$

Proof. Since f is convex,

$$f(x) \leq \frac{b - x}{b - a} f(a) + \frac{x - a}{b - a} f(b) \qquad a \leq x \leq b.$$

Therefore

$$Ef(X) \leq E\left(\frac{b - X}{b - a} f(a) + \frac{X - a}{b - a} f(b)\right) = Af(a) + Bf(b),$$

with A, B as above. Using the equality $EX = aA + bB$ it follows that

$$Ef(X) - f(EX) \leq Af(a) + Bf(b) - f(Aa + Bb).$$

On the other hand, Lemma 2 of [31] shows that

$$Af(a) + Bf(b) - f(Aa + Bb) \le \max(A, B)S_f(a, b),$$

which proves the result. $\square$

For example, if $U \sim U(a, b)$, then $EU = (a + b)/2$ and $A = B = 1/2$. So $0 \le Ef(U) - f(EU) \le S_f(a, b)/2$, which yields

$$f(\frac{a+b}{2}) \le Ef(U) \le \frac{f(a) + f(b)}{2}.$$

To obtain another upperbound, we can assume that f has a derivative $f^{(1)}$ and satisfies the condition

$$f(x) - f(y) \ge (x - y)f^{(1)}(y) \qquad \forall x, y \in [a, b]. \tag{2.2}$$

Note that (2.2) holds for any differentiable convex function. Let g denote a function so that f is well-defined on the range of g. Following the approach of [32] or [28], we can prove that:

Proposition 2.2. *If* (2.2) *holds, then*

$$Ef(g(X)) - f(Eg(X)) \le Cov(g(X), f^{(1)}(g(X)),$$

where in general $Cov(A, B) = EAB - EAEB$ *and*

$$\begin{aligned} &Ef(g(X)) - f(Eg(X)) \\ &\ge \left| E\left|f(g(X)) - f(Eg(X))\right| - E\left|g(X) - Eg(X)\right| \left|f^{(1)}(Eg(X))\right| \right| \ge 0. \end{aligned}$$

Especially when $g(x) = x$, *we have*

$$Ef(X) - f(EX) \le Cov(X, f^{(1)}(X)),$$

and

$$Ef(X) - f(EX) \ge \left| E\left|f(X) - f(EX)\right| - E\left|X - EX\right| \left|f^{(1)}(EX)\right| \right|.$$

Proof. We use (2.2) twice. In the first case, if $x = g(X)$ and $y = Eg(X)$ are replaced in (2.2), then we find that

$$\begin{aligned} &f(g(X)) - f(Eg(X)) - (g(X) - Eg(X))f^{(1)}(Eg(X)) \\ &= \left|f(g(X)) - f(Eg(X)) - (g(X) - Eg(X))f^{(1)}(Eg(X))\right| \\ &\ge \left| \left|f(g(X)) - f(Eg(X))\right| - \left|g(X) - Eg(X)\right| \left|f^{(1)}(Eg(X))\right| \right|. \end{aligned} \tag{2.3}$$

By taking the expected value from (2.3) we obtain

$$\begin{aligned} &Ef(g(X)) - f\left(Eg(X)\right) \\ &\ge E\left| \left|f(g(X)) - f(Eg(X))\right| - \left|g(X) - Eg(X)\right| \left|f^{(1)}(Eg(X))\right| \right| \\ &\ge \left| E\left|f(g(X)) - f(Eg(X))\right| - E\left|g(X) - Eg(X)\right| \left|f^{(1)}(Eg(X))\right| \right|. \end{aligned}$$

To prove the first inequality, if $x = Eg(X)$ and $y = g(X)$ are replaced in (2.2), then

$$f(Eg(X)) - f(g(X)) \ge (Eg(X) - g(X))f^{(1)}(g(X)). \tag{2.4}$$

Again, taking the expected value from (2.4) yields

$$f(Eg(X)) - Ef(g(X)) \ge E(Eg(X) - g(X))f^{(1)}(g(X)) = -Cov(g(X), f^{(1)}(g(X))).$$

□

For example, if $g(x) = x$ and $X = U \sim U(a, b)$, then

$$Ef^{(1)}(U) = \frac{1}{b-a}\int_a^b f^{(1)}(t)\,dy = \frac{f(b)-f(a)}{b-a},$$

and

$$EUf^{(1)}(U) = \frac{1}{b-a}\int_a^b tf^{(1)}(t)\,dt = \frac{bf(b)-af(a)}{b-a} - Ef(U).$$

Therefore we have

$$0 \le Ef(U) - f(EU) = \frac{bf(b)-af(a)}{b-a} - Ef(U) - \frac{a+b}{2}\frac{f(b)-f(a)}{b-a},$$

which leads to

$$0 \le Ef(U) - f(EU) \le \frac{f(a)+f(b)}{2} - Ef(U),$$

and

$$0 \le Ef(U) - f(\frac{a+b}{2}) \le \frac{1}{2}(\frac{f(a)+f(b)}{2} - f(\frac{a+b}{2})).$$

3. Estimates for $Ef(X) - f(EX)$ Using Taylor expansions

3.1. **First derivative.** Suppose that $f(.)$ has a well-defined derivative $f^{(1)}(.)$. It is clear that for $a \le x$ and $y \le b$ we have

$$f(x) = f(y) + \int_y^x f^{(1)}(t)\,dt = f(y) + (x-y)\int_0^1 f^{(1)}(y+u(x-y))\,du,$$

and for any real number z we have

$$f(x) - f(y) - z(x-y) = R(x-y, z),$$

where

$$R(x-y,z) = (x-y)\int_0^1 (f^{(1)}(y+u(x-y)) - z)\,du.$$

If $\left\|f^{(1)}\right\| < \infty$, then

$$|f(x) - f(y) - z(x-y)| \le |x-y| \sup_t \left|f^{(1)}(t) - z\right|. \tag{3.1}$$

By replacing x by X and taking the expected value from both sides of (3.1) we obtain

$$|Ef(X) - f(y) - z(EX-y)| \le E\,|X-y| \sup_t \left|f^{(1)}(t) - z\right|. \tag{3.2}$$

For example, if $X = U \sim U(a, b)$, then (3.2) is reduced to

$$E\,|U-y| = \frac{1}{2(b-a)}((y-a)^2 + (y-b)^2) = \frac{1}{b-a}((y - \frac{a+b}{2})^2 + \frac{(b-a)^2}{4}),$$

and

$$|Ef(U) - f(y) - z(EU-y)| \le \frac{1}{b-a}((y-\frac{a+b}{2})^2 + \frac{(b-a)^2}{4}) \sup_t \left|f^{(1)}(t) - z\right|. \tag{3.3}$$

a) If $z = 0$ in (3.3), we find back a result of Ostrowski [29].

b) If $y = EU$ in (3.3), then we have

$$|Ef(U) - f(EU)| \leq \frac{b-a}{4} \sup_t \left| f^{(1)}(t) - z \right|.$$

c) If $z = Ef^{(1)}(U)$, (3.3) reduces to

$$\begin{aligned} &\left| Ef(U) - f(y) - Ef^{(1)}(U)(EU - y) \right| \\ &\leq \frac{1}{b-a}((y - \frac{a+b}{2})^2 + \frac{(b-a)^2}{4}) \sup_t \left| f^{(1)}(t) - Ef^{(1)}(U) \right|. \end{aligned} \tag{3.4}$$

On the other hand, if $m \leq f^{(1)}(t) \leq M$ for any $a \leq t \leq b$, then $m \leq Ef^{(1)}(U) \leq M$ and

$$\left| f^{(1)}(t) - Ef^{(1)}(U) \right| \leq M - m.$$

Therefore, the latter result (3.4) reads as

$$\left| Ef(U) - f(y) - Ef^{(1)}(U)(EU - y) \right| \leq \frac{M-m}{b-a}((y - \frac{a+b}{2})^2 + \frac{(b-a)^2}{4}),$$

which resembles a result of [8].

d) If $z = f^{(1)}(y)$ and the modulus of continuity of $f^{(1)}$ is given by ω, then by definition we first have

$$\left| f^{(1)}(y + u(x-y)) - f^{(1)}(y) \right| \leq \omega(u\,|x-y|).$$

For a Lipschitz or Hölder function, we have $\omega(t) = \alpha t^\beta$ where $\alpha > 0$ and $\beta \geq 0$. Therefore

$$\left| R(x-y, f^{(1)}(y)) \right| \leq |x-y| \int_0^1 \alpha u^\beta \,|x-y|^\beta \; du = \frac{\alpha}{1+\beta}\,|x-y|^{\beta+1},$$

which follows that

$$\left| Ef(X) - f(y) - f^{(1)}(y)(EX - y) \right| \leq \frac{\alpha}{1+\beta} E\,|X-y|^{1+\beta}.$$

For instance, substituting $\beta = 1$ in the above inequality yields

$$\left| Ef(X) - f(y) - f^{(1)}(y)(EX - y) \right| \leq \frac{\alpha}{2} E(X-y)^2.$$

3.2. **Second derivative.** Now assume that f has a second derivative $f^{(2)}$. Then we have

$$f(x) - f(y) - f^{(1)}(y)(x-y) - \frac{1}{2} z(x-y)^2 = R(x-y, z),$$

where

$$R(x-y, z) = (x-y)^2 \int_0^1 (1-u) \left(f^{(2)}(y + u(x-y)) - z \right) \, du.$$

If $\left\| f^{(2)} \right\| < \infty$, it is clear that

$$|R(x-y, z)| \leq \frac{1}{2}(x-y)^2 \sup_t \left| f^{(2)}(t) - z \right|. \tag{3.5}$$

By replacing x by X and taking the expected value from both sides of (3.5) we obtain

$$\begin{aligned}&\left|Ef(X)-f(y)-f^{(1)}(y)(EX-y)-\frac{1}{2}zE(X-y)^2\right| \\ &\leq \frac{1}{2}E(X-y)^2\sup_t\left|f^{(2)}(t)-z\right|.\end{aligned} \tag{3.6}$$

For example, if $X=U\sim U(a,b)$ in (3.6), then the equality

$$E(U-y)^2=(y-\frac{a+b}{2})^2+\frac{(b-a)^2}{12},$$

implies that

$$\begin{aligned}&\left|Ef(U)-f(y)-f^{(1)}(y)(EU-y)-\frac{1}{2}zE(U-y)^2\right| \\ &\leq \frac{1}{2}((y-\frac{a+b}{2})^2+\frac{(b-a)^2}{12})\sup_t\left|f^{(2)}(t)-z\right|.\end{aligned} \tag{3.7}$$

a) If $z=Ef^{(2)}(U)$ in (3.6), the above inequality (3.7) reads as

$$\begin{aligned}&\left|Ef(U)-f(y)-f^{(1)}(y)(EU-y)-\frac{1}{2}Ef^{(2)}(U)E(U-y)^2\right| \\ &\leq \frac{1}{2}((y-\frac{a+b}{2})^2+\frac{(b-a)^2}{12})\sup_t\left|f^{(2)}(t)-Ef^{(2)}(U)\right|.\end{aligned}$$

Such type of inequalities have been derived in [3] and [4].

b) If $f^{(2)}(.)$ satisfies

$$\left|f^{(2)}(x)-f^{(2)}(y)\right|\leq\omega(|x-y|),$$

with $\omega(t)=\alpha t^\beta$, $\alpha,\beta>0$, then taking $z=f^{(2)}(y)$ yields

$$\left|R(x-y,f^{(2)}(y))\right|\leq\alpha\left|x-y\right|^{2+\beta}\int_0^1(1-u)u^\beta\,du=\frac{\alpha}{(\beta+1)(\beta+2)}\left|x-y\right|^{2+\beta},$$

and we find that

$$\begin{aligned}&\left|Ef(X)-f(y)-f^{(1)}(y)(EX-y)-\frac{1}{2}f^{(2)}(y)E(X-y)^2\right| \\ &\leq \frac{\alpha}{2(\beta+1)(\beta+2)}E\left|X-y\right|^{2+\beta}.\end{aligned}$$

3.3. **Higher order derivatives.** For higher derivatives, the following proposition may be applied whose proof can be easily derived by using the Taylor expansion.

Proposition 3.1. *Suppose that f has a bounded $m-th$ derivative $f^{(m)}(.)$ and $E\left|X\right|^m<\infty$. Then*

$$\left|Ef(X)-f(y)-\sum_{i=1}^{m-1}f^{(i)}(y)\frac{E(X-y)^i}{i!})\right|\leq\left\|f^{(m)}\right\|\frac{E\left|X-y\right|^m}{m!}.$$

4. Estimates Using Covariance and correlation

Recall that the correlation coefficient between two random variables is defined as

$$\rho(X,Y) = Cov(X,Y)/\sigma(X)\sigma(Y),$$

where $\sigma(X)$ denotes the standard deviation of X and $Cov(X,Y) = EXY - EXEY$ is the covariance between X and Y. Since $|\rho(X,Y)| \le 1$ so

$$|Cov(X,Y)| \le \sigma(X)\sigma(Y).$$

The equality appears whenever X, Y satisfy - with probability 1 - a linear relation of the form $\alpha X + \beta Y = \gamma$. Replacing X by $f(X)$ and Y by $g(Y)$, we find that

$$|Cov(f(X), g(Y)| \le \sigma(f(X))\sigma(g(Y)), \tag{4.1}$$

where $\sigma^2(f(X))$ is the variance of $f(X)$.

In the case of $X = Y = U \sim U(a,b)$, (4.1) is transformed to

$$\left| \frac{1}{b-a}\int_a^b f(t)g(t)\,dt - \frac{1}{(b-a)^2}\int_a^b f(t)\,dt \int_a^b g(t)\,dt \right| \le \sigma(f(U))\sigma(g(U)).$$

4.1. **Grüss type inequalities.** There are other upperbounds for the covariance under different conditions. The following result was initiated by Popoviciu (1935) and Grüss (1935). We here present a statistical proof for it.

Lemma 4.1. *(i) Let A be a random variable with the mean value $EA = \mu$ such that $m \le A \le M$. Then we have*

$$\sigma^2(A) \le \delta^2_{m,M}(\mu)\frac{(M-m)^2}{4} \le \frac{(M-m)^2}{4},$$

in which $\delta^2_{m,M}(\mu) = 4(M-\mu)(\mu-m)/(M-m)^2$.

(ii) Let X be a random variable with values in the set Ω and $f(.)$ be a bounded function with $m \le f(x) \le M$ for $x \in \Omega$. If $\mu = Ef(X)$, then

$$\sigma^2(f(X)) \le \delta^2_{m,M}(\mu)\frac{(M-m)^2}{4} \le \frac{(M-m)^2}{4}.$$

(iii) Let X and Y be two random variables with values in the set Ω and $f(.)$, $g(.)$ be bounded functions such that $m \le f(x) \le M$ and $\alpha \le g(x) \le \beta$ for $x \in \Omega$. Then we have

$$|Cov(f(X), g(Y))| \le \frac{1}{4}\delta_{m,M}(\mu)\delta_{\alpha,\beta}(\nu)(M-m)(\beta-\alpha),$$

where $\mu = Ef(X)$ and $\nu = Eg(Y)$.

Proof. (i) By noting the equality

$$A = \frac{A-m}{M-m}M + \frac{M-A}{M-m}m,$$

and this fact that $g(x) = (x-\mu)^2$ is a convex function, we first have

$$g(A) \le \frac{A-m}{M-m}g(M) + \frac{M-A}{M-m}g(m). \tag{4.2}$$

Now taking the expected value from both sides of (4.2) yields

$$\sigma^2(A) \leq \frac{\mu - m}{M - m} g(M) + \frac{M - \mu}{M - m} g(m) = \delta^2_{m,M}(\mu) \frac{(M-m)^2}{4},$$

where $\delta^2_{m,M}(\mu) = 4(M-\mu)(\mu-m)/(M-m)^2$. On the other hand, since $(x - M)(m-x)$ is maximum for $x = (m+M)/2$, it follows that $\delta^2_{m,M}(\mu) \leq 1$.
The second result (ii) follows from (i) by taking $A = f(X)$ and the third result (iii) follows from (ii) and (4.1). □

Notice that the bound in Lemma 4.1(ii) is sharp and cannot be improved. For example, taking $U \sim U(0,1)$ and

$$f(x) = \begin{cases} -1 & 0 \leq x \leq \frac{1}{2}, \\ +1 & \frac{1}{2} < x \leq 1, \end{cases}$$

yield $Ef(U) = 0$ and $\sigma^2(f(U)) = (M-m)^2/4 = 1$.

Some remarks on Lemma 4.1.

1) The advantage of $\delta^2_{m,M}(\mu)$ in Lemma 4.1 is when we have information about μ. When there is no information available, we have no choice and just we can replace $\delta_{m,M}(\mu)$ by its upperbound 1. More discussions are in [36].

2) Since $g(x) = |x - \mu|$ is also a convex function, considering it in Lemma 4.1(i) yields

$$E\,|f(X) - Ef(X)| \leq \frac{2(M-\mu)(\mu-m)}{M-m} = \delta^2_{m,M}(\mu) \frac{M-m}{2}.$$

3) Lemma 4.1(iii) is usually formulated for $X = Y = U \sim U(a,b)$, see e.g. ([12], Theorem 4) or ([14], Corollary 4).

4.2. **Dependency function.** Lemma 4.1(iii) can be improved by using the dependency function of random variables. Let $f(.)$ and $g(.)$ be two real functions of bounded variation defined on the intervals $[a,b]$ and $[c,d]$ respectively, where $-\infty \leq a,b,c,d \leq \infty$. Also let X and Y be random variables defined on $[a,b]$ and $[c,d]$ respectively and assume that $Ef(X), Eg(Y)$ and $Ef(X)g(Y)$ are finite values. Cuadras in [7] obtained the following extension of an old result of Hoeffding [11]:

$$Cov(f(X), g(Y)) = \int_a^b \int_c^d C(u,v) df(u) dg(v), \tag{4.3}$$

where

$$C(x,y) = P(X \leq x, Y \leq y) - P(X \leq x)P(Y \leq y),$$

is the dependency function of (X,Y). For $f(x) = g(x) = x$ in (4.3) Hoeffding's result is derived. It is well-known that $|C(x,y)| \leq 1/4$. The following Lemma is a variant of Lemma 4.1(iii), cf. [9].

Lemma 4.2. *If* $|C(x,y)| \leq \theta/4$ *for* $0 \leq \theta \leq 1$, *then*

$$|Cov(f(X), g(Y))| \leq \frac{1}{4} \min(\theta, \delta_{m,M}(\mu)\delta_{\alpha,\beta}(\nu))(M-m)(\beta-\alpha)),$$

where $m = f(a), M = f(b), \alpha = g(c)$ *and* $\beta = g(d)$.

For example, let $f(x) = g(x) = x$ and X, Y be defined on $[0, 1]$. Also assume that the joint distribution is given by

$$P(X \leq x, Y \leq y) = Kxy(\frac{x^\alpha}{1+\alpha} + \frac{y^\beta}{1+\beta}),$$

where $K = (1+\alpha)(1+\beta)/(2+\alpha+\beta)$. Clearly we have

$$P(X \leq x) = Kx(\frac{x^\alpha}{1+\alpha} + \frac{1}{1+\beta}), \qquad EX = \mu = K\frac{\alpha+2\beta+4}{2(2+\alpha)(1+\beta)},$$

$$P(Y \leq x) = Kx(\frac{1}{1+\alpha} + \frac{y^\beta}{1+\beta}), \qquad EY = \nu = K\frac{\beta+2\alpha+4}{2(2+\beta)(1+\alpha)}.$$

Therefore, the dependency function is computed as

$$C(x, y) = \frac{Kxy}{2+\alpha+\beta}(x^\alpha - 1)(1 - y^\beta),$$

and

$$Cov(X, Y) = \int_0^1 \int_0^1 C(x, y)\, dx\, dy = \frac{-K\alpha\beta}{4(2+\alpha)(2+\beta)(2+\alpha+\beta)}.$$

The function $|C(x, y)|$ attains its maximum at the point

$$(x^\circ, y^\circ) = ((1+\alpha)^{-1/\alpha}, (1+\beta)^{-1/\beta}),$$

and we have

$$|C(x, y)| \leq \frac{\alpha\beta x^\circ y^\circ}{(2+\alpha+\beta)^2}.$$

Now, by noting Lemma 4.2 consider the 2 terms

$$\theta = \frac{4\alpha\beta x^\circ y^\circ}{(2+\alpha+\beta)^2}, \tag{4.4}$$

and

$$\delta_{0,1}(\mu)\delta_{0,1}(\nu) = 4\sqrt{(1-\mu)\mu(1-\nu)\nu}\,. \tag{4.5}$$

Table 1. Computation of (4.4), (4.5) and $Cov(X, Y)$ for some values of α and β.

α	β	$\delta_{0,1}(\mu)\delta_{0,1}(\nu)$	θ	$\|Cov(X, Y)\|$	$\theta/4$
1	1	0.9722	0.0625	0.00694	0.0156
1	2	0.96	0.0924	0.01	0.0231
1	3	0.955	0.1049	0.011	0.0262
2	2	0.9375	0.1481	0.0156	0.0370
2	3	0.926	0.178	0.018	0.0145
3	3	0.91	0.2232	0.0225	0.0558
10	10	0.826	0.512	0.0434	0.1279

As Table 1 shows, the covariance bound of Lemma 4.2 is given by $|Cov(X, Y)| \leq \theta/4$.

Now, suppose that X is a random variable on $[a, b]$ and f satisfies $m \leq f(t) \leq M$. Also, let $g(x)$ be an arbitrary function with $Eg(Y) = 0$. Matic ([20], Theorem 4) proved the following result, which has appeared also in [6] and as a Lemma in [16].

Lemma 4.3. *Let $f(.)$ and $g(.)$ be two real functions such that $m \leq f(x) \leq M$ and $Eg(Y) = 0$. Then*

$$|Cov(f(X), g(Y))| = |Ef(X)g(Y)| \leq E\,|g(Y)|\,\frac{M-m}{2}.$$

Proof. Let us define the sets

$$G(+) = \{x : g(x) \geq 0\} \quad \text{and} \quad G(-) = \{x : g(x) < 0\},$$

and assume that $I_{G(+)}$ and $I_{G(-)}$ denote the indicators of these sets. We have

$$Ef(X)g(Y) = Eg(Y)I_{G(+)}f(X) + Eg(Y)I_{G(-)}f(X).$$

Since $m \leq f(x) \leq M$, so

$$Ef(X)g(Y) \leq MEg(Y)I_{G(+)} + mEg(Y)I_{G(-)},$$

and

$$Ef(X)g(Y) \geq mEg(Y)I_{G(+)} + MEg(Y)I_{G(-)}.$$

On the other hand, since

$$Eg(Y)I_{G(+)} + Eg(Y)I_{G(-)} = 0,$$

we obtain

$$(m-M)Eg(Y)I_{G(+)} \leq Ef(X)g(Y) \leq (M-m)Eg(Y)I_{G(+)},$$

and hence

$$|Ef(X)g(Y)| \leq (M-m)Eg(Y)I_{G(+)}.$$

Finally note that

$$E\,|g(Y)| = Eg(Y)I_{G(+)} - Eg(Y)I_{G(-)} = 2Eg(Y)I_{G(+)},$$

and the desired result is derived. □

For example, let X be a random variable defined on $[a, b]$ with the density function $h(.)$. Since

$$\sigma^2(X) = \int_a^b (t-\mu)^2 h(t)\,dt = E(X-\mu)^2,$$

by considering $f(t) = (b-a)h(t)$ and $g(t) = (t-\mu)^2$ we have

$$\sigma^2(X) = Ef(U)g(U),$$

and

$$Cov(f(U), g(U)) = Ef(U)g(U) - Ef(U)Eg(U) = \sigma^2(X) - Eg(U),$$

where $Eg(U) = E(U-\mu)^2 = (\mu - EU)^2 + \sigma^2(U)$.

Now assume that $m \leq f(.) \leq M$. Applying Lemma 4.3 yields

$$\left|\sigma^2(X) - E(U-\mu)^2\right| \leq \frac{M-m}{2} E\,|g(U) - Eg(U)|\,.$$

4.3. **Derivatives.** Many authors have obtained bounds for the covariance using the derivative(s) of the functions involved. For instance:
a) Following [17] if $\left\|f^{(1)}\right\| < \infty$, then for any arbitrary x we have

$$\begin{aligned}|Cov(f(X),g(Y))| &= |E((f(X)-f(x))(g(Y)-Eg(Y))| \\ &\leq E(|f(X)-f(x)|\,|g(Y)-Eg(Y)|).\end{aligned}$$

Now applying $|f(t)-f(x)| \leq \left\|f^{(1)}\right\| |t-x|$ gives

$$|Cov(f(X),g(Y))| \leq \left\|f^{(1)}\right\| E(|X-x|\,|g(Y)-Eg(Y)|). \tag{4.6}$$

For example, if $x = \mu = EX$, then

$$|Cov(f(X),g(Y))| \leq \left\|f^{(1)}\right\| E(|X-\mu|\,|g(Y)-Eg(Y)|),$$

is a generalization of the result obtained in [17]. Moreover, applying the Cauchy-Schwarz inequality on (4.6) leads to

$$|Cov(f(X),g(Y))| \leq \left\|f^{(1)}\right\| (E(X-x)^2)^{1/2}\sigma(g(Y)),$$

cf. Theorem 5 of [17].

Since $E(X-x)^2 = \sigma^2(X) + (EX-x)^2$, we realize that $x = \mu$ is optimum such that we have

$$|Cov(f(X),g(Y))| \leq \left\|f^{(1)}\right\| \sigma(X)\sigma(g(Y)). \tag{4.7}$$

Also, if $\left\|g^{(1)}\right\| < \infty$, we can replace (4.6) by

$$|Cov(f(X),g(Y))| \leq \left\|f^{(1)}\right\| \left\|g^{(1)}\right\| E\,|X-x|\,|Y-Eg(Y)|\,,$$

and (4.7) by

$$|Cov(f(X),g(Y))| \leq \left\|f^{(1)}\right\| \left\|g^{(1)}\right\| \sigma(X)\sigma(Y).$$

The case $f(x) = g(x) = x$ and $X = Y$ shows that this inequality cannot be improved. Such type of inequality goes back to Chebyshev [19].

b) Assume that $\left\|f^{(2)}\right\| < \infty$. By using the previous approach, replacing f by $f^{(1)}$ yields

$$|Cov(f^{(1)}(X),g(Y))| \leq \left\|f^{(2)}\right\| E(|X-EX|\,|g(Y)-Eg(Y)|),$$

and using the Cauchy-Schwarz inequality gives

$$|Cov(f^{(1)}(X),g(X))| \leq \left\|f^{(2)}\right\| (E(X-x)^2)^{1/2}\sigma(g(Y)).$$

For example, if $g(x) = x$ and $X = Y = U \sim U(a,b)$, then

$$|Cov(f^{(1)}(U),U)| \leq \left\|f^{(2)}\right\| \sigma^2(U).$$

On the other hand, it is easy to find that

$$Cov(f^{(1)}(U),U) = \frac{f(b)+f(a)}{2} - Ef(U).$$

So, the inequality can be rewritten as

$$\left| Ef(U) - \frac{f(a)+f(b)}{2} \right| \leq \left\| f^{(2)} \right\| \frac{(b-a)^2}{12},$$

which is the same as Theorem 8 of [17].

4.4. **Results based on kernels.**

4.4.1. *General kernels.* Following the paper [24], we now consider a general linear integral operator of the form

$$F_K(f;x) = EK(X;x)f'(X),$$

where X is a random variable and $K(t;x)$ is called a kernel. If

$$EK(X;x) = w(x),$$

we can define a further kernel as

$$K^\circ(t;x) = K(t;x) - w(x),$$

and consequently a linear operator as

$$F_{K^\circ}(f;x) = EK^\circ(X;x)f'(X) = Cov\left(K(X,x), f'(X)\right).$$

Note that

$$F_{K^\circ}(f;x) = F_K(f;x) - w(x)Ef'(X).$$

For example, if $K(t;x) = t$ and $X = U \sim U(a,b)$, then

$$w(x) = (a+b)/2,$$

$$F_K(f;x) = \frac{1}{b-a}\int_a^b tf'(t)\,dt = \frac{bf(b)-af(a)}{b-a} - Ef(U),$$

and

$$\begin{aligned} F_{K^\circ}(f;x) &= \frac{bf(b)-af(a)}{b-a} - Ef(U) - \frac{a+b}{2}\frac{f(b)-f(a)}{b-a} \\ &= \frac{f(b)+f(a)}{2} - Ef(U). \end{aligned}$$

By noting at the above concepts, the authors in [24] obtained the following results.

Theorem 4.4. *a) Suppose that $f'(t) \leq \beta(t), \forall t \in [a,b]$. Then we have*

$$|F_K(f;x) - EK(X;x)\beta(X)| \leq \|K(.;x)\|\, E(\beta(X) - f'(X)).$$

b) If $\alpha(t) \leq f'(t), \forall t \in [a,b]$ then

$$|F_K(f;x) - EK(X;x)\alpha(X)| \leq \|K(.;x)\|\, E(f'(X) - \alpha(X)).$$

c) If $\alpha(t) \leq f'(t) \leq \beta(t), \forall t \in [a,b]$ then

$$\left| F_K(f;x) - EK(X;x)\frac{\alpha(X)+\beta(X)}{2} \right| \leq E\left(\frac{\beta(X)-\alpha(X)}{2}\,|K(X;x)|\right).$$

Proof. To prove part a) since

$$F_K(f;x) - EK(X;x)\beta(X) = E(f'(X) - \beta(X))K(X;x),$$

so

$$|F_K(f;x) - EK(X;x)\beta(X)| \leq E(\beta(X) - f'(X))K(X;x),$$

and the result follows. The proof of b) is similar. Finally for the proof of c) we first have

$$\begin{aligned}\left|F_K(f;x) - EK(X;x)\frac{\alpha(X)+\beta(X)}{2}\right| &= \left|E\left(f'(X) - \frac{\alpha(X)+\beta(X)}{2}\right)K(X;x)\right| \\ &\leq E\left|\left(f'(X) - \frac{\alpha(X)+\beta(X)}{2}\right)K(X;x)\right|.\end{aligned}$$

On the other hand, the given condition $\alpha(t) \leq f'(t) \leq \beta(t)$ implies that

$$|f'(t) - (\alpha(t)+\beta(t))/2| \leq (\beta(t) - \alpha(t))/2.$$

Therefore

$$\left|F_K(f;x) - EK(X;x)\frac{\alpha(X)+\beta(X)}{2}\right| \leq E(\frac{\beta(X)-\alpha(X)}{2}\,|K(X;x)|).$$

□

Similar results can be derived for $F_{K^\circ}(f;x)$ as follows.

Corollary 4.5. *a) Suppose that $f'(t) \leq \beta(t), \forall t \in [a,b]$. Then we have*

$$|F_{K^\circ}(f;x) - EK(X;x)\beta(X) - w(x)Ef'(X)| \leq \|K^\circ(.;x)\|\, E(\beta(X) - f'(X)).$$

b) If $\alpha(t) \leq f'(t), \forall t \in [a,b]$ then

$$|F_K(f;x) - EK(X;x)\alpha(X) - w(x)Ef'(X)| \leq \|K^\circ(.;x)\|\, E(f'(X) - \alpha(X)).$$

c) If $\alpha(t) \leq f'(t) \leq \beta(t), \forall t \in [a,b]$ then

$$\left|F_K(f;x) - EK(X;x)\frac{\alpha(X)+\beta(X)}{2} - w(x)Ef'(X)\right| \leq E(\frac{\beta(X)-\alpha(X)}{2}\,|K^\circ(X;x)|).$$

Notice that the above corollary can be simplified if $\alpha(t)$ and $\beta(t)$ are constants. In other words:

Corollary 4.6. *a) Suppose that $f'(t) \leq \beta, \forall t \in [a,b]$. Then we have*

$$|F_K(f;x) - \beta w(x) - w(x)Ef'(X)| \leq \|K^\circ(.;x)\|\,(\beta - Ef'(X)).$$

b) If $\alpha \leq f'(t), \forall t \in [a,b]$ then

$$|F_K(f;x) - \alpha w(x) - w(x)Ef'(X)| \leq \|K^\circ(.;x)\|\,(Ef'(X) - \alpha).$$

c) If $\alpha \leq f'(t) \leq \beta, \forall t \in [a,b]$ then

$$\left|F_K(f;x) - \frac{\alpha+\beta}{2}w(x) - w(x)Ef'(X)\right| \leq \frac{\beta-\alpha}{2}E\,|K^\circ(X;x)|\,.$$

4.4.2. *Specific kernels.* Following [8], [18], [33], [34] we may define the linear kernel

$$K(t;x)=\begin{cases} t-A & a\le t\le x,\\ \\ t-B & x<t\le b,\end{cases} \tag{4.8}$$

in which A,B are real numbers and obtain some properties of $K(U;x)$ where as before $U\sim U(a,b)$.

Lemma 4.7. *Corresponding to the kernel* (4.8) *we respectively have*

(i)

$$w(x)\equiv EK(U;x)=\frac{a+b}{2}-B+\frac{B-A}{b-a}(x-a).$$

(ii)

$$Var\,(K(U;x))=\sigma^2(U)+\frac{(B-A)(x-a)(b-x)}{b-a}(\frac{B-A}{b-a}-1).$$

(iii)

$$\begin{aligned}(b-a)E\left|K(U;x)\right| \quad &= \quad sign(x-A)\frac{(x-A)^2}{2}+sign(A-a)\frac{(A-a)^2}{2}\\ &\quad +sign(B-x)\frac{(x-B)^2}{2}+sign(b-B)\frac{(B-b)^2}{2}.\end{aligned}$$

Proof. (i) First it is clear that

$$K(t;x)=t-B+(B-A)I_{\{a\le t\le x\}},$$

where I_S is the indicator function of the set S. Therefore

$$w(x)=EU-B+(B-A)EI_{\{a\le U\le x\}}=\frac{a+b}{2}-B+(B-A)P(U\le x)$$

$$=(B-A)\frac{x-a}{b-a}+\frac{a+b}{2}-B.$$

(ii) Using

$$K(U;x)=U-B+(B-A)I_{\{a\le U\le x\}},$$

we find that

$$Var(K(U;x))=Var(U)+(B-A)^2Var(I_{\{a\le U\le x\}})+2(B-A)Cov(U,I_{\{a\le U\le x\}}).$$

Since

$$Var(I_{\{a\le U\le x\}})=\frac{x-a}{b-a}-(\frac{x-a}{b-a})^2=\frac{(x-a)(b-x)}{(b-a)^2},$$

and

$$\begin{aligned}Cov(U,I_{\{a\le U\le x\}}) \quad &= \quad \frac{1}{b-a}\int_a^x t\,dt-EUEI_{\{a\le U\le x\}})\\ &= \quad \frac{(x-a)(x+a)}{2(b-a)}-\frac{b+a}{2}\frac{x-a}{b-a}=\frac{(x-a)(x-b)}{2(b-a)},\end{aligned}$$

it follows that

$$Var\,(K(U;x))=\sigma^2(U)+\frac{(B-A)(x-a)(b-x)}{b-a}(\frac{B-A}{b-a}-1).$$

(iii) First we have

$$E\left|K(U;x)\right| = \frac{1}{b-a}\left(\int_a^x |t-A|\,dt + \int_x^b |t-B|\,dt\right) = \frac{1}{b-a}(I+II).$$

Now if $A<a<x$, then

$$\begin{aligned} I &= \int_a^x (t-A)\,dt = \frac{(x-A)^2}{2} - \frac{(a-A)^2}{2} \\ &= sign(x-A)\frac{(x-A)^2}{2} + sign(A-a)\frac{(a-A)^2}{2}, \end{aligned}$$

and if $A>x>a$, we once again obtain

$$I = \int_a^x (A-t)\,dt = sign(x-A)\frac{(x-A)^2}{2} + sign(A-a)\frac{(a-A)^2}{2}.$$

This means that if $a \le A \le x$, then

$$I = sign(x-A)\frac{(x-A)^2}{2} + sign(A-a)\frac{(A-a)^2}{2}.$$

In a similar way, we can conclude that

$$II = sign(B-x)\frac{(x-B)^2}{2} + sign(b-B)\frac{(B-b)^2}{2}.$$

So the result follows. In this sense, note that

$$|K(t;x)| \le \max(|a-A|, |x-A|, |b-B|, |x-B|).$$

□

Lemma 4.8. *Corresponding to the kernel* (4.8) *we have*

$$EK(U;x)f'(U) = \frac{B-A}{b-a}f(x) + \frac{(b-B)f(b)-(a-A)f(a)}{b-a} - Ef(U),$$

and

$$\begin{aligned} &Cov(K(U;x), f'(U)) \\ &= \frac{B-A}{b-a}f(x) + \frac{(b-B-w(x))f(b)-(a-A-w(x))f(a)}{b-a} - Ef(U). \end{aligned}$$

Proof. Since

$$\int_a^b K(t;x)f'(t)\,dt = \int_a^x (t-A)f'(t)\,dt + \int_x^b (t-B)f'(t)\,dt,$$

integrating by parts yields

$$\begin{aligned} \int_a^b K(t;x)f'(t)\,dt &= (x-A)f(x)-(a-A)f(a) - \int_a^x f(t)\,dt \\ &\quad +(b-B)f(b)-(x-B)f(x) - \int_x^b f(t)\,dt. \end{aligned}$$

By rearranging the above terms, we can achieve the first result. The second result can also be derived from the first one. □

Two remarks on the kernel (4.8)**.**

1) If $\left\|f^{(1)}\right\| < \infty$, then we have

$$\begin{aligned}|EK(U;x)f'(U)| &= \left|\frac{B-A}{b-a}f(x)+\frac{(b-B)f(b)-(a-A)f(a)}{b-a}-Ef(U)\right| \\ &\leq \left\|f^{(1)}\right\| E\left|K(U;x)\right|.\end{aligned}$$

For example, substituting $A = a$ and $B = b$ yields

$$|f(x) - Ef(U)| \leq \left\|f^{(1)}\right\| \max(x-a, b-x).$$

2) Since

$$Cov(K(U;x), f'(U)) = E\left(K^{\circ}(U;x)f'(U)\right),$$

where $K^{\circ}(t;x) = K(t;x) - w(x)$, i.e.

$$K^{\circ}(t;x) = \begin{cases} t - \frac{a+b}{2} + \frac{B-A}{b-a}(b-x) & a \leq t \leq x, \\ t - \frac{a+b}{2} - \frac{B-A}{b-a}(x-a) & x < t \leq b, \end{cases} \tag{4.9}$$

we have

$$|K^{\circ}(t;x)| \leq \frac{b-a}{2} + \frac{|B-A|}{b-a}\max(x-a, b-x).$$

Moreover

$$\|K^{\circ}(.;x)\| = \max(|a-A-w(x)|, |x-A-w(x)|, |b-B-w(x)|, |x-B-w(x)|).$$

We can now present some useful upperbounds for $Cov(K(U;x), f'(U))$ via the kernel $K^{\circ}(t;x) = K(t;x) - w(x)$ given by (4.9).

a) Using (4.1), we find that

$$\begin{aligned}&\left|\frac{B-A}{b-a}f(x)+\frac{(b-B-w(x))f(b)-(a-A-w(x))f(a)}{b-a}-Ef(U)\right| \\ &\leq \sigma(K(U;x))\sigma(f'(U)).\end{aligned}$$

b) Using (4.3), we obtain

$$\begin{aligned}&\left|\frac{B-A}{b-a}f(x)+\frac{(b-B-w(x))f(b)-(a-A-w(x))f(a)}{b-a}-Ef(U)\right| \\ &\leq \left\|f^{(1)}\right\| E(|U-x|\,|K^{\circ}(U;x)|).\end{aligned}$$

For example, replacing $A = a$ and $B = b$ in the above inequality gives

$$\left|f(x) - Ef(U) - w(x)\frac{f(b)-f(a)}{b-a}\right| \leq \left\|f^{(1)}\right\| E\left(|U-x|\,|K^{\circ}(U;x)|\right),$$

which is related to Ostrowski-Grüss inequality introduced by Dragomir and Wang [8]. In this direction, if f is also differentiable such that $-\infty < m \leq f'(x) \leq M < \infty$, $\forall x \in [a,b]$, then they proved that

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - w(x)\frac{f(b)-f(a)}{b-a}\right| \leq \frac{(b-a)(M-m)}{4}.$$

c) Assume that $m \leq f'(t) \leq M$. Therefore $m \leq Ef'(U) \leq M$ and $|f(U) - Ef'(U)| \leq M - m$ which eventually yields $\sigma(f'(U)) \leq M - m$. Hence, we have

$$\left|\frac{B-A}{b-a}f(x) + \frac{(b-B-w(x))f(b)-(a-A-w(x))f(a)}{b-a} - Ef(U)\right|$$

$$\leq \sigma(K(U;x))(M-m).$$

For instance, $A = a$ and $B = b$ in the above inequality yields

$$\left|f(x) - Ef(U) - w(x)\frac{f(b)-f(a)}{b-a}\right| \leq \frac{(b-a)(M-m)}{\sqrt{12}}. \tag{4.10}$$

Let us consider a particular example for (4.10). Take $a = 0$, $b = 1$ and define

$$f(t) = \begin{cases} \frac{1}{2}t^2 & 0 \leq t \leq x, \\ \frac{1}{2}t^2 - t + x & x < t \leq 1\,. \end{cases}$$

By noting that $f(0) = 0, f(1) = x - 1/2$ and $f(x) = x^2/2$, straightforward calculations show that:

$$Ef(U) = \int_0^x \frac{1}{2}t^2\,dt + \int_x^1 (\frac{1}{2}t^2 - t + x)\,dt = -\frac{1}{2}x^2 + x - \frac{1}{3}.$$

Hence, the left hand side of (4.10) equals to

$$\left|f(x) - Ef(U) - (x - \frac{a+b}{2})\frac{f(b)-f(a)}{b-a}\right| = \frac{1}{12} \leq \frac{1}{\sqrt{12}}.$$

d) Let $m \leq f'(t) \leq M$. It can be deduced from Lemma 4.3 that

$$\left|\frac{B-A}{b-a}f(x) + \frac{(b-B-w(x))f(b)-(a-A-w(x))f(a)}{b-a} - Ef(U)\right|$$

$$\leq \frac{M-m}{2}E\left|K^\circ(U;x)\right|.$$

On the other hand, we proved that

$$|K^\circ(t;x)| \leq \frac{b-a}{2} + \frac{|B-A|}{b-a}\max(x-a, b-x).$$

Therefore

$$|Cov(K(U;x), f'(U))| \leq \frac{M-m}{2}(\frac{b-a}{2} + \frac{|B-A|}{b-a}\max(x-a, b-x)).$$

To find more precise bounds for $E\left|K^\circ(U;x)\right|$, we should consider seven different cases respectively as follows.

Case 1. If $a - A > w(x)$ and $x - B < w(x) \leq b - B$, then $K^\circ(t^*;x) = 0$ where $t^* = B + w(x)$. It is clear that

$$K^\circ(t;x) \geq 0 \quad \text{for} \quad a \leq t \leq x \;\text{ or }\; t \geq t^*,$$

and

$$K^\circ(t;x) \leq 0 \quad \text{for} \quad x \leq t \leq t^*,$$

and also
$$(\int_a^x + \int_{t^*}^b) K^\circ(t;x)\,dt = -\int_x^{t^*} K^\circ(t;x)\,dt.$$
Therefore we have
$$\begin{aligned} E\,|K^\circ(U;x)| &= \frac{1}{b-a}((\int_a^x + \int_{t^*}^b) K^\circ(t;x) - \int_x^{t^*} K^\circ(t;x)\,dt) \\ &= \frac{-2}{b-a}\int_x^{t^*} K^\circ(t;x)\,dt = \frac{(x-B-w(x))^2}{b-a}. \end{aligned}$$

Case 2. If $x-A \le w(x)$ and $x-B < w(x) \le b-B$, then $K^\circ(t^*;x)=0$ where $t^* = B + w(x)$. It is clear that
$$K^\circ(t;x) \le 0 \qquad \text{for} \quad a \le t \le x \quad \text{or} \quad x < t \le t^*,$$
and
$$K^\circ(t;x) \ge 0 \qquad \text{for} \quad t^* < t \le b,$$
and also
$$-(\int_a^x + \int_x^{t^*}) K^\circ(t;x)\,dt = \int_{t^*}^b K^\circ(t;x)\,dt.$$
Therefore
$$\begin{aligned} E\,|K^\circ(U;x)| &= \frac{1}{b-a}(-(\int_a^x + \int_x^{t^*}) K^\circ(t;x) + \int_{t^*}^b K^\circ(t;x)\,dt) \\ &= \frac{2}{b-a}\int_{t^*}^b K^\circ(t;x)\,dt = \frac{(b-B-w(x))^2}{b-a}. \end{aligned}$$

Case 3. If $a-A < w(x) \le x-A$ and $b-B \le w(x)$, then in a similar way we find that
$$|EK^\circ(U;x)| = \frac{(x-A-w(x))^2}{b-a}.$$

Case 4. If $a-A < w(x) \le x-A$ and $x-B > w(x)$, then we find that
$$E\,|K^\circ(U;x)| = \frac{(a-A-w(x))^2}{b-a}.$$

Case 5. If $a-A < w(x) \le x-A$ and $x-B < w(x) \le b-B$, then $K^\circ(t^*;x)=0$ where $t_1^* = A - w(x)$ and $t_2^* = B - w(x)$. It is clear that
$$-(\int_a^{t_1^*} + \int_x^{t_2^*}) K^\circ(t;x)\,dt = (\int_{t_1^*}^x + \int_{t_2^*}^b) K^\circ(t;x)\,dt.$$
So we get
$$E\,|K^\circ(U;x)| = \frac{(x-A-w(x))^2 + (b-B-w(x))^2}{b-a}.$$

Case 6. If $a-A > w(x)$ and $b-B < w(x)$, then
$$K^\circ(t;x) > 0 \qquad \text{for} \quad a \le t \le x,$$
and
$$K^\circ(t;x) \le 0 \qquad \text{for} \quad x < t \le b.$$

Therefore

$$
\begin{aligned}
E\,|K^{\circ}(U;x)| &= \frac{2}{b-a}\int_a^x K^{\circ}(t;x)\,dt \\
&= \frac{(x-A-w(x))^2-(a-A-w(x))^2}{b-a} = \frac{(x-a)(x+a-2A-2w(x))}{b-a}.
\end{aligned}
$$

Case 7. If $x-A<w(x)$ and $x-B>w(x)$, then

$$K^{\circ}(t;x)\le 0 \qquad \text{for} \quad a\le t\le x,$$

and

$$K^{\circ}(t;x)>0 \qquad \text{for} \quad x<t\le b.$$

So

$$
\begin{aligned}
E\,|K^{\circ}(U;x)| &= \frac{2}{b-a}\int_x^b K^{\circ}(t;x)\,dt \\
&= \frac{(b-B-w(x))^2-(x-B-w(x))^2}{b-a} = \frac{(b-x)(x+b-2B-2w(x))}{b-a}.
\end{aligned}
$$

The following table summarizes all above-mentioned cases.

TABLE 2. All conditions for evaluating $E\,|K^{\circ}(U;x)|$

case	condition 1	condition 2	$E\,\|K^{\circ}(U;x)\|$
1	$w<a-A$	$x-B<w<b-B$	(a) $=\frac{(x-B-w(x))^2}{b-a}$
2	$x-A<w$	$x-B<w<b-B$	(b) $=\frac{(b-B-w(x))^2}{b-a}$
3	$a-A<w<x-A$	$b-B<w$	(c) $=\frac{(x-A-w(x))^2}{b-a}$
4	$a-A<w<x-A$	$w<x-B$	(d) $=\frac{(a-A-w(x))^2}{(b-a)}$
5	$a-A<w<x-A$	$x-B<w<b-B$	(b) + (c)
6	$w<a-A$	$b-B<w$	(c) - (d)
7	$x-A<w$	$x-B>w$	(b) - (a)

4.4.3. *Some Special cases of the Table 2.*
1) If $A=a$ and $B=b$, then $w(x)=x-(a+b)/2$ and $E\,|K^{\circ}(U;x)|=(b-a)/4$. This case was treated by Cheng ([5], Theorem 1.5).

2) Ujevic ([33],[34]) considered the kernel $K(t,x)$ in (4.8) with $A=(2a+b)/3$,

$B = (a+2b)/3$ and $x = (a+b)/2$. In this case $A+B = a+b$ and $B-A = (b-a)/3$, which leads to $w(x) = 0$,

$$Var\,(K(U;x)) = \frac{1}{3}\sigma^2(U) = \frac{(b-a)^2}{36},$$

and

$$Cov(K(U;x), f'(U)) = \frac{1}{3}f(\frac{a+b}{2}) + \frac{f(b)+f(a)}{3} - Ef(U).$$

So we find that

$$\left|\frac{1}{3}f(\frac{a+b}{2}) + \frac{f(b)+f(a)}{3} - Ef(U)\right| \le \frac{b-a}{6}\sigma(f'(U)).$$

On the other hand, using Lemma 4.3 gives

$$\left|\frac{1}{3}f(\frac{a+b}{2}) + \frac{f(b)+f(a)}{3} - Ef(U)\right| \le \frac{M-m}{2}E\left|K^{\circ}(U;x)\right|,$$

where $K^{\circ} = K$. Therefore, by considering different cases of Table 2, only case 5 appears and we have

$$E\left|K^{\circ}(U;x)\right| = \frac{5(b-a)}{36}.$$

3) In [35] the author consideres the kernel $K(t;x)$ in (4.8) with $A = (5a+b)/6$, $B = (a+5b)/6$ and $x = (a+b)/2$. In this case $A+B = a+b$ and $B-A = 2(b-a)/3$, which respectively yields

$$w(x) = 0, \quad Var\,(K(U;x)) = \frac{(b-a)^2}{36},$$

and

$$\begin{aligned} Cov(K(U;x), f'(U)) &= \frac{2}{3}f(\frac{a+b}{2}) + (b-B)\frac{f(a)+f(b)}{b-a} - Ef(U) \\ &= \frac{2}{3}f(\frac{a+b}{2}) + \frac{f(a)+f(b)}{6} - Ef(U). \end{aligned}$$

So, by using (4.3) we find that

$$\left|\frac{2}{3}f(\frac{a+b}{2}) + \frac{f(a)+f(b)}{6} - Ef(U)\right| \le \frac{b-a}{6}\sigma(f'(U)). \tag{4.11}$$

On the other hand, using Lemma 4.3 gives

$$\left|\frac{2}{3}f(\frac{a+b}{2}) + \frac{f(a)+f(b)}{6} - Ef(U)\right| \le \frac{M-m}{2}E\left|K^{\circ}(U;x)\right|,$$

where $K^{\circ} = K$. Hence, only case 5 appears again and we obtain

$$E\left|K^{\circ}(U;x)\right| = \frac{(x-A)^2 + (b-B)^2}{b-a} = \frac{5(b-a)}{36}.$$

Let us consider a sharp example for (4.11). Take $[a,b] = [0,1]$ and define

$$f(t) = \begin{cases} \frac{1}{2}t^2 - \frac{1}{6}t & 0 \le t \le \frac{1}{2}, \\ \frac{1}{2}t^2 - \frac{5}{6}t + \frac{1}{3} & \frac{1}{2} < t \le 1. \end{cases}$$

Since $Ef(U) = 0$, $f(0) = f(1) = 0$ and $f(1/2) = 1/24$, we have

$$\left|\frac{2}{3}f(\frac{a+b}{2}) + \frac{f(a)+f(b)}{6} - Ef(U)\right| = \frac{1}{36}.$$

Also $Ef'(U) = 0$ and

$$\begin{aligned} Ef'^2(U) &= \int_0^{1/2} (t - 1/6)^2\, dt + \int_{1/2}^1 (t - 5/6)^2\, dt \\ &= \frac{(1/2 - 1/6)^3 - (-1/6)^3}{3} + \frac{(1 - 5/6)^3 - (1/2 - 5/6)^3}{3} = \frac{1}{36}. \end{aligned}$$

This means that

$$\left| \frac{2}{3} f(\frac{a+b}{2}) + \frac{f(a) + f(b)}{6} - Ef(U) \right| = \frac{b-a}{6} \sigma(f'(U)).$$

4) Consider $K(t; x)$ in (4.8) with $A = a + h(b-a)/2$ and $B = b - h(b-a)/2$ where $0 \le h \le 1$. In this case $A + B = a + b$ and $B - A = (1-h)(b-a)$, which yields

$$w(x) = (1-h)(x - \frac{a+b}{2}),$$

$$Var\,(K(U; x)) = \sigma^2(U) - h(1-h)(x-a)(b-x),$$

$$EK(U; x) f'(U) = (1-h) f(x) + h \frac{f(b) + f(a)}{2} - Ef(U),$$

and

$$Cov(K(U; x), f'(U)) = (1-h) f(x) + h \frac{f(b) + f(a)}{2} - Ef(U) - w(x) \frac{f(b) - f(a)}{b-a}.$$

Now suppose that $A \le x \le B$ where $a \le A$, $B \le b$ and $x > (a+b)/2$ so that $w(x) \ge 0$. Then we respectively obtain

$$\begin{aligned} a - A - w(x) &= h(x-b) - (x - (a+b)/2) < 0, \\ x - A - w(x) &= h(x-b) + (b-a)/2, \\ x - B - w(x) &= h(x-a) - (b-a)/2, \end{aligned}$$

and

$$b - B - w(x) = h(x-a) - (x - (a+b)/2.$$

As before, only cases 3, 4 or 5 of Table 2 are possible.

Corresponding to case 3, if $h(x-b) + (b-a)/2 > 0$ and $h(x-a) < x - (a+b)/2$, then

$$E\,|K^\circ(U; x)| = \frac{(h(x-b) - (b-a)/2))^2}{b-a}.$$

Corresponding to case 4, if $h(x-b) + (b-a)/2 > 0$ and $h(x-a) > (b-a)/2$, then

$$E\,|K^\circ(U; x)| = \frac{(h(x-b) - (x - (a+b)/2))^2}{b-a}.$$

Corresponding to case 5, if $h(x-b) + (b-a)/2 > 0$ and $(x - (a+b)/2 < h(x-a) < (b-a)/2$, then

$$E\,|K^\circ(U; x)| = \frac{(h(x-a) - (x - (a+b)/2))^2}{b-a} + \frac{(h(x-b) + (b-a)/2)^2}{b-a}.$$

Similarly, special cases include:
i) $h = 0$ which gives $A = a$, $B = b$ leading to the result of Ostrowski [29].
ii) $h = 1$ gives $A = B = (a + b)/2$ and $w(x) = 0$. So, for $x \geq (a + b)/2$ we obtain

$$E\left|K^{\circ}(U;x)\right| = \frac{(a - A - w(x))^2}{b - a} = \frac{b - a}{4}.$$

iii) $h = 1/2$ and $h = 1/3$ lead to new estimates.

5) In [15], $A = a + \theta(x - a)$ and $B = b + \theta(x - b)$ are considered where $0 \leq \theta \leq 1$. In this case we have

$$w(x) \equiv EK(U;x) = (1 - 2\theta)(x - \frac{a + b}{2}).$$

6) In ([5], Secion 2), $A = x + (a - b)/2$ and $B = x + (b - a)/2$ are considered. Hence $A + B = 2x$, $B - A = b - a$, $w(x) = 0$ and

$$Cov(K(U;x), f^{(1)}(U)) = f(x) - Ef(U) - (x - \frac{a + b}{2})Ef^{(1)}(U).$$

Again when $a \leq x \leq (a + b)/2$, we respectively find

$$\begin{aligned} a - A &= a - x - (a - b)/2 = (a + b)/2 - x \geq 0, \\ x - A &= (b - a)/2 > 0, \\ b - B &= b - x - (b - a)/2 = (a + b)/2 - x \geq 0, \end{aligned}$$

and

$$x - B = -(b - a)/2 < 0.$$

Only case 1 appears in these cases which yields $E\left|K^{\circ}(U;x)\right| = (b - a)/4$.

7) In ([5], section 3), $A = (x + a)/2$ and $B = (x + b)/2$ are considered where $a < x < b$. Hence $A + B = x + (a + b)/2$, $B - A = (b - a)/2$, $w(x) = 0$ and

$$Cov(K(U;x), f'(U)) = \frac{1}{2}f(x) + \frac{(b - x)f(b) - (a - x)f(a)}{2(b - a)} - Ef(U).$$

In this sense, Lemma 4.3 gives

$$\left|Cov(K(U;x), f'(U))\right| \leq \frac{M - m}{2}E\left|K(U;x)\right|.$$

By considering all 7 cases of Table 2, only case 5 appears and we get

$$E\left|K^{\circ}(U;x)\right| = \frac{(b - x)^2 + (x - a)^2}{4(b - a)}.$$

Remark. *Although the above-mentioned results are related to the uniform variable $U \sim U(a, b)$, one can consider them for a general random variable. Let X denote an arbitrary random variable defined on (a, b) whose distribution function is as $G(x) = P(X \leq x)$ and its density function as $g(x) = G^{(1)}(x)$. Inspired by* [18], *we now consider the kernel $K(t;x)$ as follows:*

$$K(t;x) = \begin{cases} \dfrac{G(t)}{g(t)} & t \leq x, \\[2ex] \dfrac{G(t) - 1}{g(t)} & t > x. \end{cases}$$

Since

$$EK(X;x)f^{(1)}(X) = f(x) - Ef(X),$$

and

$$\begin{aligned} EK(X;x) &= \int_a^x G(t)\,dt + \int_x^b (G(t)-1)\,dt \\ &= \int_{t=a}^x \int_{z=a}^t g(z)\,dz\,dt - \int_{t=x}^b \int_{z=t}^b g(z)\,dz\,dt \\ &= \int_{z=a}^x (x-z)g(z)\,dz - \int_{z=x}^b (z-x)g(z)\,dz = E(X-x), \end{aligned}$$

and in a similar way

$$E\left|K(X;x)\right| = \int_a^x G(t)\,dt + \int_x^b (1-G(t))\,dt = E\left|X-x\right|,$$

so we have

$$Cov(K(X;x), f^{(1)}(X)) = f(x) - Ef(X) - EK(X;x)Ef^{(1)}(X).$$

If $\left\|f^{(1)}\right\| < \infty$ *then*

$$\left|EK(X;x)f^{(1)}(X)\right| \leq \left\|f^{(1)}\right\| E\left|X-x\right|.$$

For estimating the covariance, inequalities of the previous section can be used.

Recall that

$$f(x) - Ef(X) = \int_a^b K(t;x)f^{(1)}(t)g(t)\,dt.$$

Now if $f^{(1)}(.)$ *is replaced by* $f^{(2)}(.)$, *we obtain*

$$EK(X;t)f^{(2)}(X) = f^{(1)}(t) - Ef^{(1)}(X),$$

or

$$f^{(1)}(t) = EK(X;t)f^{(2)}(X) + Ef^{(1)}(X).$$

So

$$\begin{aligned} f(x) - Ef(X) &= \int_a^b K(t;x)\left(EK(X;t)f^{(2)}(X) + Ef^{(1)}(X)\right)g(t)\,dt \\ &= \int_a^b K(t;x)EK(X;t)f^{(2)}(X)g(t)\,dt + Ef^{(1)}(X)EK(X;x). \end{aligned}$$

By rearranging terms we obtain

$$\begin{aligned} \left|f(x) - Ef(X) - EK(X;x)Ef^{(1)}(X)\right| &\leq \left\|f^{(2)}\right\| \int_a^b \left|K(t;x)\right| E\left|K(X;t)\right| g(t)\,dt \\ &\leq \left\|f^{(2)}\right\| \int_a^b \left|K(t;x)\right| E\left|X-t\right| g(t)\,dt. \end{aligned}$$

4.5. **Extensions for the kernel** $K^2(t;x)$**.** In this section, we use the same kernel as defined in (4.8) but study $EK^2(U;x)f'(U)$ and $EK^2(U;x)f''(U)$.

Lemma 4.9. *If the kernel* K *is defined as in* (4.8) *then we respectively have*

$$\begin{aligned}EK^2(U;x)f^{(1)}(U) &= 2\frac{B-A}{b-a}(x-\frac{A+B}{2})f(x)+\frac{(b-B)^2f(b)-(a-A)^2f(a)}{b-a}\\ &\quad -2EK(U;x)f(U).\\ EK^2(U;x)f^{(2)}(U) &= 2\frac{B-A}{b-a}(x-\frac{A+B}{2})f^{(1)}(x)+\frac{(b-B)^2f^{(1)}(b)-(a-A)^2f^{(1)}(a)}{b-a}\\ &\quad -2EK(U;x)f^{(1)}(U).\end{aligned}$$

Proof. Since $EK^2(U;x)f^{(1)}(U)=(I+II)/(b-a)$ where

$$I=\int_a^x (t-A)^2 f^{(1)}(t)\,dt,$$

and

$$II=\int_x^b (t-B)^2 f^{(1)}(t)\,dt,$$

using integration by parts gives

$$I=(x-A)^2f(x)-(a-A)^2f(a)-2\int_a^x (t-A)f(t)\,dt,$$

and

$$II=(b-B)^2f(b)-(x-B)^2f(x)-2\int_x^b (t-B)f(t)\,dt.$$

Therefore

$$\begin{aligned}I+II &= ((x-A)^2-(x-B)^2)f(x)+(b-B)^2f(b)-(a-A)^2f(a)\\ &\quad -2\int_a^b K(t;x)f(t)\,dt.\end{aligned}$$

The second result follows from the first one by replacing $f^{(1)}$ by $f^{(2)}$. □

Now let us consider $Cov(K^2(U;x)f^{(2)}(U))$. We first have

$$Cov(K^2(U;x)f^{(2)}(U))=EH^\circ(U;x)f^{(2)}(U),$$

where $H^\circ(t;x)=K^2(t;x)-w_2(x)$ and $w_2(x)=EK^2(U;x)$. To find $w_2(x)$, recall that

$$Var\,(K(U;x))=\sigma^2(U)+\frac{(B-A)(x-a)(b-x)}{b-a}(\frac{B-A}{b-a}-1).$$

So, by using the equality $Var\,(K(U;x))=w_2(x)-w_1^2(x)$ where

$$w_1(x)=EK(U;x)=\frac{a+b}{2}-B+\frac{B-A}{b-a}(x-a),$$

we can obtain

$$w_2(x)=\sigma^2(U)+\frac{(B-A)(x-a)(b-x)}{b-a}(\frac{B-A}{b-a}-1)+w_1^2(x).$$

For instance, if $A = a$ and $B = b$, we have

$$w_2(x) = (x - \frac{a+b}{2})^2 + \frac{(b-a)^2}{12}.$$

As before, we in this section study the values $H^\circ = K^2(t;x) - w_2(x)$ and $E\,|H^\circ(U;x)|$. First of all, note that

$$(t-A)^2 - w_2(x),$$

is a convex second degree polynomial with zero's $t_\pm = A \pm \sqrt{w_2(x)}$ and top in $t = A$ and

$$(t-B)^2 - w_2(x),$$

is convex with top in $t = B$ and zeros $z_{+,-} = B \pm \sqrt{w_2(x)}$.
Now one can consider several cases. First suppose that $A = a$ and $B = b$. Four following cases happen:

a1) If $a < t_+ < x$ then $H^\circ(t;x) \le 0$ for $a \le t \le t_+$ and $H^\circ(t;x) > 0$ for $t_+ \le t \le x$.
a2) If $x \le t_+$ then $H^\circ(t;x) \le 0$ for $a \le t \le x$.
b1) If $x < z_- < b$ then $H^\circ(t;x) > 0$ for $x < t \le z_-$ and $H^\circ(t;x) \le 0$ for $z_- \le t \le b$.
b2) If $z_- \le x$ then $H^\circ(t;x) \le 0$ for $x < t \le b$.

Consequently:

1) If a1) and b1) hold, then we have

$$\begin{aligned}
E\,|H^\circ(U;x)| &= \frac{1}{b-a}((\int_{t_+}^{x} + \int_{x}^{z_-})H^\circ(t;x)\,dt - (\int_{a}^{t_+} + \int_{z_-}^{b})H^\circ(t;x)\,dt) \\
&= \frac{2}{b-a}(\int_{t_+}^{x} + \int_{x}^{z_-})H^\circ(t;x)\,dt \\
&= \frac{2}{b-a}(\frac{(t-a)^3}{3} - w_2(x)\ |_{t_+}^{x} + \frac{(t-b)^3}{3} - w_2(x)\ |_{x}^{z_-}) \\
&= \frac{2}{b-a}(\frac{(x-a)^3-(x-b)^3}{3} - \frac{2(\sqrt{w_2(x)})^3}{3} - w_2(x)(x - t_+ + z_- - x) \\
&= \frac{2}{b-a}(\frac{(x-a)^3-(x-b)^3}{3} - \frac{2(\sqrt{w_2(x)})^3}{3} - w_2(x)(b - a - 2\sqrt{w_2(x)}) \\
&= \frac{2}{b-a}(\frac{(x-a)^3-(x-b)^3}{3} + \frac{4(\sqrt{w_2(x)})^3}{3} - w_2(x)(b-a)).
\end{aligned}$$

2) If a1) and b2) hold, then

$$\begin{aligned}
E\,|H^\circ(U;x)| &= \frac{2}{b-a}\int_{t_+}^{x} H^\circ(t;x)\,dt = \frac{2}{b-a}(\frac{(t-a)^3}{3} - w_2(x)\ |_{t_+}^{x}) \\
&= \frac{2}{b-a}(\frac{(x-a)^3 - w_2^{3/2}(x)}{3} - w_2(x)(x - a - \sqrt{w_2(x)}) \\
&= \frac{2}{b-a}(\frac{(x-a)^3}{3} + \frac{2w_2^{3/2}(x)}{3} - w_2(x)(x-a)).
\end{aligned}$$

3) If a2) and b1) hold, then

$$\begin{aligned} E\,|H^{\circ}(U;x)| &= \frac{2}{b-a}\int_{x}^{z_-} H^{\circ}(t;x)\,dt = \frac{2}{b-a}\Big(\frac{(t-b)^3}{3} - w_2(x)\ \Big|_{x}^{z_-}\Big) \\ &= \frac{2}{b-a}\Big(\frac{2w_2^{3/2}(x)-(x-b)^3}{3} - w_2(x)(b-x)\Big). \end{aligned}$$

4) If a2) and b2) hold, then the equality $EH^{\circ}(U;x)=0$ cannot hold.

Note that a1) holds if and only if $\sqrt{w_2(x)} \le x-a$, or $w_2(x) \le (x-a)^2$, i.e. by using the expression $w_2(x)$ it holds if

$$\Big(x-\frac{a+b}{2}\Big)^2 + \frac{(b-a)^2}{12} \le (x-a)^2,$$

or

$$\frac{(b-a)^2}{12} \le \frac{b-a}{2}\Big(2x-\frac{3a+b}{2}\Big),$$

which gives $(b+2a)/3 \le x$.

Similarly a2) holds if $x \le a+\sqrt{w_2(x)}$ which leads to $x \le (b+2a)/3$.

The special case $A=a$, $B=b$ and $x=(a+b)/2$ has been considered in [33] so that

$$w_2(x) = EK^2(U;x) = \frac{(b-a)^2}{12},$$

and $t_{+,-} = a \pm (b-a)/2\sqrt{3}$ and finally $z_{+,-} = a \pm (b-a)/2\sqrt{3}$.
Moreover, $a < t_+ < x$ and $x < z_- \le b$ imply that

$$E\,|H^{\circ}(U;x)| = \frac{2}{b-a}\Big(\frac{(b-a)^3}{12} + \frac{4((b-a)/2\sqrt{3})^3}{3} - \frac{(b-a)^3}{12}\Big) = \frac{(b-a)^2}{9\sqrt{3}}.$$

In general, more possibilities for A and B are as follows:
a) If $A < a$ and

$$\begin{cases} a1)\ A < t_+ \le a. \text{ In this case } H^{\circ}(t;x) \ge 0 \text{ for } a \le t \le x. \\ a2)\ a < t_+ = A + \sqrt{w(x)} < x\,, \\ \quad \text{then } H^{\circ}(x;t) \le 0 \text{ for } a \le t < t_+ \text{ and } H^{\circ}(x;t) > 0 \text{ for } t_+ < t \le x. \\ a3)\ x < t_+, \text{ then } H^{\circ}(x;t) \le 0 \text{ for } a \le t < x. \end{cases}$$

b) If $a < A < x$ and

$$\begin{cases} b1)\ t_- < a \text{ and } t_+ > x, \text{ then } H^{\circ}(t;x) < 0 \text{ for } a \le t \le x. \\ b2)\ t_- < a \text{ and } t_+ < x, \\ \quad \text{then } H^{\circ}(t;x) \le 0 \text{ for } a \le t \le t_+ \text{ and } H^{\circ}(t;x) > 0 \text{ for } t_+ \le t \le x. \\ b3)\ a < t_- \text{and } t_+ > x, \\ \quad \text{then } H^{\circ}(t;x) \le 0 \text{ for } t_- \le t \le x \text{ and } H^{\circ}(t;x) > 0 \text{ for } a \le t \le t_-. \\ b4)\ a < t_- \text{ and } t_+ < x, \\ \quad \text{then } H^{\circ}(t;x) \ge 0 \text{ for } a \le t \le t_- \text{ or } t_+ \le t \le x \text{ and } H^{\circ}(t;x) \le 0 \text{ for } t_- \le t \le t_+. \end{cases}$$

c) If $A = x$ and

$$\begin{cases} c1)\, t_- \le a, \text{then}\, H^\circ(t;x) < 0 \text{ for}\, a \le t \le x. \\ c2)\, a < t_- < x, \\ \quad \text{then}\, H^\circ(t;x) \ge 0 \text{ for}\, a \le t \le t_- \text{ and}\, H^\circ(t;x) \le 0 \text{ for}\, t_- \le t \le x. \end{cases}$$

d) If $A > x$ and

$$\begin{cases} d1)\, t_- \le a, \text{then}\, H^\circ(t;x) < 0 \text{ for}\, a \le t \le x. \\ d2)\, a < t_- < x, \\ \quad \text{then}\, H^\circ(t;x) \ge 0 \text{ for}\, a \le t \le t_- \text{ and}\, H^\circ(t;x) \le 0 \text{ for}\, t_- \le t \le x. \\ d3)\, x < t_-, \text{then}\, H^\circ(t;x) > 0 \text{ for}\, a \le t \le x. \end{cases}$$

Clearly a similar list can be made by considering the cases for the position of B. One can now use the results of previous sections to obtain upperbounds for $Cov(K^2(U;x), f''(U))$.

Final Remark. We can similarly extend the problem and obtain

$$EK^m(U;x)f^{(m)}(U) = I + II - mEK^{m-1}(U;x)f^{(m-1)}(U),$$

where

$$I = \frac{f^{(m-1)}(x)(x-A)^m - f^{(m-1)}(a)(a-A)^m}{b-a},$$

and

$$II = \frac{f^{(m-1)}(b)(b-B)^m - f^{(m-1)}(x)(x-B)^m}{b-a},$$

in order to study $Cov(K^m(U;x), f^{(m)}(U))$.

References

[1] S.A. Azar, Jensen's Inequality in Finance, *Int. Adv. Econ. Res.* **14** (2008) 433-440.

[2] R.A. Becker, The Variance Drain and Jensen's Inequality. CAEPR Working paper #2012 − 004. Indiana University.

[3] P. Cerone, S.S. Dragomir, J.Roumeliotis, An inequality of Ostrowski type for mappings whose second derivative is bounded and applications, *East Asian Math. J.* **15(1)** (1999) 1-9.

[4] P. Cerone, S.S. Dragomir, E. Kikianty, On inequalities of Jensen-Ostrowski type, *J. Inequal. Appl.* (2015) 328. DOI 10.1186/s13660-015-0849-3.

[5] X.L. Cheng, Improvement of some Ostrowski-Grüss Type Inequality, *Comput. Math. Appl.* **42** (2001) 109-114.

[6] X.L. Cheng and J. Sun, A note on the perturbed trapezoid inequality, *J. Inequal. Pure Appl. Math.* **3(2)** (2002) Article 29.

[7] C.M. Cuadras, On the covariance between functions, *J. Multivariate Anal.* **81(1)** (2002) 19-27.

[8] S.S. Dragomir, S. Wang, An inequality of Ostrowski-Grüss type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules, *Comput. Math. Appl.* **33(11)** (1997) 15-20.

[9] M. Egozcue, L.F. Garcia, W-K Wong and R. Zitikis, Grüss-type bounds for the covariance of transformed random variables, *J. Inequal. Appl.* **10** (2010) article ID 619423.

[10] G. Grüss, Uber das Maximum des absoluten Betrages von

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \frac{1}{(b-a)^2}\int_a^b f(x)\,dx \int_a^b g(x)\,dx.$$

Math. Z. **39** (1935) 215-226.

[11] W. Hoeffding, "Masstabinvariante Korrelationstheorie" in Schriften des Matematischen Instituts für Anfewandte Matematik der Universität Berlin, **5** (1940) 179-233.

[12] S. Hussain and A. Qayyum, A generalized Ostrowski-Grüss type inequality for bounded differentiable mappings and its applications, *J. Inequal. Appl.* **1** (2013).
[13] J.L.W. V. Jensens, Sur les fonctions convexes et les inégalités entre les valeur moyennes, *Acta Math.* **30** (1906) 175-193.
[14] X. Li, N. Mohapatra and R.S. Rodriguez, Grüss-Type inequalities, *J. Math. Anal. Appl.* **267** (2002) 434-443.
[15] Z. Liu, A sharp generalilzed Ostrowski-Grüss inequality, *Tamsui Oxford J. of Math. Sc.* **24**(**2**) (2008) 175-184.
[16] Z. Liu, Notes on two inequalities for random variables whose probability density functions are bounded, *Tamsui Oxford J. of Math. Sciences.* **26**(**3**) (2010) 265-727.
[17] Z. Liu, A variant of Chebyshev inequality with applications, *J. Math. Inequal.* **7**(**4**) (2013) 551-561.
[18] Z. Liu, Some weighted Ostrowski type inequalities, *Vietnam J. of Math.* **41**(**3**) (2014).
[19] M. Matic, J. Pecaric and N. Ujevic, Improvement and further generalisation of inequalities of Ostrowski-Grüss type, *Comput. Math. Appl.* **39** (2000) 161-175.
[20] M. Matic, Improvement of some inequalities of Euler-Grüss type, *Comput. Math. Appl.* **46** (2003) 1325-1336.
[21] M. Masjed-Jamei, S.S. Dragomir, A Generalization of the Ostrowski-Grüss Inequality, *Anal. Appl.* **12** (2014) 117-130.
[22] M. Masjed-Jamei, S.S. Dragomir, A new generalization of the Ostrowski inequality and applications, *Filomat*, **25** : **1** (2011) 115-123.
[23] M. Masjed-Jamei, S.S. Dragomir, An analogue of the Ostrowski Inequality and applications, *Filomat*, **28** : **2** (2014) 373-381.
[24] M. Masjed-Jamei, E. Omey and S.S. Dragomir, A main class of integral inequalities with applications, *Math. Modelling. Anal.* **21** (2016) 569-584.
[25] M. Masjed-Jamei, A certain class of weighted approximations for integrable functions and applications, *Numer. Funct. Anal. Optim.* **34** (2013) 1224-1244.
[26] N. Minculete and L. Ciurdariu, A generalized form of Grüss type inequality and other integral inequalities, *J. Inequal. Appl.* (2014) paper 119.
[27] N. Minculete, A. Ratiu and J. Pecaric, A refinement of Grüss inequaltiy via Cauchy-Schwarz's inequality for discrete random variable, *Appl. Math. Inf. Sci.* **9**(**1**) (2015) 39-45.
[28] C.P. Niculescu, An extension of Chebyshev's inequality and its connection with Jensen's inequality, *J. Inequal. Appl.* **6** (2001) 451-462.
[29] A. Ostrowski, Uber die absolutabweichung einer differentiebaren funktion van ihrem integralmittelwert, *Comment Math. Helv.* **10** (1938) 226-227.
[30] T. Popoviciu, "Sur les équations algébriques ayant toutes leurs racines réelles", *Mathematica (Cluj).* **9** (1935) 129-145.
[31] S. Simic, On a global upperbound for Jensen's inequality, *J. Math. Anal. Appl.* **343** (2008) 414-419.
[32] M.S. Slater, A companion inequality to Jensen's inequality, *J. Approx. Theory.* **32** (1981) 160-166.
[33] N. Ujevic, Perturbed trapezoid and mid-point inequalities and applications, *Soochov J. of Math.* **29**(**3**) (2003a) 249-257.
[34] N. Ujevic, New bounds for the first inequality of Ostrowski-Grüss type and applications, *Comput. Math. Appl.* **46** (2003b) 421-427.
[35] N. Ujevic, Sharp inequalities of Simpson type and Ostrowski type, *Comput. Math. Appl.* **48** (2004) 145-151.
[36] R. Zitikis, Grüss's inequality, its probabilistic interpretation and a sharper bound, *J. Math. Inequal.* **3**(**1**) (2009) 15-20.
[37] S. Zlobec, Jensen's inequality for nonconvex functions, *Math. Commun.* **9** (2004) 119-124.

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