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NEW TYPE OF RESULTS INVOLVING CLOSED BALL WITH GRAPHIC CONTRACTION

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ABSTRACT. The aim of this paper is to establish Ćirić type fixed point results for multi-valued mappings satisfying generalized α - ψ -contractive conditions on closed ball in complete metric space. As an application, we derive some new fixed point theorems for Ćirić type ψ -graphic contractions and rational ψ graphic contractions defined on metric space endowed with a graph as well as ordered metric space. Examples has been constructed to demonstrate the novelty of our results. Our results unify, extend and generalize several comparable results in the existing literature.

1. INTRODUCTION

In metric fixed point theory the contractive conditions on underlying functions play an important role for finding solution of fixed point problems. Banach contraction principle [11] is a fundamental result in metric fixed point theory. In 1969, Nadler [25], introduced a study of fixed point theorems involving multivalued mappings and proved that every multivalued contraction on a complete metric space has a fixed point.

In 2012, Samet et al. [30] introduced the concepts of α - ψ -contractive and α admissible mappings and established various fixed point theorems for such mappings in complete metric spaces. Afterwards, Hussain et al. [17], generalized the concept of α -admissible mappings and proved fixed point theorems. Subsequently, Abdeljawad [1] introduced a pair of α -admissible mappings satisfying new sufficient contractive conditions different from those in [17, 30], and obtained fixed point and common fixed point theorems. Salimi et al. [29], modified the concept of $\alpha - \psi$ contractive mappings and established fixed point results. Recently, Hussain et al. [18] proved some fixed point results for single and set-valued $\alpha - \eta - \psi$ -contractive mappings in the setting of complete metric space. Mohammadi et al. [24], introduced a new notion of $\alpha - \phi$ -contractive mappings and show that this is a real generalization for some old results. Then a lot of generalization of multivalued mappings has been given in the literature. Over the years, it has been generalized in different directions by several mathematicians(see [1]-[30]).

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Denote with Ψ the family of nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$ for all t > 0, where ψ^n is the *n*-th iterate of ψ . The following lemma is well known.

Lemma 1.1. [16] If $\psi \in \Psi$, then the following hold:

- (i) $(\psi^n(t))_{n \in \mathbb{N}}$ converges to 0 as $n \to \infty$ for all $t \in (0, +\infty)$
- (ii) $\psi(t) < t$ for all t > 0
- (iii) $\psi(t) = 0$ iff t = 0.
- Samet et al. [30] defined the notion of α -admissible mappings as follows:

Definition 1.2. [30]. Let (X, d) be a metric space and $T : X \to X$ be a given mapping. We say that T is an $\alpha - \psi -$ contractive mapping if there exist two functions $\alpha : X \times X \to [0, +\infty)$ and $\psi \in \Psi$ such that

$$\alpha(x,y)d(Tx,Ty) \le \psi(d(x,y)),$$

for all $x, y \in X$.

Definition 1.3. [30]. Let $T: X \to X$ and $\alpha: X \times X \to [0, +\infty)$. We say that T is α -admissible if $x, y \in X$, $\alpha(x, y) \ge 1 \Rightarrow \alpha(Tx, Ty) \ge 1$.

Theorem 1.4. [30]. Let (X, d) be a complete metric space and T be α -admissible mapping. Assume that

$$\alpha(x, y)d(Tx, Ty) \le \psi(d(x, y)) \tag{1.1}$$

for all $x, y \in X$, where $\psi \in \Psi$. Also, suppose that;

- (i) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$;
- (ii) either T is continuous or for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$, we have $\alpha(x_n, x) \ge 1$ for all $n \in \mathbb{N}$.

Then T has a fixed point.

Ćirić [13], introduced quasi contraction, which is one of the most general contraction conditions.

Definition 1.5. Let (X, d) be a metric space and $T : X \to X$ be a given mapping. We say that T is generalized $\alpha - \psi -$ contractive mapping if there exist two functions $\alpha : X \times X \to [0, +\infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y)d(Tx, Ty) \le \psi(M(x, y)),$$

where

$$M(x,y) = \max\left\{ d(x,y), d(x,Tx), d(y,Ty), \frac{d(y,Tx) + d(x,Ty)}{2} \right\}$$

for all $x, y \in X$.

Definition 1.6. Let (X, d) be a metric space and $T : X \to X$ be a given mapping. We say that T is generalized $\alpha - \psi -$ contractive mapping if there exist two functions $\alpha : X \times X \to [0, +\infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y)d(Tx, Ty) \le \psi(N(x, y)),$$

where

$$N(x,y) = \max\left\{ d(x,y), \frac{d(x,Tx) + d(y,Ty)}{2}, \frac{d(y,Tx) + d(x,Ty)}{2} \right\}$$

for all $x, y \in X$.

Afterwards, Asl et al. [27] generalized these notions by introducing the concepts of $\alpha_* - \psi$ contractive multifunctions, α_* -admissible and obtained some fixed point results for these multifunctions.

Definition 1.7. [27]. Let (X, d) be a metric space, $T: X \to 2^X$ be a given closedvalued multifunction. We say that T is called $\alpha_* - \psi$ - contractive multifunction if there exists two functions $\alpha: X \times X \to [0, +\infty)$ and $\psi \in \Psi$ such that

$$\alpha_*(Tx, Ty)H(Tx, Ty) \le \psi(d(x, y))$$

for all $x, y \in X$, where H is the Hausdorff generalized metric, $\alpha_*(A, B) = \inf\{\alpha(a, b) : a \in A, b \in B\}$ and 2^X denotes the family of all nonempty subsets of X.

Definition 1.8. [27]. Let (X, d) be a metric space, $T : X \to 2^X$ be a given closed-valued multifunction and $\alpha : X \times X \to [0, +\infty)$. We say that T is called α_* -admissible whenever $\alpha(x, y) \ge 1$ implies that $\alpha_*(Tx, Ty) \ge 1$.

Very recently Hussain et al. [18] modified the notions of α_* -admissible and α_* - ψ -contractive mappings as follows;

Definition 1.9. [16] Let $T: X \to 2^X$ be a multifunction, α , $\eta: X \times X \to \mathbb{R}_+$ be two functions where η is bounded. We say that T is α_* -admissible mapping with respect to η if

$$\alpha(x,y) \ge \eta(x,y)$$
 implies $\alpha_*(Tx,Ty) \ge \eta_*(Tx,Ty), x, y \in X$

where

$$\alpha_*(A,B) = \inf_{x \in A, \ y \in B} \alpha(x,y) \ and \ \eta_*(A,B) = \sup_{x \in A, \ y \in B} \eta(x,y).$$

If, $\eta(x, y) = 1$ for all $x, y \in X$, then this definition reduces to Definition 3. In case $\alpha(x, y) = 1$ for all $x, y \in X$, then T is called η_* -subadmissible mapping.

Hussain et al. [18] proved following generalization of the above mentioned results of [27].

Theorem 1.10. [16] Let (X, d) be a complete metric space and $T : X \to 2^X$ be a α_* -admissible with respect to η and closed valued multifunction on X. Assume that for $\psi \in \Psi$,

$$\forall x, y \in X, \ \alpha_*(Tx, Ty) \ge \eta_*(Tx, Ty) \Longrightarrow H(Tx, Ty) \le \psi(d(x, y)) \tag{1.2}$$

Also suppose that the following assertions holds:

- (i) there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \ge \eta(x_0, x_1)$;
- (ii) for a sequence $\{x_n\} \subset X$ converging to $x \in X$ and $\alpha(x_n, x_{n+1}) \ge \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, we have $\alpha(x_n, x) \ge \eta(x_n, x)$ for all $n \in \mathbb{N}$.

Then T has a fixed point.

Let (X,d) be a complete metric space, $x_0 \in X$ and r > 0. We denote by $B(x_0,r) = \{x \in X : d(x_0,x) < r\}$ the open ball with center x_0 and radius r and by $\overline{B(x_0,r)} = \{x \in X : d(x_0,x) \leq r\}$ the closed ball with center x_0 and radius r.

The following lemmas of Nadler will be needed in the sequel.

Lemma 1.11. [25] Let A and B be nonempty, closed and bounded subsets of a metric space (X,d) and $0 < h \in \mathbb{R}$. Then, for every $b \in B$, there exists $a \in A$ such that $d(a,b) \leq H(A,B) + h$.

Lemma 1.12. [4] Let (X, d) be a metric space and B be nonempty, closed subsets of X. and q > 1. Then, for each $x \in X$ with d(x, B) > 0 and q > 1, there exists $b \in B$ such that d(x, b) < qd(x, B).

2. Main Result

In this section, we prove some fixed point results for multi-valued mappings satisfying Ćirić type generalized α - ψ -contractive condition on the closed ball, is very useful in the sense that it requires the contractiveness of the mapping only on the closed ball instead of the whole space.

Theorem 2.1. Let (X,d) be a complete metric space and $T: X \to 2^X$ be an α_* -admissible closed valued multifunction on X. Assume that for $\psi \in \Psi$, such that

$$\alpha_*(Tx, Ty)H(Tx, Ty) \le \psi(M(x, y)) \tag{2.1}$$

where

$$M(x,y) = \max\left\{ d(x,y), d(x,Tx), d(y,Ty), \frac{d(y,Tx) + d(x,Ty)}{2} \right\},\$$

for all $x, y \in \overline{B(x_0, r)}$ and $x_0 \in X$, there exists $x_1 \in Tx_0$ such that

$$\sum_{i=0}^{n} \psi^{i}(d(x_{0}, x_{1})) < r$$
(2.2)

for all $n \in \mathbb{N}$ and r > 0. Also suppose that the following assertions hold:

(i) $\alpha(x_0, x_1) \ge 1$ for $x_0 \in X$ and $x_1 \in Tx_0$;

(ii) T is continuous or for any sequence $\{x_n\}$ in $\overline{B(x_0, r)}$ converging to $x \in \overline{B(x_0, r)}$ and $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N}$, we have $\alpha(x_n, x) \ge 1$ for all $n \in \mathbb{N}$. Then T has a fixed point.

Proof. Since $\alpha(x_0, x_1) \ge 1$ and T is α_* -admissible, so $\alpha_*(Tx_0, Tx_1) \ge 1$. From (2.2), we get

$$d(x_0, x_1) < \sum_{i=0}^n \psi^i(d(x_0, x_1)) < r.$$

It follows that,

$$x_1 \in \overline{B(x_0, r)}.$$

If $x_0 = x_1$, then finished. Assume that $x_0 \neq x_1$. By Lemmas 1 and 11, we take $x_2 \in Tx_1$ and h > 0 as $h = \psi^2(d(x_0, x_1))$. Then

$$\begin{array}{rcl} 0 & < & d(x_1, x_2) \leq H(Tx_0, Tx_1) + h \\ & \leq & \psi(M(x_0, x_1)) + \psi^2(d(x_0, x_1)). \end{array}$$

Now

$$M(x_0, x_1) = \max\left\{ d(x_0, x_1), d(x_0, Tx_0), d(x_1, Tx_1), \frac{d(x_1, Tx_0) + d(x_0, Tx_1)}{2} \right\}$$

=
$$\max\left\{ d(x_0, x_1), d(x_1, Tx_1), \frac{d(x_0, Tx_1)}{2} \right\}$$

$$\leq \max\left\{ d(x_0, x_1), d(x_1, Tx_1), \frac{d(x_0, x_1) + d(x_1, Tx_1)}{2} \right\}$$

$$\leq \max\left\{ d(x_0, x_1), d(x_1, Tx_1) \right\}$$

If $M(x_0, x_1) = d(x_1, Tx_1)$ then,

$$\begin{array}{rcl} 0 & < & d(x_1, Tx_1) \leq H(Tx_0, Tx_1) + h \\ & \leq & \psi(d(x_1, Tx_1)) + \psi^2(d(x_0, x_1)) \\ & < & d(x_1, Tx_1), \end{array}$$

which is contradiction. Therefore $M(x_0, x_1) = d(x_0, x_1)$, we have

$$0 < d(x_1, x_2) \le H(Tx_0, Tx_1) + h$$

$$\le \psi(d(x_0, x_1)) + \psi^2(d(x_0, x_1))$$

$$= \sum_{i=1}^2 \psi^i(d(x_0, x_1)) < r.$$

Note that $x_2 \in \overline{B(x_0, r)}$, since

$$\begin{aligned} d(x_0, x_2) &\leq d(x_0, x_1) + d(x_1, x_2) \\ &\leq d(x_0, x_1) + \psi(d(x_0, x_1)) + \psi^2(d(x_0, x_1)) \\ &= \sum_{i=0}^2 \psi^i(d(x_0, x_1)) < r. \end{aligned}$$

By repeating this process, we can construct a sequence x_n of points in $\overline{B(x_0, r)}$ such that $x_{n+1} \in Tx_n, x_n \neq x_{n-1}, \alpha(x_n, x_{n+1}) \geq 1$ with

$$d(x_n, x_{n+1}) \le \sum_{i=1}^{n+1} \psi^i(d(x_0, x_1)).$$
(2.3)

Now for each $n, m \in N$ with m > n using the triangular inequality, we obtain

$$d(x_n, x_m) \le \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \le \sum_{k=n}^m \psi^k(d(x_0, x_1)).$$
(2.4)

Thus we proved that $\{x_n\}$ is a Cauchy sequence. Since $\overline{B(x_0, r)}$ is closed. So there exists $x^* \in \overline{B(x_0, r)}$ such that $x_n \to x^*$ as $n \to \infty$. Now we prove that $x^* \in Tx^*$. Since T is continuous, then

$$d(x^*, Tx^*) \le \lim_{n \to \infty} d(x_{n+1}, Tx^*) \le \lim_{n \to \infty} H(Tx_n, Tx^*) = 0$$
(2.5)

Thus $x^* \in Tx^*$. On the other hand, assume that $\alpha(x_n, x^*) \ge 1$ for all n and T is α_* -admissible, so $\alpha_*(Tx_n, Tx^*) \ge 1$ for all n. Then

$$d(x^*, Tx^*) \leq \alpha_*(Tx_n, Tx^*) H(Tx_n, Tx^*) + d(x_n, x^*) \\ \leq \psi(M(x_n, x^*)) + d(x_n, x^*) \\ \leq \psi(d(x^*, Tx^*)) + d(x_n, x^*).$$

For sufficiently large n. Hence $d(x^*, Tx^*) = 0$. Thus $x^* \in Tx^*$.

Corollary 2.2. Let (X,d) be a complete metric space and $T: X \to 2^X$ be an α_* -admissible and closed valued multifunction on X. Assume that for $\psi \in \Psi$,

$$\alpha_*(Tx, Ty)H(Tx, Ty) \le \psi(N(x, y)) \tag{2.6}$$

where

$$N(x,y) = \max\left\{d(x,y), \frac{d(x,Tx) + d(y,Ty)}{2}, \frac{d(y,Tx) + d(x,Ty)}{2}\right\},\$$

for all $x, y \in \overline{B(x_0, r)}$ and for $x_0 \in X$, there exists $x_1 \in Tx_0$ such that

$$\sum_{i=0}^{n} \psi^{i}(d(x_{0}, x_{1})) < r$$

for all $n \in \mathbb{N}$ and and r > 0. Also suppose that the following assertions hold:

(i) $\alpha(x_0, x_1) \ge 1$ for $x_0 \in X$ and $x_1 \in Tx_0$;

(ii) T is continuous or for any sequence $\{x_n\}$ in $\overline{B(x_0, r)}$ converging to $x \in \overline{B(x_0, r)}$ and $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N}$, we have $\alpha(x_n, x) \ge 1$ for all $n \in \mathbb{N}$. Then T has a fixed point.

Example 2.3. Let $X = [0, \infty)$ and d(x, y) = |x - y|. Define the multi-valued mapping $T: X \to 2^X$ by

$$Tx = \begin{cases} [0, \frac{x}{2}] & \text{if } x \in [0, 1] \\ [x, x+1] & \text{if } x \in (1, \infty) \end{cases}$$

Considering, $x_0 = \frac{1}{3}$ and $x_1 = \frac{1}{6}$, $r = \frac{1}{3}$, then $\overline{B(x_0, r)} = [0, 1]$ and
 $\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [0, 1] \\ \frac{3}{2} & \text{otherwise.} \end{cases}$

Clearly T is an α - ψ -contractive mapping with $\psi(t) = \frac{t}{3}$. Now

$$d(x_0, x_1) = \frac{1}{6}$$
$$\sum_{i=1}^{n} \psi^n(d(x_0, x_1)) = \frac{1}{6} \sum_{i=0}^{n} \frac{1}{3^n} < 2(\frac{1}{6}) = \frac{1}{3} = r.$$

We prove that conditions of our Theorem 2.1 are satisfied only for $x, y \in \overline{B(x_0, r)}$. Without loss of generality, we suppose that $x \leq y$. We suppose that x < y. Then

$$\alpha_*(Tx, Ty)H(Tx, Ty) = \frac{1}{3}|y - x| = \psi(d(x, y)) \le \psi(M(x, y)).$$

Put $x_0 = \frac{1}{2}$ and $x_1 = \frac{1}{6}$. Then $\alpha(x_0, x_1) \ge 1$. Then T has a fixed point 0.

Now we prove that the contractive condition is not satisfied for $x, y \notin \overline{B(x_0, r)}$. We suppose x = 2 and y = 3, then

$$\alpha_*(Tx,Ty)H(Tx,Ty) = \frac{3}{2} \ge \frac{1}{3} = \psi(M(x,y)).$$

Similarly we can deduce the following Corollaries.

Corollary 2.4. [16] Let (X, d) be a complete metric space and $T: X \to 2^X$ be an α_* -admissible and closed valued multifunction on X. Assume that for $\psi \in \Psi$,

$$\alpha_*(Tx,Ty)H(Tx,Ty) \le \psi(d(x,y))$$

for all $x, y \in \overline{B(x_0, r)}$ and for $x_0 \in X$, there exists $x_1 \in Tx_0$ such that

$$\sum_{i=0}^{n} \psi^{i}(d(x_{0}, x_{1})) < r$$

for all $n \in \mathbb{N}$ and and r > 0. Also suppose that the following assertions hold:

(i) $\alpha(x_0, x_1) \ge 1$ for $x_0 \in X$ and $x_1 \in Tx_0$;

(ii) for any sequence $\{x_n\}$ in $B(x_0, r)$ converging to $x \in B(x_0, r)$ and $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N}$, we have $\alpha(x_n, x) \ge 1$ for all $n \in \mathbb{N}$.

Then T has a fixed point.

Theorem 2.5. Let (X,d) be a complete metric space and $T: X \to 2^X$ be an α_* -admissible and closed valued multifunction on X. Assume that for $\psi \in \Psi$, we have

$$\alpha_*(Tx, Ty)H(Tx, Ty) \le \psi(Q(x, y)), \tag{2.7}$$

where

$$Q(x,y) = \max\{d(x,y), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)}\}$$

for all $x, y \in X$. Also suppose that the following assertions hold:

(i) there exists $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0, x_1) \ge 1$;

(ii) T is continuous or for any sequence $\{x_n\}$ in $\overline{B(x_0, r)}$ converging to $x \in \overline{B(x_0, r)}$ and $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N}$, we have $\alpha(x_n, x) \ge 1$ for all $n \in \mathbb{N}$. Then T has a fixed point.

Proof. Since $\alpha(x_0, x_1) \geq 1$ and T is α_* -admissible, so $\alpha_*(Tx_0, Tx_1) \geq 1$. If $x_0 = x_1$, then we have nothing to prove. Let $x_0 \neq x_1$. If $x_1 \in Tx_1$, then x_1 is fixed point of T. Assume that $x_1 \notin Tx_1$, then from (2.7), we get

$$0 < d(x_1, Tx_1) \le H(Tx_0, Tx_1)$$

$$\le \psi(\max\{d(x_0, x_1), d(x_1, Tx_1), \frac{d(x_0, Tx_0)d(x_1, Tx_1)}{1 + d(x_0, x_1)}, \frac{d(x_0, Tx_0)d(x_1, Tx_1)}{1 + d(Tx_0, Tx_1)}\})$$

 $\leq \psi(\max\{d(x_0, x_1), d(x_1, Tx_1)\}).$

If $\max\{d(x_1, Tx_1), d(x_0, x_1)\} = d(x_1, Tx_1)$, then $d(x_1, Tx_1) \le \psi(d(x_1, Tx_1))$. Since $\psi(t) < t$ for all t > 0. Then we get a contradiction. Hence, we obtain $\max\{d(x_1, Tx_1), d(x_0, x_1)\} = d(x_0, x_1)$. So

$$d(x_1, Tx_1) \le \psi(d(x_0, x_1)).$$

Let q > 1, then from Lemma 1.12 we take $x_2 \in Tx_1$ such that

$$d(x_1, x_2) < qd(x_1, Tx_1) \le q\psi(d(x_0, x_1)).$$
(2.8)

It is clear that $x_1 \neq x_2$. Put $q_1 = \frac{\psi(q\psi(d(x_0,x_1)))}{\psi(d(x_1,x_2))}$. Then $q_1 > 1$ and $\alpha(x_1,x_2) \geq 1$. Since T is α_* -admissible, so $\alpha_*(Tx_1,Tx_2) \geq 1$. If $x_2 \in Tx_2$, then x_2 is fixed point of T. Assume that $x_2 \notin Tx_2$. Then from (2.7), we get

$$\begin{array}{lll} 0 &<& d(x_2, Tx_2) \leq \alpha_*(Tx_1, Tx_2)H(Tx_1, Tx_2) \\ &\leq& \psi(\max\{d(x_1, x_2), d(x_2, Tx_2), \frac{d(x_1, Tx_1)d(x_2, Tx_2)}{1 + d(x_1, x_2)}, \frac{d(x_1, Tx_1)d(x_2, Tx_2)}{1 + d(Tx_1, Tx_2)}\}) \\ &\leq& \psi(\max\{d(x_1, x_2), d(x_2, Tx_2)\}). \end{array}$$

If $\max\{d(x_2, Tx_2), d(x_1, x_2)\} = d(x_2, Tx_2)$, we get contradiction to the fact $d(x_2, Tx_2) < d(x_2, Tx_2)$. Hence we obtain

$$\max\{d(x_2, Tx_2), d(x_1, x_2)\} = d(x_1, x_2).$$

So $d(x_2, Tx_2) \leq \psi(d(x_1, x_2))$. Since $q_1 > 1$, so by Lemma 12 we can find $x_3 \in Tx_2$ such that

$$d(x_2, x_3) < q_1 d(x_2, Tx_2) \le q_1 \psi(d(x_1, x_2)).$$

$$d(x_2, x_3) < q_1 \psi(d(x_1, x_2)) \le q_1 \psi(d(x_1, x_2)) = \psi(q \psi(d(x_0, x_1))).$$
(2.9)

It is clear that $x_2 \neq x_3$. Put $q_2 = \frac{\psi^2(q\psi(d(x_0,x_1)))}{\psi(d(x_2,x_3))}$. Then $q_2 > 1$ and $\alpha(x_2,x_3) \geq 1$. Since T is α_* -admissible, so $\alpha_*(Tx_2,Tx_3) \geq 1$. If $x_3 \in Tx_3$, then x_3 is fixed point of T. Assume that $x_3 \notin Tx_3$. From (2.7), we have

$$\begin{array}{lcl} 0 &<& d(x_3, Tx_3) \leq \alpha_*(Tx_2, Tx_3)H(Tx_2, Tx_3) \\ &\leq & \psi(\max\{d(x_2, x_3), d(x_3, Tx_3), \frac{d(x_2, Tx_2)d(x_3, Tx_3)}{1 + d(x_2, x_3)}, \frac{d(x_2, Tx_2)d(x_3, Tx_3)}{1 + d(Tx_2, Tx_3)}\}) \\ &\leq & \psi(\max\{d(x_2, x_3), d(x_3, Tx_3)\}). \end{array}$$

If $\max\{d(x_3, Tx_3), d(x_2, x_3)\} = d(x_3, Tx_3)$. Then we get a contradiction. So $\max\{d(x_3, Tx_3), d(x_2, x_3)\} = d(x_2, x_3)$. Thus

$$d(x_3, Tx_3) \le \psi(d(x_2, x_3))$$

Since $q_2 > 1$, so by Lemma 1.12 we can find $x_4 \in Tx_3$ such that

$$d(x_3, x_4) < q_2 d(x_3, Tx_3) \le q_2 \psi(d(x_2, x_3)) = \psi^2(q\psi(d(x_0, x_1))).$$
(2.10)

Continuing in this way, we can generate a sequence $\{x_n\}$ in X such that $x_n \in Tx_{n-1}$ and $x_n \neq x_{n-1}$, and

$$d(x_n, x_{n+1}) \le \psi^{n-1}(q\psi(d(x_0, x_1)))$$
(2.11)

for all n. Now, for each m > n, we have

$$d(x_n, x_m) \le \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \le \sum_{i=n}^{m-1} \psi^{i-1}(q\psi(d(x_0, x_1))).$$

This implies that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $x^* \in X$ such that $x_n \longrightarrow x^*$ as $n \longrightarrow \infty$. We now show that $x^* \in Tx^*$. Since T is continuous, then

$$d(x^*, Tx^*) \le \lim_{n \to \infty} d(x_{n+1}, Tx^*) \le \lim_{n \to \infty} H(Tx_n, Tx^*) = 0.$$

Thus $x^* \in Tx^*$. On the other hand. Since $\alpha(x_n, x^*) \ge 1$ for all n and T is α_* -admissible, so $\alpha_*(Tx_n, Tx^*) \ge 1$ for all n. Then, we have

$$\begin{aligned} d(x^*, Tx^*) &\leq \alpha_*(Tx_n, Tx^*)H(Tx^*, Tx_n) + d(x_n, x^*) \\ &\leq \psi(\max\{d(x_n, x^*), d(x^*, Tx^*), \frac{d(x_n, Tx_n)d(x^*, Tx^*)}{1 + d(x_n, x^*)}, \frac{d(x_n, Tx_n)d(x^*, Tx^*)}{1 + d(Tx_n, Tx^*)}\}) + d(x_n, x^*) \\ &\leq \psi(\max\{d(x_n, x^*), d(x_n, Tx_n), d(x^*, Tx^*)\}) + d(x_n, x^*) \end{aligned}$$

taking limit as $n \to \infty$, we get $d(x^*, Tx^*) = 0$. Thus $x^* \in Tx^*$.

Example 2.6. Let X = [0,1] and d(x,y) = |x - y|. Define $T : X \to 2^X$ by $Tx = [0, \frac{x}{7}]$ for all $x \in X$ and

$$\alpha(x,y) = \begin{cases} \frac{1}{|x-y|} & \text{if } x \neq y \\ 1 & \text{if } x = y. \end{cases}$$

Then $\alpha(x, y) \ge 1 \Longrightarrow \alpha^*(Tx, Ty) = \inf\{\alpha(a, b) : a \in Tx, b \in Ty\} \ge 1$. Then clearly T is α^* -admissible. Now for x, y and $x \le y$, it is easy to check that

$$\alpha_*(Tx, Ty)H(Tx, Ty) \le \psi(Q(x, y))$$

where

$$Q(x,y) = \max\{d(x,y), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)}\}$$

and $\psi(t) = \frac{t}{4}$, for all $t \ge 0$. Put $x_0 = 1$ and $x_1 = \frac{1}{3}$. Then $\alpha(x_0, x_1) = 3 > 1$. Then T has fixed point 0.

Corollary 2.7. Let (X,d) be a complete metric space and $T : X \to X$ be an α -admissible mapping. Assume that for $\psi \in \Psi$, we have

$$\alpha(Tx, Ty)d(Tx, Ty) \le \psi(Q(x, y)), \tag{2.12}$$

where

$$Q(x,y) = \max\{d(x,y), d(y,Ty) \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)}\}$$

for all $x, y \in X$. Also suppose that the following assertions hold:

Theorem 2.8. (i) there exists $x_0 \in X$ with $\alpha(x_0, Tx_0) \ge 1$;

(ii) for any sequence $\{x_n\}$ in $\overline{B(x_0, r)}$ converging to $x \in \overline{B(x_0, r)}$ and $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N}$, we have $\alpha(x_n, x) \ge 1$ for all $n \in \mathbb{N}$. Then T has a fixed point

Then T has a fixed point.

3. Fixed point results for Generalized ciric type graphic contractions

Consistent with Jachymski [20], let (X, d) be a metric space and Δ denotes the diagonal of the Cartesian product $X \times X$. Consider a directed graph G such that the set V(G) of its vertices coincides with X, and the set E(G) of its edges contains all loops, i.e., $E(G) \supseteq \Delta$. We assume G has no parallel edges, so we can identify G with the pair (V(G), E(G)). Moreover, we may treat G as a weighted graph (see [20]) by assigning to each edge the distance between its vertices. If x and y are vertices in a graph G, then a path in G from x to y of length N ($N \in \mathbb{N}$) is a sequence $\{x_i\}_{i=0}^N$ of N + 1 vertices such that $x_0 = x$, $x_N = y$ and $(x_{n-1}, x_n) \in E(G)$ for i = 1, ..., N. A graph G is connected if there is a path between any two vertices. G is weakly connected if \tilde{G} is connected(see for details [12, 15, 19, 20]).

Definition 3.1. [20] We say that a mapping $T : X \to X$ is a Banach G-contraction or simply G-contraction if T preserves edges of G, i.e.,

$$\forall x , y \in X((x, y) \in E(G) \Rightarrow (T(x), T(y)) \in E(G))$$
(3.1)

and T decreases weights of edges of G in the following way:

 $\exists k \in [0,1), \forall x, y \in X((x,y) \in E(G) \Rightarrow d(T(x), T(y)) \le kd(x,y))$

Definition 3.2. [20] A mapping $T : X \to X$ is called G-continuous, if given $x \in X$ and sequence $\{x_n\}$

 $x_n \to x \text{ as } n \to \infty \text{ and } (x_n, x_{n+1}) \in E(G) \text{ for all } n \in \mathbb{N} \text{ imply } Tx_n \to Tx.$

Theorem 3.3. Let (X, d) be a complete metric space endowed with a graph G and T be a self-mapping on X. Suppose the following assertions hold:

- (i) $\forall x, y \in X, (x, y) \in E(G) \Rightarrow (T(x), T(y)) \in E(G);$
- (ii) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$;

(iii) there exists $\psi \in \Psi$ such that

$$d(Tx, Ty) \le \psi(M(x, y))$$

for all $(x, y) \in E(G)$ where

$$M(x,y) = \max\left\{ d(x,y), d(x,Tx), d(y,Ty), \frac{d(y,Tx) + d(x,Ty)}{2} \right\};$$

(iv) if $\{x_n\}$ is a sequence in X such that $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$, then $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$.

Then T has a fixed point.

 $\begin{array}{ll} \textit{Proof. Define, } \alpha: X^2 \rightarrow [0, +\infty) \text{ by } \alpha(x,y) = \left\{ \begin{array}{ll} 1, & \text{if } (x,y) \in E(G) \\ 0, & \text{otherwise} \end{array} \right. \text{ At fist we prove that } T \text{ is an } \alpha \text{-admissible mapping. Let, } \alpha(x,y) \geq 1, \text{ then } (x,y) \in E(G). \end{array} \right.$

From (i), we have, $(Tx, Ty) \in E(G)$. That is, $\alpha(Tx, Ty) \geq 1$. Thus T is an α admissible mapping. From (ii) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$. That is, $\alpha(x_0, Tx_0) \geq 1$. If $(x, y) \in E(G)$, then $(Tx, Ty) \in E(G)$ and hence $\alpha(Tx, Ty) = 1$. Thus, from (iii) we have $\alpha(Tx, Ty)d(Tx, Ty) = d(Tx, Ty) \leq \psi(M(x, y))$. Condition (iv) implies condition (ii) of Theorem 15. Hence, all conditions of Theorem 15 are satisfied and T has a fixed point. \Box

Theorem 3.4. Let (X, d) be a complete metric space endowed with a graph G and T be a self-mapping on X. Suppose the following assertions hold:

- (i) $\forall x, y \in X, (x, y) \in E(G) \Rightarrow (T(x), T(y)) \in E(G);$
- (ii) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$;
- (iii) there exists $\psi \in \Psi$ such that

$$d(Tx, Ty) \le \psi(Q(x, y))$$

for all $(x, y) \in E(G)$ where

$$Q(x,y) = \max\{d(x,y), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)}\};$$

(iv) if $\{x_n\}$ is a sequence in X such that $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$, then $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$.

Then T has a fixed point.

Corollary 3.5. Let (X, d) be a complete metric space endowed with a graph G and T be a self-mapping on X. Suppose the following assertions hold:

- (i) T is Banach G-contraction;
- (ii) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$;
- (iii) if $\{x_n\}$ is a sequence in X such that $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$, then $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$. Then T has a fixed point.

Let (X, d, \preceq) be a partially ordered metric space. Define the graph G by

$$E(G) = \{(x, y) \in X \times X : x \leq y\}$$

We derive following important results in partially ordered metric spaces.

Theorem 3.6. Let (X, d, \preceq) be a complete partially ordered metric space and T be a self-mapping on X. Suppose the following assertions hold:

- (i) T is nondecreasing map;
- (ii) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;
- (iii) there exists $\psi \in \Psi$ such that

$$d(Tx, Ty) \le \psi(M(x, y))$$

for all $x \leq y$ where

$$M(x,y) = \max\left\{ d(x,y), d(x,Tx), d(y,Ty), \frac{d(y,Tx) + d(x,Ty)}{2} \right\};$$

(iv) if $\{x_n\}$ is a sequence in X such that $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$, then $x_n \leq x$ for all $n \in \mathbb{N}$.

Then T has a fixed point.

Theorem 3.7. Let (X, d, \preceq) be a complete partially ordered metric space and T be a self-mapping on X. Suppose the following assertions hold:

- (i) T is nondecreasing map;
- (ii) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;
- (iii) there exists $\psi \in \Psi$ such that

$$d(Tx,Ty) \le \psi(Q(x,y))$$

for all $x \leq y$ where

$$Q(x,y) = \max\{d(x,y), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)}\};$$

(iv) if $\{x_n\}$ is a sequence in X such that $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$, then $x_n \leq x$ for all $n \in \mathbb{N}$.

Then T has a fixed point.

Corollary 3.8. [26] Let (X, d, \preceq) be a complete partially ordered metric space and $T: X \to X$ be nondecreasing mapping such that

$$d(Tx, Ty) \le rd(x, y)$$

for all $x, y \in X$ with $x \leq y$ where $0 \leq r < 1$. Suppose that the following assertions hold:

- (i) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;
- (ii) if $\{x_n\}$ is a sequence in X such that $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$, then $x_n \leq x$ for all $n \in \mathbb{N}$.

Then T has a fixed point.

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