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SOME FIXED POINT THEOREMS IN ORDERED DUALISTIC PARTIAL METRIC SPACES WITH APPLICATION

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Abstract. In this paper, we introduce the notion of dualistic contraction of rational type. We prove some fixed point theorems for ordered mappings satisfying above mentioned contraction. We give examples to illustrate the importance of these results. We present an application of our fixed point result to show the existence of solution of integral equations.

1. Introduction and preliminaries

Matthews [6] introduced the concept of partial metric spaces as a suitable mathematical tool for program verification and proved an analogue of Banach fixed point theorem in complete partial metric spaces. Fixed point theorems in complete partial metric spaces have been investigated in [1, 4, 9]. O'Neill [10] introduced the concept of dualistic partial metric, which is more general than partial metric and established a robust relationship between dualistic partial metric and quasi metric. Oltra and Valero [11] presented a Banach fixed point theorem on complete dualistic partial metric spaces. Valero also showed that the contractive condition in Banach fixed point theorem in complete dualistic partial metric spaces cannot be replaced by the contractive condition of Banach fixed point theorem for complete partial metric spaces. Following Oltra and Valero, Nazam et al. [7, 3, 8] established some fixed point results in dualistic partial metric spaces for Greghty type contraction and monotone mappings and discussed an application of fixed point theorem to show the existence of solution of integral equation.
For the sake of completeness, we recall Geraghty's Theorem. For this purpose, we first remind the class $\mathbf{S}$ of all functions $\beta : [0,\infty) \to [0,1)$ which satisfy the condition:

$$\lim_{n\to\infty} \beta(t_n) = 1 \text{ implies } \lim_{n\to\infty} t_n = 0.$$

In [5], Geraghty presented a new class of mappings $T : M \to M$, known as Geraghty contraction, which satisfies the following condition:

$$d(T(j), T(k)) \leq \beta(d(j,k))d(j,k), \tag{1.1}$$

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for all $j, k \in M$, where $\beta \in \mathbf{S}$. For this new family of mappings, Geraghty [5] proved following fixed point theorem.

Theorem 1.1. [5] Let (M, d) be a complete metric space and $T : M \to M$ be a mapping. Assume that there exists $\beta \in \mathbf{S}$ such that, for all $k, l \in M$,

$$d(T(j), T(k)) \leq \beta(d(j, k))d(j, k).$$

Then T has a unique fixed point $\upsilon \in M$ and, for any choice of the initial point $j_0 \in M$, the sequence $\{j_n\}$ defined by $j_n = T(j_{n-1})$ for each $n \geq 1$ converges to the point υ.

Following Geraghty, Amini-Harandi and Emami [2] generalized Theorem 1.1 in context of ordered metric spaces, as follows.

Theorem 1.2. [2] Let $(M, \preceq)$ be an ordered set and suppose that there exists a metric d in M such that (M, d) is a complete metric space. Let $T : M \to M$ be an increasing mapping such that there exists $j_0 \in M$ with $j_0 \preceq T(j_0)$. Suppose that there exists $\beta \in \mathbf{S}$ such that

$$d(T(j), T(k)) \leq \beta(d(j, k))d(j, k) \text{ for all } j, k \in M \text{ with } j \succeq k.$$

Assume that either T is continuous or M is such that if an increasing sequence $\{j_n\}$ converges to u, then $j_n \preceq u$ for each $n \geq 1$.
Besides, if

for all $j, k \in M$, there exists $z \in M$ which is comparable to both j and k, then T has a unique fixed point in M.

La.Rosa and Vetro [12] have extended the notion of Geraghty contraction mappings to the context of partial metric spaces.

In this paper, we shall present Theorems 1.1 and 1.2 in dualistic partial metric spaces. We shall show that our results generalize Theorems 1.1 and 1.2 in many ways. In last section we shall apply our fixed point theorem to show the existence of solution of a particular class of integral equations.

$$j(w) = g(w) + \int_0^1 G_n(w, s, j(s))\, ds \ \ \forall\ w \in [0, 1].$$

We need some mathematical basics of dualistic partial metric space and results to make this paper self sufficient.

Throughout this paper, the letters $\mathbb{R}_0^+$, $\mathbb{R}$ and $\mathbb{N}$ will represent the set of nonnegative real numbers, set of real numbers and set of natural numbers, respectively.

O'Neill [10] introduced the notion of dualistic partial metric as a generalization of partial metric in order to expand the connections between partial metrics and semantics via valuation spaces.

According to O'Neill a dualistic partial metric can be defined as follows:

Definition 1.3. [10] Let M be a nonempty set. If a function $D : M \times M \to \mathbb{R}$ satisfies, for all $j, k, l \in M$, the following properties:

(1) $j = k \Leftrightarrow D(j, j) = D(k, k) = D(j, k)$.
(2) $D(j, j) \leq D(j, k)$.
(3) $D(j, k) = D(k, j)$.
(4) $D(j, l) + D(k, k) \leq D(j, k) + D(k, l)$.

Then D is called dualistic partial metric and the pair (M, D) is known as dualistic partial metric space.

Following [10], each dualistic partial metric D on M generates a T_0 topology $\tau(D)$ on M. The base of $\tau(D)$ consists of family of open balls $\{B_D(j, \epsilon) : j \in M, \epsilon > 0\}$, where $B_D(j, \epsilon) = \{k \in M : D(j, k) < \epsilon + D(j, j)\}$.
A sequence $\{j_n\}_{n\in\mathbb{N}}$ in (M, D) converges to a point $j \in M$ if and only if $D(j, j) = \lim_{n\to\infty} D(j, j_n)$.
If (M, D) is a dualistic partial metric space, then $d_D : M \times M \to \mathbb{R}_0^+$ defined by

$$d_D(j, k) = D(j, k) - D(j, j).$$

is called a quasi metric on M such that $\tau(D) = \tau(d_D)$ for all $j, k \in M$. Moreover, if d_D is a quasi metric on M, then $d_D^s(j, k) = \max\{d_D(j, k), d_D(k, j)\}$ defines a metric on M.

Remark. It is obvious that every partial metric is dualistic partial metric but converse is not true. To support this comment, define $D_\vee : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$D_\vee(j, k) = j \vee k = \sup\{j, k\} \text{ for all } j, k \in \mathbb{R}.$$

It is clear that $D_\vee$ is a dualistic partial metric. Note that $D_\vee$ is not a partial metric, because $D_\vee(-1, -2) = -1 \notin \mathbb{R}_0^+$. However, the restriction of $D_\vee$ to $\mathbb{R}_0^+$, $D_\vee|_{\mathbb{R}_0^+}$, is a partial metric.

Example 1.4. If (M, d) is a metric space and $c \in \mathbb{R}$ is an arbitrary constant, then

$$D(j, k) = d(j, k) + c.$$

defines a dualistic partial metric on M.

Definition 1.5. [10] Let (M, D) be a dualistic partial metric space, then

(1) A sequence $\{j_n\}_{n\in\mathbb{N}}$ in (M, D) is called a Cauchy sequence if $\lim_{n,m\to\infty} D(j_n, j_m)$ exists and is finite.
(2) A dualistic partial metric space (M, D) is said to be complete if every Cauchy sequence $\{j_n\}_{n\in\mathbb{N}}$ in M converges, with respect to $\tau(D)$, to a point $j \in M$ such that $D(j, j) = \lim_{n,m\to\infty} D(j_n, j_m)$.

Following lemma will be helpful in the sequel.

Lemma 1.6. [10, 11]

(1) A dualistic partial metric (M, D) is complete if and only if the metric space (M, d_D^s) is complete.
(2) A sequence $\{j_n\}_{n\in\mathbb{N}}$ in M converges to a point $j \in M$, with respect to $\tau(d_D^s)$ if and only if $\lim_{n\to\infty} D(j, j_n) = D(j, j) = \lim_{n\to\infty} D(j_n, j_m)$.
(3) If $\lim_{n\to\infty} j_n = \upsilon$ such that $D(\upsilon, \upsilon) = 0$ then $\lim_{n\to\infty} D(j_n, k) = D(\upsilon, k)$ for every $k \in M$.

Oltra and Valero [11] established a Banach fixed point theorem for dualistic partial metric spaces in such a way that the Matthews fixed point theorem is obtained as a particular case.

2. The results

In this section we shall prove fixed point theorems 1.1 and 1.2 in ordered dualistic partial metric spaces. To this end we need to define following notions:

Definition 2.1. Let M be a nonempty set. Then $(M, \preceq, D)$ is said to be an ordered dualistic partial metric space if:

(1) $(M, \preceq)$ is a partially ordered set.

(2) (M, D) is a dualistic partial metric space.

Definition 2.2. Let $(M, \preceq)$ be a partially ordered set and suppose that (M, D) is a dualistic partial metric space, a mapping $T : M \to M$ is called a generalized dualistic contraction of rational type if there exists $\beta \in \mathbf{S}$ such that,

$$|D(T(j), T(k))| \leq \beta(N(j, k))N(j, k), \tag{2.1}$$

for all comparable $j, k \in M$.
Where $N(j,k) = \max\left\{|D(j,k)|, \left|\dfrac{D(k,T(k))(1 + D(j,T(j))}{1 + D(j,k)}\right|\right\}$.

Now we present our main result:

Theorem 2.3. *Let $(M, \preceq)$ be a partially ordered set and suppose that (M, D) is a complete dualistic partial metric space and let $T : M \to M$ be a mapping such that,*

(1) *T is a dominated mapping.*
(2) *T is a generalized dualistic contraction of rational type.*
(3) *either T is continuous or if $\{j_n\}$ is a non increasing sequence in M such that $\{j_n\} \to \upsilon$, then $j_n \succeq \upsilon$ for all n.*

Then T has a fixed point $\upsilon \in M$ and the Picard iterative sequence $\{T^n(j)\}_{n\in\mathbb{N}}$ converges to υ with respect to $\tau(d^s_D)$, for every $j \in M$. Moreover, $D(\upsilon, \upsilon) = 0$.

Proof. Let $j_0 \in M$ be an initial element and $j_n = T(j_{n-1})$ for all $n \geq 1$, if there exists a positive integer r such that $j_{r+1} = j_r$ then $j_r = T(j_r)$, so we are done. Suppose that $j_n \neq j_{n+1}$ for all $n \in \mathbb{N}$, then since T is dominated mapping, we have $j_0 \succeq T(j_0) = j_1$ that is $j_0 \succeq j_1$ and $j_1 \succeq T(j_1)$ implies $j_1 \succeq j_2$, moreover, $j_2 \succeq T(j_2)$ implies $j_2 \succeq j_3$, continuing in similar way, we get

$$j_0 \succeq j_1 \succeq j_2 \succeq j_3 \succeq ... \succeq j_n \succeq j_{n+1} \succeq j_{n+2} \succeq ...$$

Since $j_n \succeq j_{n+1}$, from contractive condition (2.1), we have

$$\begin{aligned} |D(j_{n+1}, j_{n+2})| &= |D(T(j_n), T(j_{n+1}))|, \\ &\leq \beta(N(j_n, j_{n+1}))N(j_n, j_{n+1}) < N(j_n, j_{n+1}). \end{aligned}$$

Where

$$\begin{aligned} N(j_n, j_{n+1}) &= \max\left\{|D(j_n, j_{n+1})|, \left|\frac{D(j_{n+1}, j_{n+2})(1 + D(j_n, j_{n+1})}{1 + D(j_n, j_{n+1})}\right|\right\} \\ &= \max\left\{|D(j_n, j_{n+1})|, |D(j_{n+1}, j_{n+2})|\right\}. \end{aligned}$$

If $|D(j_n, j_{n+1})| \leq |D(j_{n+1}, j_{n+2})|$, then $N(j_n, j_{n+1}) = |D(j_{n+1}, j_{n+2})|$.
From contractive condition (2.1) we have,

$$|D(j_{n+1}, j_{n+2})| \leq \beta(N(j_n, j_{n+1}))N(j_n, j_{n+1}) < N(j_n, j_{n+1}) = |D(j_{n+1}, j_{n+2})|,$$

which is a contradiction. Consequently, $N(j_n, j_{n+1}) = |D(j_n, j_{n+1})|$. So in this case (2.1) gives,

$$|D(j_{n+1}, j_{n+2})| < |D(j_n, j_{n+1})|.$$

This implies that $\{|D(j_n, j_{n+1})|\}_{n=1}^{\infty}$ is a monotone and bounded above sequence, it is convergent and converges to a point α, *i.e*

$$\lim_{n\to\infty} |D(j_n, j_{n+1})| = \alpha \geq 0.$$

If $\alpha = 0$. Then we have done but if $\alpha > 0$, then from (2.1) we have

$$|D(j_{n+1}, j_{n+2})| \leq \beta(N(j_n, j_{n+1}))N(j_n, j_{n+1}).$$

This implies that

$$\frac{|D(j_{n+1}, j_{n+2})|}{N(j_n, j_{n+1})} \leq \beta(N(j_n, j_{n+1})).$$

Taking limit we have

$$\lim_{n\to\infty} \beta(N(j_n, j_{n+1})) = 1.$$

Since $\beta \in S$ and $N(j_n, j_{n+1}) = |D(j_n, j_{n+1})|$, $\lim_{n\to\infty} |D(j_n, j_{n+1})| = 0$, which entails $\alpha = 0$.
Hence

$$\lim_{n\to\infty} D(j_n, j_{n+1}) = 0. \tag{2.2}$$

Similarly we can prove that

$$\lim_{n\to\infty} D(j_n, j_n) = 0.$$

Now since

$$d_D(j_n, j_{n+1}) = D(j_n, j_{n+1}) - D(j_n, j_n)$$

we deduce that

$$\lim_{n\to\infty} d_D(j_n, j_{n+1}) = 0 \text{ for all } n \geq 1.$$

Now, we show that sequence $\{j_n\}$ is Cauchy sequence (M, d_D^s). Suppose on contrary that $\{j_n\}$ is not a Cauchy sequence. Then given $\epsilon > 0$, we will construct a pair of subsequences $\{j_{m_r}\}$ and $\{j_{n_r}\}$ violating the following condition for least integer n_r such that $m_r > n_r > r$, where $r \in \mathbb{N}$

$$d_D(j_{m_r}, j_{n_r}) \geq \epsilon \tag{2.3}$$

In addition, upon choosing the smallest possible m_r, we may assume that

$$d_D(j_{m_r}, j_{n_{r-1}}) < \epsilon. \tag{2.4}$$

By the triangle inequality, we have

$$\begin{aligned} \epsilon &\leq d_D(j_{m_r}, j_{n_r}) \\ &\leq d_D(j_{m_r}, j_{n_{r-1}}) + d_D(j_{n_{r-1}}, j_{n_r}) \\ &< \epsilon + d_D(j_{n_{r-1}}, j_{n_r}). \end{aligned}$$

That is,

$$\epsilon < \epsilon + d_D(j_{n_{r-1}}, j_{n_r}) \tag{2.5}$$

for all $r \in \mathbb{N}$. In the view of (2.5), (2.11), we have

$$\lim_{r\to\infty} d_D(j_{m_r}, j_{n_r}) = \epsilon. \tag{2.6}$$

Again using triangle inequality, we have

$$d_D(j_{m_r}, j_{n_r}) \leq d_D(j_{m_r}, j_{m_{r+1}}) + d_D(j_{m_{r+1}}, j_{n_{r+1}}) + d_D(j_{n_{r+1}}, j_{n_r}).$$

and

$$d_D(j_{m_{r+1}}, j_{n_{r+1}}) \leq d_D(j_{m_{r+1}}, j_{m_r}) + d_D(j_{m_r}, j_{n_r}) + d_D(j_{n_r}, j_{n_{r+1}}).$$

Taking limit as $r \to +\infty$ and using (2.11) and (2.6), we obtain

$$\lim_{r\to+\infty} d_D(j_{m_{r+1}}, j_{n_{r+1}}) = \epsilon. \tag{2.7}$$

Now from contractive condition (2.1), we have

$$\begin{aligned} |D(j_{n_{r+1}}, j_{m_{r+2}})| &= |D(T(j_{n_r}), T(j_{m_{r+1}}))|, \\ &\leq \beta(N(j_{n_r}, j_{m_{r+1}}))N(j_{n_r}, j_{m_{r+1}}). \end{aligned}$$

We conclude that

$$\frac{|D(j_{n_{r+1}}, j_{m_{r+2}})|}{N(j_{n_r}, j_{m_{r+1}})} \leq \beta(N(j_{n_r}, j_{m_{r+1}})).$$

By using (2.11), letting $r \to +\infty$ in the above inequality, we obtain

$$\lim_{r\to\infty} \beta(N(j_{n_r}, j_{m_{r+1}})) = 1. \tag{2.8}$$

Since $\beta \in \mathbf{S}$ and $N(j_{n_r}, j_{m_{r+1}}) = |D(j_{n_r}, j_{m_{r+1}})|$, $\lim_{r\to\infty} |D(j_{n_r}, j_{m_{r+1}})| = 0$ and hence $\lim_{r\to\infty} d_D(j_{n_r}, j_{m_{r+1}}) = 0 < \epsilon$, which contradicts our assumption (2.3). Arguing like above , we can have $\lim_{r\to\infty} d_D(j_{m_r}, j_{n_{r+1}}) = 0 < \epsilon$. Hence $\{j_n\}$ is a Cauchy sequence in (M, d_D^s) that is $\lim_{n,m\to\infty} d_D^s(j_n, j_m) = 0$. Since (M, d_D^s) is a complete metric space, $\{j_n\}$ converges to a point υ in M, *i.e* $\lim_{n\to\infty} d_D^s(j_n, \upsilon) = 0$, then from Lemma 1.6, we get

$$\lim_{n\to\infty} D(\upsilon, j_n) = D(\upsilon, \upsilon) = \lim_{n,m\to\infty} D(j_n, j_m) = 0. \tag{2.9}$$

We are left to prove that υ is a fixed point of T. For this purpose we have to deal with two cases:

Case 1: If T is continuous.

Then,

$$\upsilon = \lim_{n\to\infty} j_n = \lim_{n\to\infty} T^n(j_0) = \lim_{n\to\infty} T^{n+1}(j_0) = T(\lim_{n\to\infty} T^n(j_0)) = T(\upsilon).$$

Hence $\upsilon = T(\upsilon)$ that is υ is fixed point of T.

Case 2: If $\{j_n\}$ is a non increasing sequence in M such that $\{j_n\} \to \upsilon$, then $j_n \succeq \upsilon \ \forall \ n$.

We note that $D(\upsilon, T(\upsilon)) \geq 0$. Indeed $D(\upsilon, T(\upsilon)) - D(\upsilon, \upsilon) = d_D(\upsilon, T(\upsilon)) \geq 0$. For the case when $D(\upsilon, T(\upsilon)) > 0$, contractive condition (2.1) and (2.9) implies

$$\begin{aligned} |D(j_{n+1}, T(\upsilon))| &= |D(T(j_n), T(\upsilon))|. \\ &\leq \beta(N(j_n, \upsilon))N(j_n, \upsilon). \\ \lim_{n\to\infty} |D(j_{n+1}, T(\upsilon))| &\leq \lim_{n\to\infty} \beta(N(j_n, \upsilon))N(j_n, \upsilon). \\ \text{Thus } D(\upsilon, T(\upsilon)) &< D(\upsilon, T(\upsilon)). \end{aligned}$$

A contradiction. This shows that $D(\upsilon, T(\upsilon)) = 0$. So from (D_1) and (D_2), we deduce that $\upsilon = T(\upsilon)$ and hence υ is a fixed point of T. $\square$

Note that in the above result fixed point may not be unique, in order to prove uniqueness of the fixed point, we need some more conditions and for this purpose we have following Theorem.

Theorem 2.4. *Let $(M, \preceq, D)$ be an ordered complete dualistic partial metric space. Let $T : M \to M$ be a mapping satisfying all the conditions of Theorem 2.3. Besides, if for each $j, k \in M$ there exists $z \in M$ which is comparable to both j and k. Then T has a unique fixed point.*

Proof. Following the proof of Theorem 2.3, we are only left to prove the uniqueness of the fixed point υ. Let υ_1 be another fixed point of T then $T(\upsilon_1) = \upsilon_1$. Two cases arises, first if υ, υ_1 are comparable then from (2.1) it follows that $\upsilon = \upsilon_1$. Secondly, if υ, υ_1 are not comparable then there exists $z \in M$ which is comparable to both υ, υ_1 that is $\upsilon \succeq z$ and $\upsilon_1 \succeq z$. Since T is dominated mapping, we deduce that $\upsilon \succeq T^n(z)$ and $\upsilon_1 \succeq T^n(z)$. Moreover consider $|D(\upsilon, T^n(z))| = |D(T^n(\upsilon), T^n(z))|$ and by contractive condition (2.1) we obtain,

$$|D(T^n(\upsilon), T^n(z))| \leq \beta(N(T^{n-1}(\upsilon), T^{n-1}(z)))|D(T^{n-1}(\upsilon), T^{n-1}(z))|. \tag{2.10}$$

This implies

$$|D(\upsilon, T^n(z))| \leq |D(\upsilon, T^{n-1}(z))|.$$

It shows that $\{|D(\upsilon, T^n(z))|\}_{n=1}^{\infty}$ is a non-negative and decreasing sequence, so for $\lambda \geq 0$ we get

$$\lim_{n\to\infty} |D(\upsilon, T^n(z))| = \lambda.$$

We claim that $\lambda = 0$. Suppose on contrary that $\lambda > 0$.
By passing to subsequences, if necessary, we may assume that

$$\lim_{n\to\infty} \beta(N(\upsilon, T^n(z))) = \gamma.$$

Then by (2.10) we have $\lambda \leq \gamma\lambda \Rightarrow \ \gamma = 1$. Since $\beta \in \mathbf{S}$, we have

$$\lim_{n\to\infty} |D(\upsilon, T^n(z))| = 0.$$

Hence

$$\lim_{n\to\infty} D(\upsilon, T^n(z)) = 0.$$

Similarly, we can prove that

$$\lim_{n\to\infty} D(\upsilon_1, T^n(z)) = 0.$$

Finally by D_4 we have

$$\begin{aligned} D(\upsilon, \upsilon_1) &\leq D(\upsilon, T^n(z)) + D(T^n(z), \upsilon_1) - D(T^n(z), T^n(z)), \\ &\leq D(\upsilon_1, T^n(z)) + D(T^n(z), \upsilon) - D(T^n(z), \upsilon) - D(\upsilon, T^n(z)) + D(\upsilon, \upsilon). \end{aligned}$$

Taking limit we get $D(\upsilon, \upsilon_1) \leq 0$. Since $d_D(\upsilon, \upsilon_1) = D(\upsilon, \upsilon_1) - D(\upsilon_1, \upsilon_1)$, which implies $D(\upsilon, \upsilon_1) \geq 0$. Hence $D(\upsilon, \upsilon_1) = 0$. From D_1 and D_2 we deduce that is $\upsilon = \upsilon_1$ which proves the uniqueness of υ. □

For monotone mappings we present following result.

Theorem 2.5. *Let $(M, \preceq)$ be a partially ordered set and suppose that (M, D) is a complete dualistic partial metric space and let $T : M \to M$ be a mapping such that,*

1. *T is an increasing map with $j_0 \preceq T(j_0)$ for some $j_0 \in M$.*
2. *T is a generalized dualistic contraction of rational type.*

(3) *either T is continuous or M is such that if an increasing sequence $\{j_n\} \to u \in M$ then $j_n \preceq u$.*

Besides, if for each $j, k \in M$ there exists $z \in M$ which is comparable to both j and k. Then T has a unique fixed point $\upsilon \in M$ and the Picard iterative sequence $\{T^n(j_0)\}_{n\in\mathbb{N}}$ converges to υ with respect to $\tau(d_D^s)$, for any $j_0 \in M$. Moreover, $D(\upsilon, \upsilon) = 0$.

Proof. We begin by defining a Picard iterative sequence in M by $j_n = T(j_{n-1})$ for all $n \in \mathbb{N}$. Given $j_0 \preceq T(j_0) = j_1$ so $j_0 \preceq j_1$. Since T is increasing, $j_0 \preceq j_1$ implies $T(j_0) \preceq T(j_1)$ that is $j_1 \preceq j_2$, this in turn gives $T(j_1) \preceq T(j_2)$ which implies $j_2 \preceq j_3$. Continuing in a similar way we get

$$j_0 \preceq j_1 \preceq j_2 \preceq j_3 \preceq ... \preceq j_n \preceq j_{n+1}....$$

Since $j_n \preceq j_{n+1}$ for each $n \in \mathbb{N}$, from contractive condition (2.1) we have

$$\begin{aligned} |D(j_{n+1}, j_{n+2})| &= |D(T(j_n), T(j_{n+1}))|, \\ &\leq \beta(N(j_n, j_{n+1}))N(j_n, j_{n+1}), \\ \text{implies } |D(j_{n+1}, j_{n+2})| &\leq |D(j_n, j_{n+1})| \ \forall\ n \geq 1. \end{aligned}$$

This implies that $\{|D(j_n, j_{n+1})|\}_{n=1}^{\infty}$ is a monotone and bounded below sequence, it is convergent and converges to a point α, *i.e*

$$\lim_{n\to\infty} |D(j_n, j_{n+1})| = \alpha \geq 0.$$

If $\alpha = 0$. Then we have done but if $\alpha > 0$, then from (2.1) we have

$$|D(j_{n+1}, j_{n+2})| \leq \beta(N(j_n, j_{n+1}))N(j_n, j_{n+1}).$$

This implies that

$$\frac{|D(j_{n+1}, j_{n+2})|}{N(j_n, j_{n+1})} \leq \beta(N(j_n, j_{n+1})).$$

Taking limit we have

$$\lim_{n\to\infty} \beta(N(j_n, j_{n+1})) = 1.$$

Since $\beta \in \mathbf{S}$, $\lim_{n\to\infty} |D(j_n, j_{n+1})| = 0$, which implies $\alpha = 0$.
Hence

$$\lim_{n\to\infty} D(j_n, j_{n+1}) = 0. \tag{2.11}$$

Similarly, we can prove that

$$\lim_{n\to\infty} D(j_n, j_n) = 0.$$

and the desired conclusion follows arguing like in the proofs of Theorem 2.3 and Theorem 2.4. □

A natural question that can be raised is, whether the contractive condition in the statements of Theorems 2.3 and 2.5 can be replaced by the contractive condition in Theorems 1.1 and 1.2, The following easy example provides a negative answer to such a question.

Example 2.6. *Consider the complete dualistic partial metric $(\mathbb{R}, D_\vee)$ and the mapping $T_0 : \mathbb{R} \to \mathbb{R}$ defined by,*

$$T_0(j) = \begin{cases} 0 & \text{if } j \neq 0 \\ -1 & \text{if } j = 0 \end{cases}.$$

It is easy to check that the contractive condition in the statement of Theorems 1.1 and 1.2 holds

$$D_\vee(T_0(j), T_0(k)) \leq \beta(N_\vee^+(j,k))N_\vee^+(j,k)$$

for all $j,k \in \mathbb{R}$, where

$$N_\vee^+(j,k) = \max\left\{D(j,k), \frac{D(k,T_0(k))(1+D(j,T_0(j))}{1+D(j,k)}\right\}.$$

However, T_0 does not have a fixed point. Observe that T_0 does not satisfy the contractive condition in the statements of Theorem 2.3 and Theorem 2.5. Indeed,

$$1 = |D_\vee(-1,-1)| = |D_\vee(T_0(0), T_0(0))| > \beta(N_\vee(0,0))N_\vee(0,0) = 0.$$

The analogues of Theorems 2.3 and 2.5 are given below without proofs as they can be obtained easily by following proofs of above theorems.

Theorem 2.7. *Let $(M, \preceq)$ be a partially ordered set and suppose that (M, D) is a complete dualistic partial metric space and let $T : M \to M$ be a mapping such that,*

1. *T is a dominating map.*
2. *T is a generalized dualistic contraction of rational type.*
3. *either T is continuous or M is such that if an increasing sequence $\{j_n\} \to u \in M$ then $j_n \preceq u$.*

Besides, if for each $j,k \in M$ there exists $z \in M$ which is comparable to both j and k, then T has a unique fixed point $\upsilon \in M$ and the Picard iterative sequence $\{T^n(j_0)\}_{n\in\mathbb{N}}$ converges to υ with respect to $\tau(d_D^s)$, for any $j_0 \in M$. Moreover, $D(\upsilon,\upsilon) = 0$.

Theorem 2.8. *Let $(M, \preceq)$ be a partially ordered set and suppose that (M, D) is a complete dualistic partial metric space and let $T : M \to M$ be a mapping such that,*

1. *T is a decreasing map with $T(x_0) \preceq x_0$.*
2. *T is a generalized dualistic contraction of rational type.*
3. *either T is continuous or M is such that if a decreasing sequence $\{j_n\} \to u \in M$ then $j_n \succeq u$.*

Besides, if for each $j,k \in M$ there exists $z \in M$ which is comparable to both j and k, then T has a unique fixed point $\upsilon \in M$ and the Picard iterative sequence $\{T^n(j_0)\}_{n\in\mathbb{N}}$ converges to υ with respect to $\tau(d_D^s)$, for any $j_0 \in M$. Moreover, $D(\upsilon,\upsilon) = 0$.

Observations:

If we set $D(j,j) = 0$ in Theorem 2.5, we retrieve Theorem 1.2 as a particular case. If we set $D(j,k) \in \mathbb{R}_0^+$ in Theorems 2.5 and 2.3, we retrieve corresponding theorems in partial metric spaces.

Following theorem generalizes Theorem 2.3 presented by Valero in [11].

Corollary 2.9. *Let $(M, \preceq)$ be a partially ordered set and suppose that (M, D) is a complete dualistic partial metric space and let $T : M \to M$ be a mapping such that,*

1. *T is an increasing map with $j_0 \preceq T(j_0)$ for some $j_0 \in M$.*
2. *T satisfies $|D(T(j),T(k)) \leq \beta(|D(j,k)|)|D(j,k)|$, for all comparable $j,k \in M$.*
3. *either T is continuous or M is such that if an increasing sequence $\{j_n\} \to u \in M$ then $j_n \preceq u$.*

Then T has a fixed point.

Proof. Set $N(j,k) = |D(j,k)|$ in (2.1). Proof follows from Theorem 2.5 $\square$

3. Application to integral equations

In this section we shall show how Theorem 2.5 can be applied to prove the existence of solution of integral equation (3.1).
Let Φ represents the class of functions $\phi : [0,\infty) \to [0,\infty)$ with properties;

(1) ϕ is increasing.
(2) For each $t > 0$, $\phi(t) < t$
(3) $\int_0^1 \phi(t)\,dt \leq \phi(\int_0^1 t\,dt)$.
(4) $\beta(t) = \frac{\phi(t)}{t} \in \mathbf{S}$.

For example, $\phi(t) = \frac{1}{5}t$, $\phi(t) = \frac{t}{t+1}$ are elements of Φ .

Let us consider the following integral equation:

$$j(w) = g(w) + \int_0^1 G(w,s,j(s))\,ds \;\; \forall\; w \in [0,1]. \tag{3.1}$$

To show the existence of solution of integral equation (3.1), we need following lemma

Lemma 3.1. *Let $\mathbb{B} = \bar{B}(0,\rho) = \{j : \; j \in L^2([0,1],\mathbb{R}) \; ; \; \|j\| \leq \rho\}$.
Assume following hypotheses are satisfied:*

(1) $g \in L^2([0,1],\mathbb{R})$

(2) $G : [0,1] \times [0,1] \times L^2([0,1],\mathbb{R}) \to \mathbb{R}$.

(4) $|G_n(w,s,j)| \leq f(w,s) + \upsilon|j|$ *where* $f \in L^2([0,1]\times[0,1])$ *and* $\upsilon < \frac{1}{2}$.

Then operator T defined by

$$(Tk)(w) = g(w) + \int_0^1 \tilde{G}(w)(k)(s)\,ds$$

satisfies $T(\mathbb{B}) \subset \mathbb{B}$.

Proof. We begin by defining the operator $\tilde{G}(w)(k)(s) = G_n(w,s,k(s))$.

$$\begin{aligned}
\|Tj\|^2_{L^2([0,1],\mathbb{R})} &= \int_0^1 |Tj(w)|^2\,dw. \\
&= \int_0^1 (|g(w) + \int_0^1 \tilde{G}(w)(j)(s)\,ds|^2)\,dw. \\
&\leq 2\int_0^1 |g(w)|^2\,dw + 2\int_0^1\int_0^1 |\tilde{G}(w)(j)(s)|^2\,dsdw. \\
&\leq 2\int_0^1 |g(w)|^2\,dw + 2\int_0^1\int_0^1 |f(w,s) + \upsilon|j(s)||^2\,dsdw \\
&\leq 2\int_0^1 |g(w)|^2\,dw + 4\int_0^1\int_0^1 |f(w,s)|^2\,dsdw \\
&+ 4\upsilon^2\|j\|^2_{L^2([0,1],\mathbb{R})}. \\
&\leq 2\int_0^1 |g(w)|^2\,dw + 4\int_0^1\int_0^1 f^2(w,s)\,dsdw + 4\upsilon^2\rho^2.
\end{aligned}$$

Since $\upsilon < \frac{1}{2}$, choose ρ such that

$$\frac{2}{1-4\upsilon^2}\int_0^1 |g(w)|^2\,dw + \frac{4}{1-4\upsilon^2}\int_0^1\int_0^1 f^2(w,s)\,dsdw \leq \rho^2$$

This implies that $T(j) \in \mathbb{B}$, hence $T(\mathbb{B}) \subset \mathbb{B}$. □

Now we are in position to state our result regarding application

Theorem 3.2. *Assume that the following hypotheses are satisfied:*

(1) *The conditions supposed in Lemma 3.1.*

(2) *As* $n \to \infty$ $G_n(w,s,j) - G_n(w,s,k) \leq \phi(j-k)$, *for all comparable* $j,k \in M$,

Then integral equation (3.1) has a solution.

Proof. Let $M = L^2([0,1],\mathbb{R})$ and $D(j,k) = d(j,k) + c_n$ for all $j,k \in M$, where $d(j,k) = \|j-k\|_M$ and $\{c_n\}$ is a sequence of real numbers satisfying, $|c_n| \to 0$ as $n \to \infty$. Suppose that $T : M \to M$ be a mapping defined by

$$(Tk)(w) = g(w) + \int_0^1 \tilde{G}(w)(k)(s)\,ds.$$

We introduce a partial ordering on M, setting

$$u_1 \preceq u_2 \iff u_1(w) \leq u_2(w) \text{ for all } w \in [0,1].$$

Then $(M, \preceq, D)$ is a complete ordered dualistic partial metric space. Notice that T is well-defined and (3.1) has a solution if and only if the operator T has a fixed point. Precisely, we have to show that our Theorem 2.5 is applicable to the operator T. Then, for all comparable $j,k \in M$, we write

$$\begin{aligned}
|D(T(j),T(k))|^2 &= |d(T(j),T(k)) + c_n|^2 \\
&\leq |d(T(j),T(k))|^2 + |c_n|^2 + 2|d(T(j),T(k))||c_n| \\
&\leq \|T(j)-T(k)\|^2 + |c_n|^2 + 2|d(T(j),T(k))||c_n| \\
&\leq \int_0^1 (\int_0^1 \tilde{G}(w)(j)(s) - \tilde{G}(w)(k)(s)\,ds)^2 dw \\
&+ |c_n|^2 + 2|d(Tj,Tk)||c_n| \\
&\leq \int_0^1 (\int_0^1 G_n(w,s,j) - G_n(w,s,k)\,ds)^2 dw + |c_n|^2 \\
&+ 2|d(T(j),T(k))||c_n| \\
&\leq \int_0^1 (\int_0^1 \phi(j(s)-k(s))\,ds)^2 dw \text{ as } n \to \infty \\
&\leq \phi^2(\int_0^1 (j(s)-k(s))^2\,ds). \\
\text{It follows } |D(T(j),T(k))|^2 &\leq [\phi(|D(j,k)|)]^2 \\
|D(T(j),T(k))| &\leq \phi(|D(j,k)|) \\
&\leq \phi(N(j,k)) = \frac{\phi(N(j,k))}{N(j,k)} N(j,k)
\end{aligned}$$

This implies

$$|D(T(j),T(k))| \leq \beta(N(j,k))N(j,k)$$

. Hence T satisfies all the conditions of Theorem 2.5, so it has a fixed point and hence (3.1) has a solution. □

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