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FURTHER GENERALIZATION OF KOBAYASHI'S GAMMA k-FUNCTIONS

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Abstract. In this present paper, we deal with the more generalized functions involving Gauss hypergeometric k-functions $F_k((a;k),(b;k);(c,k);z)$

$$D_k\left(\begin{array}{c}(a;k),(b;k);(c,k),p\\ u,v,\delta\end{array}\right)$$
$$= v^{\frac{-a}{k}}\int_0^\infty t^{u-1}(1-k\frac{t}{v})^{\frac{\delta}{k}-1}e^{-\frac{pt^k}{k}}F_k((a;k),(b;k);(c,k);-\frac{t}{v})dt,$$

where $|\arg v|<\pi$, $Re(u)>0$. This function reduces to a generalized gamma k-function recently defined by "[Mubeen et al.; Jour. Ineq. Spec. Funct. Volume 7 Issue 1(2016), pp.53-60]" when $\delta=k$. Also, it reduces to gamma k-function Γ_k when $p=1$, $\delta=k$ and $b=c$. In this paper, we obtain several properties for the newly defined generalized function D_k, its recurrence relations and asymptotic expansion for D_k where $k>0$. We establish further generalization of Kobayashi's gamma k-function i.e., $\Gamma_{m,k}(u,v)$, which is an improved version of Kobayashi's gamma function.

1. Introduction

Kobayashi [11] has considered plane wave diffraction theory by a strip using the Wiener-Hopf techniques and further he has established a complete high frequency asymptotic solution which is uniformly valid everywhere in space and which has no limitations on incidence and observation angles. The asymptotic solutions established by him can predict a high frequency field behavior everywhere in space. Whereas dealing with the asymptotic expansion of a class of branch-cut integrals arises in diffraction theory related to Wiener-Hopf techniques, Kobayashi [12] also established integrals of the following form

$$\Gamma_m(u,v)=\int_0^\infty \frac{t^{u-1}e^{-t}}{(t+v)^m}dt, \quad |\arg v|<\pi, Re(u)>0, \tag{1.1}$$

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where $m \in \mathbb{Z}^+$, the function $\Gamma_m(u,v)$ is the generalized gamma function. When $m = 0$, (1.1) reduces to the well known gamma function

$$\Gamma(u) = \int\limits_0^\infty t^{u-1}e^{-t}dt, Re(u) > 0 \tag{1.2}$$

and he proved some analytical properties of $\Gamma_m(u,v)$ including regularity, asymptotic expansion and analytic continuation.

In (1998), Al-Musallam and Kalla [3] defined a more general function involving Gauss hypergeometric function $F(a,b;c;z)$ which is given by

$$D\begin{pmatrix} a,b;c,p \\ u,v \end{pmatrix} = v^{-a}\int\limits_0^\infty t^{u-1}e^{-pt}F(a,b;c,-\frac{t}{v})dt, |\arg v| < \pi, Re(u) > 0. \tag{1.3}$$

In the same paper, they investigated several properties for the function D including analytical continuation, recurrence formulae and computation for special values of parameters and some new properties of the generalized gamma function $\Gamma_m(u,v)$. For further detail about the function D, see ([1, 2, 5],[7],[8],[10],[13, 14, 17]).

In (2001), Galue *et al.* [9], have introduced further generalization of Kobayashi's gamma function and they defined

$$D\begin{pmatrix} a,b;c,p \\ u,v,\delta \end{pmatrix} = v^{-a}\int\limits_0^\infty t^{u-1}(1-\frac{t}{v})^{\delta-1}e^{-pt}F(a,b;c,-\frac{t}{v})dt, \tag{1.4}$$

where $|\arg v| < \pi$, $Re(p) > 0$ and $Re(u) > 0$. This reduces to Kobayashi's [11] generalized gamma function when $\delta = 1$, $p = 1$ and $b = c$. Also, it reduces to a more generalized function defined by the researchers [3, 4]. In the same paper, they have defined the generalized incomplete and the complementary incomplete functions associated with $D\begin{pmatrix} a,b;c,p \\ u,v,\delta \end{pmatrix}$ and introduced their asymptotic expansions.

Diaz and Pariguan [6] introduced the generalized k-gamma function as

$$\Gamma_k(x) = \lim_{n\to\infty}\frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}},\ k > 0, x \in \mathbb{C}\setminus kZ^- \tag{1.5}$$

and also gave the properties of said function. The Γ_k is one parameter deformation of the classical gamma function such that $\Gamma_k \to \Gamma$ as $k \to 1$. The Γ_k is based on the repeated appearance of the expression of the following form

$$\alpha(\alpha+k)(\alpha+2k)(\alpha+3k)...(\alpha+(n-1)k). \tag{1.6}$$

The function of the variable α given by the statement (1.6), denoted by $(\alpha)_{n,k}$, is called the Pochhammer k-symbol. We obtain the usual Pochhammer symbol $(\alpha)_n$ by taking $k = 1$. The definition given in relation (1.5), is the generalization of $\Gamma(x)$ and the integral form of Γ_k is given by

$$\Gamma_k(x) = \int_0^\infty t^{x-1}e^{-\frac{t^k}{k}}dt,\ Re(x) > 0. \tag{1.7}$$

From relation (1.7), we can easily show that

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma(\frac{x}{k}). \tag{1.8}$$

Recently Mubeen *et al.* [15] have defined the following contiguous function relations for hypergeometric k-functions as

$$(a-b)F_k(a,b;c;z) = aF_k(a+k,b;c;z) - bF_k(a,b+k;c;z), \tag{1.9}$$

$$(a-c+k)F_k(a,b;c;z) = aF_k(a+k,b;c;z) - (c-k)F_k(a,b;c-k;z). \tag{1.10}$$

$$(a+b-c)F_k(a,b;c;z) = a(1-kz)F_k(a+k,b;c;z) - (c-b)F_k(a,b-k;c;z). \tag{1.11}$$

$$(c-a-b)F_k(a,b;c;z) = (c-a)F_k(a-k,b;c;z) - b(1-kz)F_k(a,b+k;c;z). \tag{1.12}$$

$$(b-a)(1-kz)F_k(a,b;c;z) = (c-a)F_k(a-k,b;c;z) - (c-b)F_k(a,b-k;c;z). \tag{1.13}$$

$$(b-c+k)F_k(a,b;c;z) = bF_k(a,b+k;c;z) - (c-k)F_k(a,b;c-k;z). \tag{1.14}$$

Very recently Mubeen *et al.* [16] have defined the generalized gamma k-function

$$D_k\left(\begin{array}{c}(a;k),(b;k);(c;k),p\\ u,v\end{array}\right) = v^{\frac{-a}{k}}\int_0^\infty t^{u-1}e^{-p\frac{t^k}{k}}F_k[(a;k),(b;k);(c;k),-\frac{t}{v}]dt, \tag{1.15}$$

and investigate some of its properties. In the same paper, they have defined an improved version of Kobayashi's gamma k-function of the form

$$\Gamma_{a,k}(u,v) = \int_0^\infty \frac{t^{u-1}e^{-\frac{t^k}{k}}}{(v+kt)^{\frac{a}{k}}}dt, \quad |\arg v| < \pi, Re(u) > 0, k > 0.$$

2. Further Generalization of Gamma k-Functions

In this section, we introduce further generalization of gamma k-functions and some of their properties where $k > 0$.

Definition 2.1. Let a, b, c and p be complex parameters with $c \neq 0, -1, -2, \cdots$, $Re(p) > 0$ and ${}_2F_{1,k}(a,b;c;z)$ is Gauss hypergeometric k-function, then we define

$$D_k\left(\begin{array}{c}(a;k),(b;k);(c;k),p\\ u,v,\delta\end{array}\right)$$
$$= v^{\frac{-a}{k}}\int_0^\infty t^{u-1}(1-k\frac{t}{v})^{\frac{\delta}{k}-1}e^{-p\frac{t^k}{k}}F_k[(a;k),(b;k);(c;k),-\frac{t}{v}]dt, \tag{2.1}$$

where $v \in W_v = \{v \in \mathbb{C} : |\arg v| < \pi\}$, $u \in W_u = \{u \in \mathbb{C} : Re(u) > 0\}$ and $F_k[(a;k),(b;k);(c;k),-\frac{t}{v}]$ is the Gauss hypergeometric k-functions. Obviously if $\delta = k$, then it reduces to $D_k\left(\begin{array}{c}(a;k),(b;k);(c;k),p\\ u,v\end{array}\right)$ (see [16]). Now we prove the

generalize Kobayashi's gamma k-function $\Gamma_{m,k}(u,v)$.
Theorem 2.1. If $k>0$, then

$$\Gamma_{a,k}(u,v)=\int_0^\infty \frac{t^{u-1}e^{-\frac{t^k}{k}}}{(v+kt)^{\frac{a}{k}}}dt, \quad |\arg v|<\pi, Re(u)>0. \tag{2.2}$$

Proof. Consider

$$D_k\left(\begin{array}{c}(a;k),(b;k);(c;k),p\\ u,v,\delta\end{array}\right)=v^{\frac{-a}{k}}\int_0^\infty t^{u-1}(1-k\frac{t}{v})^{\frac{\delta}{k}-1}e^{-p\frac{t^k}{k}}F_k[(a;k),(b;k);(c;k),-\frac{t}{v}]dt.$$

Now by putting $p=1$, $\delta=k$ and $b=c$, we get

$$D_k\left(\begin{array}{c}(a;k),(b;k);(b;k),1\\ u,v,k\end{array}\right)=v^{\frac{-a}{k}}\int_0^\infty t^{u-1}e^{-\frac{t^k}{k}}F_k[(a;k),(b;k);(b;k),-\frac{t}{v}]dt. \tag{2.3}$$

Now since $F_k[(a;k),(b;k);(b;k),-\frac{t}{v}]=(1+k\frac{t}{v})^{\frac{-a}{k}}$. By substituting this result in (2.3), we have

$$\begin{aligned} D_k\left(\begin{array}{c}(a;k),(b;k);(b;k),1\\ u,v,k\end{array}\right) &= v^{\frac{-a}{k}}\int_0^\infty t^{u-1}e^{-\frac{t^k}{k}}(1+k\frac{t}{v})^{\frac{-a}{k}}dt\\ &= \int_0^\infty \frac{t^{u-1}e^{-\frac{t^k}{k}}}{(v+kt)^{\frac{a}{k}}}dt\\ &= \Gamma_{a,k}(u,v),\end{aligned}$$

which is an improved version of Kobayashi's gamma function. The substitution of $k=1$ in (2.2) will reduces to Kobayashi's gamma function $\Gamma_a(u,v)$.
Remark. If $a=0$, then (2.2) becomes gamma k-function which is given by

$$\Gamma_k(u)=\int_0^\infty t^{u-1}e^{-\frac{t^k}{k}}dt. \tag{2.4}$$

Similarly if $a=0$ and $k=1$, then (2.2) becomes the classical gamma function defined by (1.2).
Proposition 2.1. Let a,b,c and p be complex numbers, then D can be represented as

$$D_k\left(\begin{array}{c}(a;k),(b;k);(c;k),p\\ u,v,\delta\end{array}\right)=v^{u-\frac{a}{k}}\int_0^\infty s^{u-1}(1-ks)^{\frac{\delta}{k}-1}e^{-p\frac{(vs)^k}{k}}F_k[(a,k),(b,k);(c,k);-s]ds, \tag{2.5}$$

where $v\in W_v$ and $u\in W_u$.
Proof. Assume that $v>0$ and by changing the variable of integration $s=\frac{t}{v}$ in

(2.1), we have

$$\begin{aligned} D_k\begin{pmatrix} (a;k),(b;k);(c;k),p \\ u,v,\delta \end{pmatrix} &= v^{-\frac{a}{k}}\int_0^\infty (sv)^{u-1}(1-ks)^{\frac{\delta}{k}-1}e^{-p\frac{(vs)^k}{k}}F_k[(a,k),(b,k);(c,k);-s]vds \\ &= v^{u-\frac{a}{k}}\int_0^\infty (s)^{u-1}(1-ks)^{\frac{\delta}{k}-1}e^{-p\frac{(vs)^k}{k}}F_k[(a,k),(b,k);(c,k);-s]ds, \end{aligned}$$

which completes the proof.

Theorem 2.2 Let a, b, c and p be complex numbers, then

$$\frac{d^n}{du^n}D_k = v^{\frac{-a}{k}}\int_0^\infty t^{u-1}(1-k\frac{t}{v})^{\frac{\delta}{k}-1}e^{-p\frac{t^k}{k}}(\ln t)^n F_k[(a;k),(b;k);(c;k),-\frac{t}{v}]dt$$

Proof Consider

$$\frac{d^n}{du^n}D_k = v^{\frac{-a}{k}}\int_0^\infty \frac{d^n}{du^n}t^{u-1}(1-k\frac{t}{v})^{\frac{\delta}{k}-1}e^{-p\frac{t^k}{k}}F_k[(a;k),(b;k);(c;k),-\frac{t}{v}]dt. \quad (2.6)$$

Now since $\frac{d^n}{du^n}t^{u-1} = t^{u-1}(\ln t)^n$, thus by substituting this result in (2.6), we get the required result.

3. The Generalized Incomplete and Complementary Incomplete Functions of D_k.

In this section, we introduce the generalized incomplete and complementary incomplete gamma k-functions of $D_k\begin{pmatrix} (a;k),(b;k);(c;k),p \\ u,v,\delta \end{pmatrix}$. If $\omega > 0$, then we defined the generalized incomplete gamma k-function as

$$\begin{aligned} &D_{0,k}^{\omega}\begin{pmatrix} (a;k),(b;k);(c;k),p \\ u,v,\delta \end{pmatrix} \\ &\qquad = v^{\frac{-a}{k}}\int_0^\omega t^{u-1}(1-k\frac{t}{v})^{\frac{\delta}{k}-1}e^{-p\frac{t^k}{k}}F_k[(a;k),(b;k);(c;k),-\frac{t}{v}]dt, \quad (3.1) \end{aligned}$$

where $v \in W_v = \{v \in \mathbb{C} : |\arg v| < \pi\}$, $u \in W_u = \{u \in \mathbb{C} : Re(u) > 0\}$, c are complex parameters with $c \neq 0, 1, 2, \cdot$ and $F_k[(a;k),(b;k);(c;k),-\frac{t}{v}]$ is the Gauss hypergeometric k-functions, and the generalized complementary incomplete gamma k-function is given by

$$\begin{aligned} &D_{\omega,k}^{\infty}\begin{pmatrix} (a;k),(b;k);(c;k),p \\ u,v,\delta \end{pmatrix} \\ &\qquad = v^{\frac{-a}{k}}\int_\omega^\infty t^{u-1}(1-k\frac{t}{v})^{\frac{\delta}{k}-1}e^{-p\frac{t^k}{k}}F_k[(a;k),(b;k);(c;k),-\frac{t}{v}]dt, \quad (3.2) \end{aligned}$$

with the same conditions given in (3.1).

Thus, this implies that

$$D_k\begin{pmatrix} (a;k),(b;k);(c;k),p \\ u,v,\delta \end{pmatrix} = D_{0,k}^{\omega}\begin{pmatrix} (a;k),(b;k);(c;k),p \\ u,v,\delta \end{pmatrix} + D_{\omega,k}^{\infty}\begin{pmatrix} (a;k),(b;k);(c;k),p \\ u,v,\delta \end{pmatrix}.$$

For $\delta = k$, $p = 1$, $b = c$ and $a = m$ where $m \in \mathbb{Z}^+$ and using the relation for Gauss hypergeometric function [6]

$$F_k[(a;k),(b;k);(b;k),z] = (1-kz)^{\frac{-a}{k}}, \quad \arg|1-kz| < \pi,$$

then (3.2) reduces to the generalized incomplete gamma k-function discussed in section 2 .

$$\Gamma_{m,k}(u,v,\omega) = \int\limits_{\omega}^{\infty} \frac{t^{u-1}e^{-\frac{t^k}{k}}}{(v+kt)^{\frac{m}{k}}} dt. \tag{3.3}$$

If $m = 0$, then this result reduces to the well known complementary incomplete gamma k-function [6]

$$\Gamma_k(u,\omega) = \int\limits_{\omega}^{\infty} t^{u-1}e^{-\frac{t^k}{k}} dt, \tag{3.4}$$

and if $k = 1$ this result reduces to the classical incomplete gamma function

$$\Gamma(u,\omega) = \int\limits_{\omega}^{\infty} t^{u-1}e^{-t} dt.$$

On the other hand if $\delta = k$, $p = 1$, $b = c$ and $a = 0$, then (3.1) reduces to the classical incomplete gamma function

$$\gamma(u,\omega) = \int\limits_{0}^{\omega} t^{u-1}e^{-t} dt.$$

4. Relations Involving The Function D_k

In this section, we derive some recurrence relations which containing more generalized gamma k-function $D_k\begin{pmatrix} (a;k),(b;k);(c;k),p \\ u,v,\delta \end{pmatrix}$. The function $D_k\begin{pmatrix} (a;k),(b;k);(c;k),p \\ u,v,\delta \end{pmatrix}$ is not symmetric with respect to the parameters a and b. In fact,

$$D_k\begin{pmatrix} (b;k),(a;k);(c;k),p \\ u,v,\delta \end{pmatrix} = v^{\frac{a-b}{k}} D_k\begin{pmatrix} (a;k),(b;k);(c;k),p \\ u,v,\delta \end{pmatrix}. \tag{4.1}$$

Proposition 3.1. The following relations hold for $k > 0$;

$$(b-a)D_k\begin{pmatrix} (a;k),(b;k);(c;k),p \\ u,v,\delta \end{pmatrix} + avD_k\begin{pmatrix} (a+k;k),(b;k);(c;k),p \\ u,v,\delta \end{pmatrix} - bD_k\begin{pmatrix} (a;k),(b+k;k);(c;k),p \\ u,v,\delta \end{pmatrix} = 0, \tag{4.2}$$

$$(c-a-k)D_k\begin{pmatrix} (a;k),(b;k);(c;k),p \\ u,v,\delta \end{pmatrix} + avD_k\begin{pmatrix} (a+k;k),(b;k);(c;k),p \\ u,v,\delta \end{pmatrix} - (c-k)D_k\begin{pmatrix} (a;k),(b;k);(c-k;k),p \\ u,v,\delta \end{pmatrix} = 0, \tag{4.3}$$

$$(c-a-b)D_k\left(\begin{array}{c}(a;k),(b;k);(c;k),p\\ u,v,\delta\end{array}\right)-(c-b)D_k\left(\begin{array}{c}(a;k),(b-k;k);(c;k),p\\ u,v,\delta\end{array}\right)$$
$$+avD_k\left(\begin{array}{c}(a+k;k),(b;k);(c;k),p\\ u,v,\delta\end{array}\right)+akD_k\left(\begin{array}{c}(a+k;k),(b;k);(c;k),p\\ u+1,v,\delta\end{array}\right)=0, \tag{4.4}$$

$$(c-b-a)D_k\left(\begin{array}{c}(a;k),(b;k);(c;k),p\\ u,v\end{array}\right)-(c-a)D_k\left(\begin{array}{c}(a-k;k),(b;k);(c;k),p\\ u,v,\delta\end{array}\right)$$
$$+bvD_k\left(\begin{array}{c}(a;k),(b+k;k);(c;k),p\\ u,v,\delta\end{array}\right)+bkD_k\left(\begin{array}{c}(a;k),(b+k;k);(c;k),p\\ u+1,v,\delta\end{array}\right)=0, \tag{4.5}$$

$$(b-a)vD_k\left(\begin{array}{c}(a;k),(b;k);(c;k),p\\ u,v,\delta\end{array}\right)-(c-a)D_k\left(\begin{array}{c}(a-k;k),(b;k);(c;k),p\\ u,v,\delta\end{array}\right)$$
$$+(b-a)kD_k\left(\begin{array}{c}(a;k),(b;k);(c;k),p\\ u+1,v,\delta\end{array}\right)+(c-b)vD_k\left(\begin{array}{c}(a;k),(b-k;k);(c;k),p\\ u,v\delta\end{array}\right)=0, \tag{4.6}$$

$$(c-b-k)D_k\left(\begin{array}{c}(a;k),(b;k);(c;k),p\\ u,v,\delta\end{array}\right)+bvD_k\left(\begin{array}{c}(a;k),(b+k;k);(c;k),p\\ u,v,\delta\end{array}\right)$$
$$-(c-k)D_k\left(\begin{array}{c}(a;k),(b;k);(c-k;k),p\\ u,v,\delta\end{array}\right)=0. \tag{4.7}$$

Proof. To prove the relation (4.2), we use (1.9) by taking $F_k[a,b;c,z]=F_k[(a;k),(b;k);(c;k),z]$ as

$$(b-a)F_k[(a;k),(b;k);(c;k),z]+aF_k[(a+k;k),(b;k);(c;k),z]-bF_k[(a;k),(b+k;k);(c;k),z]=0.$$

By putting $z=\frac{-t}{v}$ in above relation, we have

$$(b-a)D_k\left(\begin{array}{c}(a;k),(b;k);(c;k),p\\ u,v,\delta\end{array}\right)$$

$$\begin{aligned}
&= v^{\frac{-a}{k}}\int_0^\infty t^{u-1}(1-k\frac{t}{v})^{\frac{\delta}{k}-1}e^{\frac{-pt^k}{k}}\times[-aF_k[(a+k,k),(b,k);(c,k);-\frac{t}{v}]\\
&+ bF_k[(a,k),(b+k,k);(c,k);-\frac{t}{v}]]dt\\
&= -a[v^{-(\frac{a}{k}-1)}\int_0^\infty t^{u-1}(1-k\frac{t}{v})^{\frac{\delta}{k}-1}e^{\frac{-pt^k}{k}}\times F_k[(a+k;k),(b;k);(c;k),-\frac{t}{v}]]dt\\
&+ bv^{\frac{-a}{k}}\int_0^\infty t^{u-1}(1-k\frac{t}{v})^{\frac{\delta}{k}-1}e^{\frac{-pt^k}{k}}F_k[(a;k),(b+k;k);(c;k),-\frac{t}{v}]dt\\
&= -avD_k\left(\begin{array}{c}(a+k;k),(b;k);(c;k),p\\ u,v\end{array}\right)+bD_k\left(\begin{array}{c}(a;k),(b+k;k);(c;k),p\\ u,v\end{array}\right),
\end{aligned}$$

which is the required relation. On the similar way we can prove relation (4.3). To prove relation (4.4), we use (1.11) as;

$$\begin{aligned}(a+b-c)F_k[(a;k),(b;k);(c;k);z] &= a(1-kz)F_k[(a+k;k),(b;k);(c;k);z) \\ &- (c-b)F_k[(a;k),(b-k;k);(c;k);z].\end{aligned}$$

with $z=-\frac{t}{v}$. Now, we have

$$(c-a-b)D_k\left(\begin{array}{c}(a;k),(b;k);(c;k),p\\ u,v,\delta\end{array}\right)$$

$$\begin{aligned}
&= v^{\frac{-a}{k}}\int\limits_0^\infty t^{u-1}(1-k\frac{t}{v})^{\frac{\delta}{k}-1}e^{\frac{-pt^k}{k}}\times(c-a-b)F_k[(a+k,k),(b,k);(c,k);-\frac{t}{v}]dt\\
&= v^{\frac{-a}{k}}\int\limits_0^\infty t^{u-1}(1-k\frac{t}{v})^{\frac{\delta}{k}-1}e^{\frac{-pt^k}{k}}(c-b)F_k[(a,k),(b-k,k);(c,k);-\frac{t}{v}]\\
&- a(1+k\frac{t}{v})bF_k[(a+k,k),(b,k);(c,k);-\frac{t}{v}]]dt\\
&= (c-b)[v^{\frac{-a}{k}}\int\limits_0^\infty t^{u-1}(1-k\frac{t}{v})^{\frac{\delta}{k}-1}e^{\frac{-pt^k}{k}}\times F_k[(a;k),(b-k;k);(c;k),-\frac{t}{v}]]dt\\
&+ av^{-(\frac{a}{k}-1)}\int\limits_0^\infty t^{u-1}(1-k\frac{t}{v})^{\frac{\delta}{k}-1}e^{\frac{-pt^k}{k}}F_k[(a+k;k),(b;k);(c;k),-\frac{t}{v}]dt\\
&- akv^{-(\frac{a}{k})}\int\limits_0^\infty t^{u}(1-k\frac{t}{v})^{\frac{\delta}{k}-1}e^{\frac{-pt^k}{k}}F_k[(a+k;k),(b;k);(c;k),-\frac{t}{v}]dt\\
&= (c-b)vD_k\left(\begin{array}{c}(a;k),(b-;k);(c;k),p\\ u,v,\delta\end{array}\right)-avD_k\left(\begin{array}{c}(a+k;k),(b;k);(c;k),p\\ u,v,\delta\end{array}\right)\\
&- akD_k\left(\begin{array}{c}(a+k;k),(b;k);(c;k),p\\ u+1,v,\delta\end{array}\right),
\end{aligned}$$

which completes the proof. On the similar method, we can prove relation (4.5), (4.6)and (4.7) by using (1.12), (1.13) and (1.14) respectively.

5. Asymptotic Expansions for Generalized Gamma k-Functions

In this section, we introduce the asymptotic expansion for $D_k\left(\begin{array}{c}(a;k),(b;k);(c;k),p\\ u,v,\delta\end{array}\right)$, $D^{\omega}_{0,k}\left(\begin{array}{c}(a;k),(b;k);(c;k),p\\ u,v,\delta\end{array}\right)$ and $D^{\infty}_{\omega,k}\left(\begin{array}{c}(a;k),(b;k);(c;k),p\\ u,v,\delta\end{array}\right)$. We assume that $v\to\infty$ and using the expansion in series of the Gauss hypergeometric k-function

$${}_2F_{1,k}[a,b;c;z]=\sum_{n=0}^{\infty}\frac{(a)_{n,k}(b)_{n,k}}{(c)_{n,k}}\frac{z^n}{n!},\quad |z|<1,k>0,$$

where $z = -\frac{t}{v}$. After substitution in (2.1) and interchanging the order of integral and sum, we obtain

$$D_k\left(\begin{array}{c}(a;k),(b;k);(c;k),p\\ u,v,\delta\end{array}\right) = v^{-\frac{a}{k}}\sum_{n=0}^{\infty}\frac{(-1)^n(a)_{n,k}(b)_{n,k}}{(c)_{n,k}n!v^n}\int_0^{\infty}t^{n+u-1}(1-k\frac{t}{v})^{\frac{\delta}{k}-1}e^{-p\frac{t^k}{k}}dt. \tag{5.1}$$

Now since

$$(1-kz)^{-\frac{a}{k}} = \sum_{m=0}^{\infty}\frac{(a)_{m,k}}{m!}z^m, \quad |z|<1$$

by substituting in (5.1), we have

$$\begin{aligned}D_k\left(\begin{array}{c}(a;k),(b;k);(c;k),p\\ u,v,\delta\end{array}\right) &= v^{-\frac{a}{k}}\sum_{n=0}^{\infty}\frac{(-1)^n(a)_{n,k}(b)_{n,k}}{(c)_{n,k}n!v^n}\sum_{m=0}^{\infty}\frac{(k-\delta)_{m,k}}{m!v^m}\\ &\times\int_0^{\infty}t^{n+m+u-1}e^{-p\frac{t^k}{k}}dt.\end{aligned}$$

Substituting $y = pt^k$ and $dt = \frac{1}{kp^{\frac{1}{k}}}y^{\frac{1}{k}-1}$ in above equation, we get

$$\begin{aligned}D_k\left(\begin{array}{c}(a;k),(b;k);(c;k),p\\ u,v,\delta\end{array}\right) = \frac{v^{-\frac{a}{k}}}{p^{\frac{u}{k}}}\sum_{n=0}^{\infty}\frac{(-1)^n(a)_{n,k}(b)_{n,k}}{(c)_{n,k}n!}\sum_{m=0}^{\infty}\frac{(k-\delta)_{m,k}}{m!p^{\frac{m+n}{k}}v^{n+m}}\\ \times\frac{1}{k}\int_0^{\infty}y^{\frac{n+m+u}{k}-1}e^{-\frac{y}{k}}dy.\end{aligned} \tag{5.2}$$

Again by variable substitution the integral part becomes

$$\begin{aligned}\frac{1}{k}\int_0^{\infty}y^{\frac{n+m+u}{k}-1}e^{-\frac{y}{k}}dy &= \int_0^{\infty}t^{n+m+u-1}e^{-\frac{t^k}{k}}dt\\ &= \Gamma_k(n+m+u).\end{aligned}$$

Using this result in (5.2), we get

$$D_k\left(\begin{array}{c}(a;k),(b;k);(c;k),p\\ u,v,\delta\end{array}\right) = \frac{v^{-\frac{a}{k}}}{p^{\frac{u}{k}}}\sum_{n=0}^{\infty}\frac{(-1)^n(a)_{n,k}(b)_{n,k}}{(c)_{n,k}n!v^np^{\frac{n}{k}}}\sum_{m=0}^{\infty}\frac{(k-\delta)_{m,k}}{m!p^{\frac{m}{k}}v^m}\Gamma_k(n+m+u), \tag{5.3}$$

which is the required asymptotic expansion for the generalized gamma k-function. Following the same procedure, we obtain the asymptotic expansion of generalized incomplete gamma k-function from definition (3.1)

$$\begin{aligned}D_k\left(\begin{array}{c}(a;k),(b;k);(c;k),p\\ u,v,\delta\end{array}\right) = \frac{v^{-\frac{a}{k}}}{p^{\frac{u}{k}}}\sum_{n=0}^{\infty}\frac{(-1)^n(a)_{n,k}(b)_{n,k}}{(c)_{n,k}n!}\sum_{m=0}^{\infty}\frac{(k-\delta)_{m,k}}{m!p^{\frac{m+n}{k}}v^{n+m}}\\ \times\int_0^{p\omega}t^{n+m+u-1}e^{-\frac{t^k}{k}}dt,\end{aligned} \tag{5.4}$$

and by using the result (5.5), this can be written as

$$D_{0,k}^{\omega}\left(\begin{array}{c}(a;k),(b;k);(c;k),p\\ u,v,\delta\end{array}\right)=\frac{v^{-\frac{a}{k}}}{p^{\frac{u}{k}}}\sum_{n=0}^{\infty}\frac{(-1)^n(a)_{n,k}(b)_{n,k}}{(c)_{n,k}n!v^np^{\frac{n}{k}}}\sum_{m=0}^{\infty}\frac{(k-\delta)_{m,k}}{m!p^{\frac{m}{k}}v^m}\gamma_k(n+m+u,p\omega). \tag{5.5}$$

Similarly, we can find the asymptotic expansion of complementary incomplete gamma k-function. From definition (3.2), we have

$$D_{\omega,k}^{\infty}\left(\begin{array}{c}(a;k),(b;k);(c;k),p\\ u,v,\delta\end{array}\right)=\frac{v^{-\frac{a}{k}}}{p^{\frac{u}{k}}}\sum_{n=0}^{\infty}\frac{(-1)^n(a)_{n,k}(b)_{n,k}}{(c)_{n,k}n!}\sum_{m=0}^{\infty}\frac{(k-\delta)_{m,k}}{m!p^{\frac{m+n}{k}}v^{n+m}}\times\int_{p\omega}^{\infty}t^{n+m+u-1}e^{-\frac{t^k}{k}}dt$$

$$=\frac{v^{-\frac{a}{k}}}{p^{\frac{u}{k}}}\sum_{n=0}^{\infty}\frac{(-1)^n(a)_{n,k}(b)_{n,k}}{(c)_{n,k}n!v^np^{\frac{n}{k}}}\sum_{m=0}^{\infty}\frac{(k-\delta)_{m,k}}{m!p^{\frac{m}{k}}v^m}\Gamma_k(n+m+u,p\omega). \tag{5.6}$$

6. Conclusion and Summary

In this paper, the aim of the authors is to consider further generalization of the gamma k-function. Also, to obtain some properties, recurrence relations involving generalized gamma k-function and establish the asymptotic expansions for the said functions in term of k where $k>0$. Obviously the substitution of $k=1$ will lead to the classical results of the said functions.

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References

[1] I. Ali, S.L. Kalla, H.G. Khajah, *A generalized inverse gaussian ditribution with τ-confluent hypergeometric function*, Integral Transforms and Special Function, 12(2), (2001), pp. 101-114.

[2] S.K. Agarwal and S.L. Kalla, *A generalized gamma distribution and its applications in reliability*, Commun. Statist. Theory Meth. 25(1), (1996), pp. 201-210 .

[3] F. AL-Musallam, S.L. Kalla, *Further result on generalized gamma function occurring in diffraction theory*, Integral Transforms and special functions, 7, 1998, pp. 175-190.

[4] F. AL-Musallam, S.L. Kalla, *Asymptotic expansion for generalized gamma and incomplete gamma functions*, Appli. Anal. 66(1997), pp. 173-187.

[5] Al-Zamel, S.L. Kalla , *On a relation between integral transforms and the Kober operator of fractional integration*, Hadronic Journal 18(1995), pp. 563-575.

[6] R. Diaz and E. Pariguan, *On hypergeometric functions and k-Pochhammer symbol*, Divulgaciones Mathematics Vol, 15 No.2(2007), pp. 179-192.

[7] A. Erdelyi, *Higher Transcendental Functions,* I, McGraw-Hill, New York, 1953.

[8] S.L. Kalla, *On the evaluation of the Gauss hypergeometric functions*, Comp. Rend de l'Acad. Bulgare Des Sci. 45(1992), pp. 35-36.

[9] L. Galue, G. Alobaidi, S.L. Kalla, *Further generalization of Kobayash'i gamma function*, Serdica Math. J. 27(2001), pp. 303-316.

[10] H.G. Khaja, *Tau method approximation of the Gauss hypergeometric function* , Comp. Rend de l'Acad. Bulgare Des Sci. 50(1997), pp. 13-16.

[11] K. Kobayashi, *On generalized gamma function occurring in diffraction theory*, J. Phy. Soc. Japan, 60, 5(1991), pp. 1501-1512. (1989).
[12] K. Kobayashi, *Plane wave diffraction theory by a strip: exact and asymptotic solutions*, J. Phy. Soc. Japan, 60, 6(1991), pp. 1891-1905.
[13] N.N. Lebedev, *Special Functions and their applications*, Dover Publications, Inc. New York,1965.
[14] Y.L. Luke, *Mathematical functions and their approximations*, Academic Press, 1975. (1989).
[15] S. Mubeen, G. Rahman, A. Rehman, and M. Naz, *Contiguous function relations for k-hypergeometric functions*,ISRN Mathematical Analysis Volume 2014, Article ID 410801, 6 pages.
[16] S. Mubeen, M. Arshad, G. Rahman, *k-Analogue of generalized gamma function*, J. Ineq. Spec. Funct. ISSN: 2217-4303, URL: http://www.ilirias.com Volume 7 Issue 1(2016), pp. 53-60.
[17] N. Virchenko, S.L. Kalla, Al-Zamel, *some results on a generalized hypergeometric function*, Integral Transforms and Special functions, 12(1), (2001) pp. 89-100.

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