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ANALOGUES OF SEVERAL IDENTITIES AND SUPERCONGRUENCES FOR THE CATALAN–QI NUMBERS

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ABSTRACT. In the paper, the author presents q-analogues of some identities of the Catalan–Qi numbers and finds supercongruences for some special cases of the Catalan–Qi numbers.

1. INTRODUCTION

The basic notation of this paper are the shifted factorial

$$(a)_0 = 1, \quad (a)_n = a(a+1)\cdots(a+n-1), \quad n = 1, 2, \dots$$

and the quantum factorial

$$(x;q)_0 = 1,$$
 $(x;q)_n := \prod_{k=0}^{n-1} (1 - xq^k),$ $n = 1, 2, \dots$

In combinatorial mathematics, the Catalan numbers form a sequence of natural numbers that occur in various counting problems, often involving recursivelydefined objects. The nth Catalan number can be given in terms of binomial coefficients by

$$C_n = \frac{(2n)!}{(n+1)!n!} = \frac{1}{n+1} \binom{2n}{n}, \quad n \ge 0.$$

Up to now, there are many generalizations of the Catalan numbers, including q-analogues of the Catalan numbers. For example, Andrews [1] defined

$$C_n(q) = \frac{(1-q)}{(1-q^{n+1})} \begin{bmatrix} 2n\\n \end{bmatrix}_q,$$

where

$$\begin{bmatrix} n \\ j \end{bmatrix}_q = \begin{cases} 0, & \text{if } j < 0 \text{ or } j > n \\ \frac{(q;q)_n}{(q;q)_j(q;q)_{n-j}}, & 0 \leqslant j \leqslant n \end{cases}$$

represents Gaussian polynomials (also known as $q\mbox{-binomial}$ coefficients).

It is easy to prove

$$\lim_{q \to 1} C_n(q) = C_n.$$

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Thus, $C_n(q)$ can be viewed as one kind of q-analogues of C_n . Moreover, Andrews [2] obtained

$$C_{n+1}(q) = \sum_{r=0}^{n} q^{2r^2 + 2r} \begin{bmatrix} n\\2r \end{bmatrix}_q C_r(q) \frac{(-q^{r+2};q)_{n-r}}{(-q;q)_r}$$

which is the q-analogue of the identity

$$C_{n+1} = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} {n \choose 2r} 2^{n-2r} C_r$$
(1.1)

by Touchard [3]. Basing on the works of Andrews, Zou [4] obtained several congruences from (1.1).

In this paper, we mainly focus on Qi's generalization of the Catalan numbers

$$C(a,b;n) = \left(\frac{b}{a}\right)^n \frac{(a)_n}{(b)_n}.$$
(1.2)

It is clear that

$$C\left(\frac{1}{2},2;n\right) = C_n, \quad n \ge 0.$$

Actually, the expression (1.2) is a special case of the function

$$C(a,b;z) = \frac{\Gamma(b)}{\Gamma(a)} \left(\frac{b}{a}\right)^z \frac{\Gamma(z+a)}{\Gamma(z+b)}, \qquad \Re(a), \Re(b) > 0, \quad \Re(z) \ge 0, \tag{1.3}$$

where $\Gamma(z)$ represents the gamma function which is defined by

$$\Gamma(z) = \int_0^\infty x^{z-1} \mathrm{e}^{-x} \mathrm{d}x, \quad \Re(z) > 0.$$

The formula (1.3) was firstly considered by Qi in [5]. Accordingly, the numbers C(a, b; n) defined by (1.2) were named as the Catalan–Qi numbers. There have been more and more researches on the Catalan–Qi numbers C(a, b; n) during recent years. See [6, 7, 8, 9, 10] and closely related references therein.

In this paper, we will discuss the q-analogues

$$C_n(a,b;q) := \left(\frac{b}{a}\right)^n \frac{(q^a;q)_n}{(q^b;q)_n}, \quad n \ge 0$$

of the Catalan–Qi numbers C(a, b; n). It is not difficult to prove that

$$\lim_{q \to 1} C_n(a, b; q) = C(a, b, n).$$

In Section 2, we will present q-analogues of some identities for the Catalan–Qi numbers C(a, b; n). In Section 3, we will establish some supercongruences concerning special Catalan–Qi numbers C(a, b; n).

2. The q-analogues of identities for Catalan-Qi numbers

In this section, we present q-analogues of several identities for the Catalan–Qi numbers C(a, b; n). It is straightforward to verify that

$$C(a,b;n) = \frac{1}{C(b,a;n)}$$
 and $C(a,b;n+1) = \frac{b}{a}\frac{a+n}{b+n}C(a,b;n).$

The first formula has been stated in [8, p. 2].

Similar to the Catalan–Qi numbers C(a, b; n), the q-analogues satisfy

$$C_n(a,b;q) = \frac{1}{C_n(b,a;q)}$$
 and $C_{n+1}(a,b;q) = \frac{b}{a} \frac{1-q^{a+n}}{1-q^{b+n}} C_n(a,b;q)$.

The proofs of these two identities are also straightforward.

Except for the two formulas mentioned above, we also have three identities

$${}_{2}F_{1}\left(a,1;b;\frac{bt}{a}\right) = \sum_{n=0}^{\infty} C(a,b;n)t^{n},$$
(2.1)

$${}_{2}F_{1}\left(a,-n;b;-\frac{b}{a}\right) = \sum_{r=0}^{n} \binom{n}{r} C(a,b;r),$$
(2.2)

and

$${}_{3}F_{2}\left(a,\frac{1-n}{2},-\frac{n}{2};b,\frac{1}{2};1\right) = \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor} \binom{n}{2k} \binom{a}{b}^{k} C(a,b;k)$$
(2.3)

concerning the Catalan–Qi numbers C(a, b; n), where $a, b > 0, n \ge 0$, and

$${}_{p}F_{q}(a_{1},\cdots,a_{p};b_{1},\cdots,b_{q};x) = \sum_{n=0}^{\infty} \frac{(a_{1})_{n}\cdots(a_{p})_{n}}{(b_{1})_{n}\cdots(b_{q})_{n}} \frac{x^{n}}{n!}.$$

The proofs of these three identities can be found in [8, Theorem 10] and [9, Theorem 1].

The identity (2.1) shows that the function ${}_{2}F_{1}\left(a, 1; b; \frac{bt}{a}\right)$ is a generating function of the Catalan–Qi numbers C(a, b; n) and that the identity (2.3) can be regarded as a generalization of the identity (1.1).

The above three identities can be generalized as follows.

Theorem 2.1. For a, b > 0, $n \ge 0$ and |q| < 1, we have

$${}_{2}\phi_{1}\left(q^{a},q;q^{b};q,\frac{bt}{a}\right) = \sum_{n=0}^{\infty} C_{n}(a,b;q)t^{n},$$
(2.4)

$${}_{2}\phi_{1}\left(q^{a},q^{-n};q^{b};q,-\frac{b}{a}\right) = \sum_{r=0}^{n} {n \brack r}_{q} C_{r}(a,b;q)q^{r(r-1)/2-nr},$$
(2.5)

and

$${}_{3}\phi_{2}(q^{a},q^{(1-n)/2},q^{-n/2};q^{b},q^{1/2};q,1) = \sum_{k=0}^{\lfloor n/2 \rfloor} {n \brack 2k}_{q} \left(\frac{a}{b}\right)^{k} C_{k}(a,b;q) \frac{(q^{-n/2};q^{1/2})_{2k}(q;q)_{2k}}{(q^{1/2};q^{1/2})_{2k}(q^{n-2k+1};q)_{2k}}, \quad (2.6)$$

where

$${}_{r}\phi_{s}(a_{1},a_{2},\cdots,a_{r};b_{1},b_{2},\cdots,b_{s};q,z) = \sum_{n=0}^{\infty} \frac{(a_{1},a_{2},\ldots,a_{r};q)_{n}}{(q,b_{1},b_{2},\ldots,b_{s};q)_{n}} \left[(-1)^{n} q^{\binom{n}{2}} \right]^{1+s-r} z^{n}$$

is the basic hypergeometric series.

Proof. According to the definition of basic hypergeometric series, we have

$${}_{2}\phi_{1}\left(q^{a},q;q^{b};q,\frac{bt}{a}\right) = \sum_{n=0}^{\infty} \frac{(q^{a};q)_{n}(q;q)_{n}}{(q^{b};q)_{n}(q;q)_{n}} \left(\frac{bt}{a}\right)^{n}$$

$$=\sum_{n=0}^{\infty}\frac{(q^a;q)_n}{(q^b;q)_n}\left(\frac{b}{a}\right)^n t^n = \sum_{n=0}^{\infty}C_n(a,b;q)t^n$$

and

$$\begin{split} {}_{2}\phi_{1}\left(q^{a},q^{-n};q^{b};q,-\frac{b}{a}\right) \\ &=\sum_{r=0}^{\infty}\frac{(q^{a};q)_{r}(q^{-n};q)_{r}}{(q^{b};q)_{r}(q;q)_{r}}\left(-\frac{b}{a}\right)^{r} \\ &=\sum_{r=0}^{\infty}\left(\frac{b}{a}\right)^{r}\frac{(q^{a};q)_{r}}{(q^{b};q)_{r}}(-1)^{r}\frac{(q^{-n};q)_{r}}{(q;q)_{r}} \\ &=\sum_{r=0}^{\infty}C_{r}(a,b;q)\frac{(-1)^{r}(1-q^{-n})(1-q^{-n+1})\cdots(1-q^{-n+r-1})}{(q;q)_{r}} \\ &=\sum_{r=0}^{\infty}C_{r}(a,b;q)\frac{(1-q^{n})(1-q^{n-1})\cdots(1-q^{n-r+1})}{(q;q)_{r}}q^{r(r-1)/2-nr} \\ &=\sum_{r=0}^{\infty}\left[\frac{n}{r}\right]_{q}C_{r}(a,b;q)q^{r(r-1)/2-nr} \\ &=\sum_{r=0}^{n}\left[\frac{n}{r}\right]_{q}C_{r}(a,b;q)q^{r(r-1)/2-nr}. \quad (\text{since } (q^{-n};q)_{r}=0 \text{ when } r>n) \end{split}$$

The formulas (2.4) and (2.5) follow.

By the definition of basic hypergeometric series, we can derive

$${}_{3}\phi_{2}(q^{a},q^{(1-n)/2},q^{-n/2};q^{b},q^{1/2};q,1) = \sum_{k=0}^{\infty} \frac{(q^{a};q)_{k}(q^{(1-n)/2};q)_{k}(q^{-n/2};q)_{k}}{(q;q)_{k}(q^{b};q)_{k}(q^{1/2};q)_{k}}.$$

Using the relations

$$(q^{(1-n)/2};q)_k = 0, \quad k > \lfloor \frac{n}{2} \rfloor, \quad n = 1, 3, 5, \dots$$

 $\quad \text{and} \quad$

$$(q^{-n/2};q)_k = 0, \quad k > \left\lfloor \frac{n}{2} \right\rfloor, \quad n = 2, 4, 6, \dots$$

we have

$${}_{3}\phi_{2}(q^{a},q^{(1-n)/2},q^{-n/2};q^{b},q^{1/2};q,1) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(q^{a};q)_{k}(q^{(1-n)/2};q)_{k}(q^{-n/2};q)_{k}}{(q;q)_{k}(q^{b};q)_{k}(q^{1/2};q)_{k}}.$$

For another side of (2.6),

$$\begin{split} &\sum_{k=0}^{\lfloor n/2 \rfloor} \left[\frac{n}{2k} \right]_q \left(\frac{a}{b} \right)^k C_k(a,b;q) \frac{(q^{-n/2};q^{1/2})_{2k}(q;q)_{2k}}{(q^{1/2};q^{1/2})_{2k}(q^{n-2k+1};q)_{2k}} \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(q;q)_n (q^{-n/2};q^{1/2})_{2k}(q;q)_{2k}(q^a;q)_k}{(q;q)_{2k}(q;q)_{n-2k}(q^{1/2};q^{1/2})_{2k}(q^{n-2k+1};q)_{2k}(q^b;q)_k} \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(q;q)_n (q^{-n/2};q)_k (q^{(1-n)/2};q)_k (q^a;q)_k}{(q;q)_{k-2k}(q^{1/2};q)_{k}(q^{n-2k+1};q)_{2k}(q^b;q)_k} \end{split}$$

$$=\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(q^{-n/2};q)_k (q^{(1-n)/2};q)_k (q^a;q)_k}{(q;q)_k (q^b;q)_k (q^{1/2};q)_k} \\ = {}_3\phi_2 (q^a, q^{(1-n)/2}, q^{-n/2};q^b, q^{1/2};q,1).$$

The proof of this theorem is complete.

3. Supercongruences of special Catalan–Qi numbers

Let us first look at four supercongruences as follows. If $p \geqslant 5$ is a prime, then

$$_{2}F_{1}\left(\frac{1}{2},\frac{1}{2};1;1\right)_{tr(p)} \equiv \left(\frac{-4}{p}\right) \pmod{p^{2}},$$
(3.1)

$$_{2}F_{1}\left(\frac{1}{3},\frac{2}{3};1;1\right)_{tr(p)} \equiv \left(\frac{-3}{p}\right) \pmod{p^{2}},$$
(3.2)

$$_{2}F_{1}\left(\frac{1}{4},\frac{3}{4};1;1\right)_{tr(p)} \equiv \left(\frac{-2}{p}\right) \pmod{p^{2}},$$
(3.3)

$$_{2}F_{1}\left(\frac{1}{6}, \frac{5}{6}; 1; 1\right)_{tr(p)} \equiv \left(\frac{-1}{p}\right) \pmod{p^{2}},$$
 (3.4)

where $\left(\frac{\cdot}{p}\right)$ represents the Legendre symbol and

$$_{p}F_{q}(a_{1},\cdots,a_{p};b_{1},\cdots,b_{q};x)_{tr(p)} = \sum_{n=0}^{p-1} \frac{(a_{1})_{n}\cdots(a_{p})_{n}}{(b_{1})_{n}\cdots(b_{q})_{n}} \frac{x^{n}}{n!}.$$

The proofs of these four supercongruences was first given by Eric Mortenson in [11]. Actually, these four supercongruences were conjectured by Rodriguez–Villegas for hypergeometric Calabi-Yau manifolds of dimension d = 1. The conjecture of Rodriguez-Villegas has far-reaching impact on mathematics, especially on q-series. The recent work should be the one given by Guo and Zeng [12].

Now, we can give the main theorem of this section.

Theorem 3.1. Let

$$A_n = \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdots \frac{1}{2} \cdot 1,$$

$$B_n = \frac{3n-1}{3n} \cdot \frac{3n-4}{3n-3} \cdots \frac{2}{3} \cdot 1,$$

$$D_n = \frac{4n-1}{4n} \cdot \frac{4n-5}{4n-4} \cdots \frac{3}{4} \cdot 1,$$

and

$$E_n = \frac{6n-1}{6n} \cdot \frac{6n-7}{6n-6} \cdots \frac{5}{6} \cdot 1.$$

If $p \ge 5$ is a prime, then

$$1 + \sum_{n=1}^{p-1} C\left(\frac{1}{2}, 1; n\right) \frac{A_n}{2^n} \equiv \left(\frac{-4}{p}\right) \pmod{p^2},$$

$$1 + \sum_{n=1}^{p-1} C\left(\frac{1}{3}, 1; n\right) \frac{B_n}{3^n} \equiv \left(\frac{-3}{p}\right) \pmod{p^2},$$

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$$1 + \sum_{n=1}^{p-1} C\left(\frac{1}{4}, 1; n\right) \frac{D_n}{4^n} \equiv \left(\frac{-2}{p}\right) \pmod{p^2},$$

$$1 + \sum_{n=1}^{p-1} C\left(\frac{1}{6}, 1; n\right) \frac{E_n}{6^n} \equiv \left(\frac{-1}{p}\right) \pmod{p^2}.$$

Proof. Firstly, we prove that

$$_{2}F_{1}\left(\frac{1}{2},\frac{1}{2};1;1\right) = 1 + \sum_{n=1}^{\infty} C\left(\frac{1}{2},1;n\right) \frac{A_{n}}{2^{n}}.$$

According to the definition, we can derive

$${}_{2}F_{1}\left(\frac{1}{2},\frac{1}{2};1;1\right) = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_{n}\left(\frac{1}{2}\right)_{n}}{(1)_{n}}\frac{1}{n!} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_{n}}{(1)_{n}} \cdot 2^{n} \cdot \frac{\left(\frac{1}{2}\right)_{n}}{2^{n} \cdot n!}$$
$$= 1 + \sum_{n=1}^{\infty} C\left(\frac{1}{2},1;n\right) \frac{(2n-1)!!}{(2n)!!}\frac{1}{2^{n}} = 1 + \sum_{n=1}^{\infty} C\left(\frac{1}{2},1;n\right) \frac{A_{n}}{2^{n}}.$$

Similarly, we can prove

$${}_{2}F_{1}\left(\frac{1}{3},\frac{2}{3};1;1\right) = 1 + \sum_{n=1}^{\infty} C\left(\frac{1}{3},1;n\right) \frac{B_{n}}{3^{n}},$$
$${}_{2}F_{1}\left(\frac{1}{4},\frac{3}{4};1;1\right) = 1 + \sum_{n=1}^{\infty} C\left(\frac{1}{4},1;n\right) \frac{D_{n}}{4^{n}},$$
$${}_{2}F_{1}\left(\frac{1}{6},\frac{5}{6};1;1\right) = 1 + \sum_{n=1}^{\infty} C\left(\frac{1}{6},1;n\right) \frac{E_{n}}{6^{n}}.$$

Then, combining the above four formulas with (3.1)–(3.4) proves the desired results. $\hfill\square$

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