# ANALOGUES OF SEVERAL IDENTITIES AND SUPERCONGRUENCES FOR THE CATALAN-QI NUMBERS 

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#### Abstract

In the paper, the author presents $q$-analogues of some identities of the Catalan-Qi numbers and finds supercongruences for some special cases of the Catalan-Qi numbers.


## 1. Introduction

The basic notation of this paper are the shifted factorial

$$
(a)_{0}=1, \quad(a)_{n}=a(a+1) \cdots(a+n-1), \quad n=1,2, \ldots
$$

and the quantum factorial

$$
(x ; q)_{0}=1, \quad(x ; q)_{n}:=\prod_{k=0}^{n-1}\left(1-x q^{k}\right), \quad n=1,2, \ldots
$$

In combinatorial mathematics, the Catalan numbers form a sequence of natural numbers that occur in various counting problems, often involving recursivelydefined objects. The $n$th Catalan number can be given in terms of binomial coefficients by

$$
C_{n}=\frac{(2 n)!}{(n+1)!n!}=\frac{1}{n+1}\binom{2 n}{n}, \quad n \geqslant 0
$$

Up to now, there are many generalizations of the Catalan numbers, including $q$-analogues of the Catalan numbers. For example, Andrews [1] defined

$$
C_{n}(q)=\frac{(1-q)}{\left(1-q^{n+1}\right)}\left[\begin{array}{c}
2 n \\
n
\end{array}\right]_{q}
$$

where

$$
\left[\begin{array}{c}
n \\
j
\end{array}\right]_{q}= \begin{cases}0, & \text { if } j<0 \text { or } j>n \\
\frac{(q ; q)_{n}}{(q ; q)_{j}(q ; q)_{n-j}}, & 0 \leqslant j \leqslant n\end{cases}
$$

represents Gaussian polynomials (also known as $q$-binomial coefficients).
It is easy to prove

$$
\lim _{q \rightarrow 1} C_{n}(q)=C_{n}
$$

[^0]Thus, $C_{n}(q)$ can be viewed as one kind of $q$-analogues of $C_{n}$. Moreover, Andrews [2] obtained

$$
C_{n+1}(q)=\sum_{r=0}^{n} q^{2 r^{2}+2 r}\left[\begin{array}{c}
n \\
2 r
\end{array}\right]_{q} C_{r}(q) \frac{\left(-q^{r+2} ; q\right)_{n-r}}{(-q ; q)_{r}}
$$

which is the $q$-analogue of the identity

$$
\begin{equation*}
C_{n+1}=\sum_{r=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n}{2 r} 2^{n-2 r} C_{r} \tag{1.1}
\end{equation*}
$$

by Touchard 3. Basing on the works of Andrews, Zou 4] obtained several congruences from 1.1.

In this paper, we mainly focus on Qi's generalization of the Catalan numbers

$$
\begin{equation*}
C(a, b ; n)=\left(\frac{b}{a}\right)^{n} \frac{(a)_{n}}{(b)_{n}} \tag{1.2}
\end{equation*}
$$

It is clear that

$$
C\left(\frac{1}{2}, 2 ; n\right)=C_{n}, \quad n \geqslant 0
$$

Actually, the expresion $(1.2)$ is a special case of the function

$$
\begin{equation*}
C(a, b ; z)=\frac{\Gamma(b)}{\Gamma(a)}\left(\frac{b}{a}\right)^{z} \frac{\Gamma(z+a)}{\Gamma(z+b)}, \quad \Re(a), \Re(b)>0, \quad \Re(z) \geqslant 0 \tag{1.3}
\end{equation*}
$$

where $\Gamma(z)$ represents the gamma function which is defined by

$$
\Gamma(z)=\int_{0}^{\infty} x^{z-1} \mathrm{e}^{-x} \mathrm{~d} x, \quad \Re(z)>0
$$

The formula 1.3 was firstly considered by Qi in [5]. Accordingly, the numbers $C(a, b ; n)$ defined by 1.2 were named as the Catalan-Qi numbers. There have been more and more researches on the Catalan-Qi numbers $C(a, b ; n)$ during recent years. See [6, 7, 8, 9, 10] and closely related references therein.

In this paper, we will discuss the $q$-analogues

$$
C_{n}(a, b ; q):=\left(\frac{b}{a}\right)^{n} \frac{\left(q^{a} ; q\right)_{n}}{\left(q^{b} ; q\right)_{n}}, \quad n \geqslant 0
$$

of the Catalan-Qi numbers $C(a, b ; n)$. It is not difficult to prove that

$$
\lim _{q \rightarrow 1} C_{n}(a, b ; q)=C(a, b, n)
$$

In Section 2, we will present $q$-analogues of some identities for the Catalan-Qi numbers $C(a, b ; n)$. In Section 3 , we will establish some supercongruences concerning special Catalan-Qi numbers $C(a, b ; n)$.

## 2. The $q$-Analogues of identities for Catalan-Qi numbers

In this section, we present $q$-analogues of several identities for the Catalan-Qi numbers $C(a, b ; n)$. It is straightforward to verify that

$$
C(a, b ; n)=\frac{1}{C(b, a ; n)} \quad \text { and } \quad C(a, b ; n+1)=\frac{b}{a} \frac{a+n}{b+n} C(a, b ; n)
$$

The first formula has been stated in [8, p. 2].

Similar to the Catalan-Qi numbers $C(a, b ; n)$, the $q$-analogues satisfy

$$
C_{n}(a, b ; q)=\frac{1}{C_{n}(b, a ; q)} \quad \text { and } \quad C_{n+1}(a, b ; q)=\frac{b}{a} \frac{1-q^{a+n}}{1-q^{b+n}} C_{n}(a, b ; q)
$$

The proofs of these two identities are also straightforward.
Except for the two formulas mentioned above, we also have three identities

$$
\begin{align*}
{ }_{2} F_{1}\left(a, 1 ; b ; \frac{b t}{a}\right) & =\sum_{n=0}^{\infty} C(a, b ; n) t^{n},  \tag{2.1}\\
{ }_{2} F_{1}\left(a,-n ; b ;-\frac{b}{a}\right) & =\sum_{r=0}^{n}\binom{n}{r} C(a, b ; r), \tag{2.2}
\end{align*}
$$

and

$$
\begin{equation*}
{ }_{3} F_{2}\left(a, \frac{1-n}{2},-\frac{n}{2} ; b, \frac{1}{2} ; 1\right)=\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n}{2 k}\left(\frac{a}{b}\right)^{k} C(a, b ; k) \tag{2.3}
\end{equation*}
$$

concerning the Catalan-Qi numbers $C(a, b ; n)$, where $a, b>0, n \geqslant 0$, and

$$
{ }_{p} F_{q}\left(a_{1}, \cdots, a_{p} ; b_{1}, \cdots, b_{q} ; x\right)=\sum_{n=0}^{\infty} \frac{\left(a_{1}\right)_{n} \cdots\left(a_{p}\right)_{n}}{\left(b_{1}\right)_{n} \cdots\left(b_{q}\right)_{n}} \frac{x^{n}}{n!} .
$$

The proofs of these three identities can be found in [8, Theorem 10] and [9, Theorem 1].

The identity (2.1) shows that the function ${ }_{2} F_{1}\left(a, 1 ; b ; \frac{b t}{a}\right)$ is a generating function of the Catalan-Qi numbers $C(a, b ; n)$ and that the identity 2.3 can be regarded as a generalization of the identity (1.1).

The above three identities can be generalized as follows.
Theorem 2.1. For $a, b>0, n \geqslant 0$ and $|q|<1$, we have

$$
\begin{align*}
{ }_{2} \phi_{1}\left(q^{a}, q ; q^{b} ; q, \frac{b t}{a}\right) & =\sum_{n=0}^{\infty} C_{n}(a, b ; q) t^{n},  \tag{2.4}\\
{ }_{2} \phi_{1}\left(q^{a}, q^{-n} ; q^{b} ; q,-\frac{b}{a}\right) & =\sum_{r=0}^{n}\left[\begin{array}{l}
n \\
r
\end{array}\right]_{q} C_{r}(a, b ; q) q^{r(r-1) / 2-n r}, \tag{2.5}
\end{align*}
$$

and

$$
\begin{align*}
{ }_{3} \phi_{2}\left(q^{a}, q^{(1-n) / 2},\right. & \left.q^{-n / 2} ; q^{b}, q^{1 / 2} ; q, 1\right) \\
& =\sum_{k=0}^{\lfloor n / 2\rfloor}\left[\begin{array}{c}
n \\
2 k
\end{array}\right]_{q}\left(\frac{a}{b}\right)^{k} C_{k}(a, b ; q) \frac{\left(q^{-n / 2} ; q^{1 / 2}\right)_{2 k}(q ; q)_{2 k}}{\left(q^{1 / 2} ; q^{1 / 2}\right)_{2 k}\left(q^{n-2 k+1} ; q\right)_{2 k}} \tag{2.6}
\end{align*}
$$

where
${ }_{r} \phi_{s}\left(a_{1}, a_{2}, \cdots, a_{r} ; b_{1}, b_{2}, \cdots, b_{s} ; q, z\right)=\sum_{n=0}^{\infty} \frac{\left(a_{1}, a_{2}, \ldots, a_{r} ; q\right)_{n}}{\left(q, b_{1}, b_{2}, \ldots, b_{s} ; q\right)_{n}}\left[(-1)^{n} q^{\binom{n}{2}}\right]^{1+s-r} z^{n}$
is the basic hypergeometric series.
Proof. According to the definition of basic hypergeometric series, we have

$$
{ }_{2} \phi_{1}\left(q^{a}, q ; q^{b} ; q, \frac{b t}{a}\right)=\sum_{n=0}^{\infty} \frac{\left(q^{a} ; q\right)_{n}(q ; q)_{n}}{\left(q^{b} ; q\right)_{n}(q ; q)_{n}}\left(\frac{b t}{a}\right)^{n}
$$

$$
=\sum_{n=0}^{\infty} \frac{\left(q^{a} ; q\right)_{n}}{\left(q^{b} ; q\right)_{n}}\left(\frac{b}{a}\right)^{n} t^{n}=\sum_{n=0}^{\infty} C_{n}(a, b ; q) t^{n}
$$

and

$$
\begin{aligned}
& { }_{2} \phi_{1}\left(q^{a}, q^{-n} ; q^{b} ; q,-\frac{b}{a}\right) \\
= & \sum_{r=0}^{\infty} \frac{\left(q^{a} ; q\right)_{r}\left(q^{-n} ; q\right)_{r}}{\left(q^{b} ; q\right)_{r}(q ; q)_{r}}\left(-\frac{b}{a}\right)^{r} \\
= & \sum_{r=0}^{\infty}\left(\frac{b}{a}\right)^{r} \frac{\left(q^{a} ; q\right)_{r}}{\left(q^{b} ; q\right)_{r}}(-1)^{r} \frac{\left(q^{-n} ; q\right)_{r}}{(q ; q)_{r}} \\
= & \sum_{r=0}^{\infty} C_{r}(a, b ; q) \frac{(-1)^{r}\left(1-q^{-n}\right)\left(1-q^{-n+1}\right) \cdots\left(1-q^{-n+r-1}\right)}{(q ; q)_{r}} \\
= & \sum_{r=0}^{\infty} C_{r}(a, b ; q) \frac{\left(1-q^{n}\right)\left(1-q^{n-1}\right) \cdots\left(1-q^{n-r+1}\right)}{(q ; q)_{r}} q^{r(r-1) / 2-n r} \\
= & \sum_{r=0}^{\infty}\left[\begin{array}{l}
n \\
r
\end{array}\right]_{q} C_{r}(a, b ; q) q^{r(r-1) / 2-n r} \\
= & \sum_{r=0}^{n}\left[\begin{array}{l}
n \\
r
\end{array}\right]_{q} C_{r}(a, b ; q) q^{r(r-1) / 2-n r .} \quad\left(\text { since }\left(q^{-n} ; q\right)_{r}=0 \text { when } r>n\right)
\end{aligned}
$$

The formulas $(2.4)$ and $(2.5)$ follow.
By the definition of basic hypergeometric series, we can derive

$$
{ }_{3} \phi_{2}\left(q^{a}, q^{(1-n) / 2}, q^{-n / 2} ; q^{b}, q^{1 / 2} ; q, 1\right)=\sum_{k=0}^{\infty} \frac{\left(q^{a} ; q\right)_{k}\left(q^{(1-n) / 2} ; q\right)_{k}\left(q^{-n / 2} ; q\right)_{k}}{(q ; q)_{k}\left(q^{b} ; q\right)_{k}\left(q^{1 / 2} ; q\right)_{k}} .
$$

Using the relations

$$
\left(q^{(1-n) / 2} ; q\right)_{k}=0, \quad k>\left\lfloor\frac{n}{2}\right\rfloor, \quad n=1,3,5, \ldots
$$

and

$$
\left(q^{-n / 2} ; q\right)_{k}=0, \quad k>\left\lfloor\frac{n}{2}\right\rfloor, \quad n=2,4,6, \ldots
$$

we have

$$
{ }_{3} \phi_{2}\left(q^{a}, q^{(1-n) / 2}, q^{-n / 2} ; q^{b}, q^{1 / 2} ; q, 1\right)=\sum_{k=0}^{\lfloor n / 2\rfloor} \frac{\left(q^{a} ; q\right)_{k}\left(q^{(1-n) / 2} ; q\right)_{k}\left(q^{-n / 2} ; q\right)_{k}}{(q ; q)_{k}\left(q^{b} ; q\right)_{k}\left(q^{1 / 2} ; q\right)_{k}} .
$$

For another side of (2.6),

$$
\begin{aligned}
& \sum_{k=0}^{\lfloor n / 2\rfloor}\left[\begin{array}{c}
n \\
2 k
\end{array}\right]_{q}\left(\frac{a}{b}\right)^{k} C_{k}(a, b ; q) \frac{\left(q^{-n / 2} ; q^{1 / 2}\right)_{2 k}(q ; q)_{2 k}}{\left(q^{1 / 2} ; q^{1 / 2}\right)_{2 k}\left(q^{n-2 k+1} ; q\right)_{2 k}} \\
= & \sum_{k=0}^{\lfloor n / 2\rfloor} \frac{(q ; q)_{n}\left(q^{-n / 2} ; q^{1 / 2}\right)_{2 k}(q ; q)_{2 k}\left(q^{a} ; q\right)_{k}}{(q ; q)_{2 k}(q ; q)_{n-2 k}\left(q^{1 / 2} ; q^{1 / 2}\right)_{2 k}\left(q^{n-2 k+1} ; q\right)_{2 k}\left(q^{b} ; q\right)_{k}} \\
= & \sum_{k=0}^{\lfloor n / 2\rfloor} \frac{(q ; q)_{n}\left(q^{-n / 2} ; q\right)_{k}\left(q^{(1-n) / 2} ; q\right)_{k}\left(q^{a} ; q\right)_{k}}{(q ; q)_{k}(q ; q)_{n-2 k}\left(q^{1 / 2} ; q\right)_{k}\left(q^{n-2 k+1} ; q\right)_{2 k}\left(q^{b} ; q\right)_{k}}
\end{aligned}
$$

$$
\begin{aligned}
& =\sum_{k=0}^{\lfloor n / 2\rfloor} \frac{\left(q^{-n / 2} ; q\right)_{k}\left(q^{(1-n) / 2} ; q\right)_{k}\left(q^{a} ; q\right)_{k}}{(q ; q)_{k}\left(q^{b} ; q\right)_{k}\left(q^{1 / 2} ; q\right)_{k}} \\
& ={ }_{3} \phi_{2}\left(q^{a}, q^{(1-n) / 2}, q^{-n / 2} ; q^{b}, q^{1 / 2} ; q, 1\right) .
\end{aligned}
$$

The proof of this theorem is complete.

## 3. Supercongruences of special Catalan-Qi numbers

Let us first look at four supercongruences as follows.
If $p \geqslant 5$ is a prime, then

$$
\begin{align*}
& { }_{2} F_{1}\left(\frac{1}{2}, \frac{1}{2} ; 1 ; 1\right)_{\operatorname{tr}(p)} \equiv\left(\frac{-4}{p}\right) \quad\left(\bmod p^{2}\right),  \tag{3.1}\\
& { }_{2} F_{1}\left(\frac{1}{3}, \frac{2}{3} ; 1 ; 1\right)_{\operatorname{tr}(p)} \equiv\left(\frac{-3}{p}\right) \quad\left(\bmod p^{2}\right)  \tag{3.2}\\
& { }_{2} F_{1}\left(\frac{1}{4}, \frac{3}{4} ; 1 ; 1\right)_{t r(p)} \equiv\left(\frac{-2}{p}\right) \quad\left(\bmod p^{2}\right),  \tag{3.3}\\
& { }_{2} F_{1}\left(\frac{1}{6}, \frac{5}{6} ; 1 ; 1\right)_{t r(p)} \equiv\left(\frac{-1}{p}\right) \quad\left(\bmod p^{2}\right), \tag{3.4}
\end{align*}
$$

where $(\dot{\bar{p}})$ represents the Legendre symbol and

$$
{ }_{p} F_{q}\left(a_{1}, \cdots, a_{p} ; b_{1}, \cdots, b_{q} ; x\right)_{\operatorname{tr}(p)}=\sum_{n=0}^{p-1} \frac{\left(a_{1}\right)_{n} \cdots\left(a_{p}\right)_{n}}{\left(b_{1}\right)_{n} \cdots\left(b_{q}\right)_{n}} \frac{x^{n}}{n!}
$$

The proofs of these four supercongruences was first given by Eric Mortenson in [11]. Actually, these four supercongruences were conjectured by Rodriguez-Villegas for hypergeometric Calabi-Yau manifolds of dimension $d=1$. The conjecture of Rodriguez-Villegas has far-reaching impact on mathematics, especially on $q$-series. The recent work should be the one given by Guo and Zeng [12.

Now, we can give the main theorem of this section.
Theorem 3.1. Let

$$
\begin{aligned}
& A_{n}=\frac{2 n-1}{2 n} \cdot \frac{2 n-3}{2 n-2} \cdots \frac{1}{2} \cdot 1, \\
& B_{n}=\frac{3 n-1}{3 n} \cdot \frac{3 n-4}{3 n-3} \cdots \frac{2}{3} \cdot 1, \\
& D_{n}=\frac{4 n-1}{4 n} \cdot \frac{4 n-5}{4 n-4} \cdots \frac{3}{4} \cdot 1,
\end{aligned}
$$

and

$$
E_{n}=\frac{6 n-1}{6 n} \cdot \frac{6 n-7}{6 n-6} \cdots \frac{5}{6} \cdot 1
$$

If $p \geqslant 5$ is a prime, then

$$
\begin{aligned}
& 1+\sum_{n=1}^{p-1} C\left(\frac{1}{2}, 1 ; n\right) \frac{A_{n}}{2^{n}} \equiv\left(\frac{-4}{p}\right) \quad\left(\bmod p^{2}\right) \\
& 1+\sum_{n=1}^{p-1} C\left(\frac{1}{3}, 1 ; n\right) \frac{B_{n}}{3^{n}} \equiv\left(\frac{-3}{p}\right) \quad\left(\bmod p^{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
& 1+\sum_{n=1}^{p-1} C\left(\frac{1}{4}, 1 ; n\right) \frac{D_{n}}{4^{n}} \equiv\left(\frac{-2}{p}\right) \quad\left(\bmod p^{2}\right) \\
& 1+\sum_{n=1}^{p-1} C\left(\frac{1}{6}, 1 ; n\right) \frac{E_{n}}{6^{n}} \equiv\left(\frac{-1}{p}\right) \quad\left(\bmod p^{2}\right)
\end{aligned}
$$

Proof. Firstly, we prove that

$$
{ }_{2} F_{1}\left(\frac{1}{2}, \frac{1}{2} ; 1 ; 1\right)=1+\sum_{n=1}^{\infty} C\left(\frac{1}{2}, 1 ; n\right) \frac{A_{n}}{2^{n}} .
$$

According to the definition, we can derive

$$
\begin{aligned}
&{ }_{2} F_{1}\left(\frac{1}{2}, \frac{1}{2} ; 1 ; 1\right)=\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_{n}\left(\frac{1}{2}\right)_{n}}{(1)_{n}} \frac{1}{n!}=\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_{n}}{(1)_{n}} \cdot 2^{n} \cdot \frac{\left(\frac{1}{2}\right)_{n}}{2^{n} \cdot n!} \\
&= 1+\sum_{n=1}^{\infty} C\left(\frac{1}{2}, 1 ; n\right) \frac{(2 n-1)!!}{(2 n)!!} \frac{1}{2^{n}}=1+\sum_{n=1}^{\infty} C\left(\frac{1}{2}, 1 ; n\right) \frac{A_{n}}{2^{n}} .
\end{aligned}
$$

Similarly, we can prove

$$
\begin{aligned}
& { }_{2} F_{1}\left(\frac{1}{3}, \frac{2}{3} ; 1 ; 1\right)=1+\sum_{n=1}^{\infty} C\left(\frac{1}{3}, 1 ; n\right) \frac{B_{n}}{3^{n}}, \\
& { }_{2} F_{1}\left(\frac{1}{4}, \frac{3}{4} ; 1 ; 1\right)=1+\sum_{n=1}^{\infty} C\left(\frac{1}{4}, 1 ; n\right) \frac{D_{n}}{4^{n}}, \\
& { }_{2} F_{1}\left(\frac{1}{6}, \frac{5}{6} ; 1 ; 1\right)=1+\sum_{n=1}^{\infty} C\left(\frac{1}{6}, 1 ; n\right) \frac{E_{n}}{6^{n}} .
\end{aligned}
$$

Then, combining the above four formulas with $(3.1-3.4$ proves the desired results.

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