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EQUIVALENCE AMONG THREE 2-NORMS ON THE SPACE OF p-SUMMABLE SEQUENCES

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Abstract. There are two known 2-norms defined on the space of p-summable sequences of real numbers. The first 2-norm is a special case of Gähler's formula [Mathematische Nachrichten, 1964], while the second is due to Gunawan [Bulletin of the Australian Mathematical Society, 2001]. The aim of this paper is to define a new 2-norm on ℓ^p and prove the equivalence among these three 2-norms.

1. Introduction

The theory of 2-normed spaces was first introduced and developed to the theory of n-normed spaces by Gähler (see, [1]-[5]) in the mid 1960's, while that of n-normed spaces was studied later by Misiak [15]. Related works may be found in, e.g., [6]-[14], [16]-[17].

We shall study the space ℓ^p, $1 \leq p < \infty$, containing all sequences of real numbers $x = (x_j)$ for which $\sum_j |x_j|^p < \infty$, and usual norm defined on it is $\|x\|_p := (\sum_j |x_j|^p)^{\frac{1}{p}}$. Throughout this note, we assume that p lies in the interval $1 \leq p < \infty$ unless otherwise stated.

Let n be a nonnegative integer and X be a real vector space of dimension $d \geq n$ (d may be infinite). A real-valued function $\|.,...,.\|$ on X^n satisfying the following four properties is called an n-norm on X and the pair $(X, \|.,...,.\|)$ is called an n-normed space

(1) $\|x_1,...,x_n\| = 0$ if and only if $x_1,...,x_n$ are linearly dependent;
(2) $\|x_1,...,x_n\|$ is invariant under permutation;
(3) $\|\alpha x_1,...,x_n\| = |\alpha| \, \|x_1,...,x_n\|$ for $\alpha \in \mathbb{R}$;
(4) $\|x_1 + x_1', x_2,...,x_n\| \leq \|x_1, x_2,...,x_n\| + \|x_1', x_2,...,x_n\|$.

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If X is a normed space, then, according to Gähler, the following formula defines an n-norm on X:

$$\|x_1,...,x_n\|^G := \sup_{\substack{f_i\in X',\ \|f_i\|\le 1\\ i=1,\dots,n}} \begin{vmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \dots & f_n(x_n) \end{vmatrix}. \tag{1.1}$$

Here X' denotes the dual of X, which consists of bounded linear functionals on X. For $X = \ell^p$, the space of p-summable sequences of real numbers, the above formula reduces to

$$\|x_1,...,x_n\|_p^G := \sup_{\substack{z_i\in \ell^{p\prime},\ \|z_i\|_q\le 1\\ i=1,\dots,n}} \begin{vmatrix} \sum_j x_{1j}z_{1j} & \cdots & \sum_j x_{1j}z_{nj} \\ \vdots & \ddots & \vdots \\ \sum_j x_{nj}z_{1j} & \dots & \sum_j x_{nj}z_{nj} \end{vmatrix}, \tag{1.2}$$

where $\|.\|_q$ denotes the usual norm on $X' = \ell^q$ and each of the sums is taken over $j \in \mathbb{N}$. Here q denotes the dual exponent of p, so that $\frac{1}{p}+\frac{1}{q}=1$ $(1 < p < \infty)$.

In 2001, Gunawan [7] defined a different n-norm on ℓ^p $(1 \le p < \infty)$ by

$$\|x_1,...,x_n\|_p^H := \left[\frac{1}{n!}\sum_{j_1}\cdots\sum_{j_n} abs\begin{vmatrix} x_{1j_1} & \cdots & x_{nj_1} \\ \vdots & \ddots & \vdots \\ x_{1j_n} & \dots & x_{nj_n} \end{vmatrix}^p\right]^{\frac{1}{p}}, \tag{1.3}$$

where $x_i = (x_{ij})$, $i = 1, ..., n$, $j \in \mathbb{N}$. Thus, on ℓ^p, we have two definitions of n-norms, one is derived from Gähler's formula given by equation (1.2) and the other is due to Gunawan given by equation (1.3). For $p = 2$, one may verify that the two n-noms are identical (see [6]). The equation (1.1) reduces to the equation (1.4) for $n = 2$ and $x, y \in \ell^p$

$$\|x,y\|^G := \sup_{\substack{f_u,f_v\in X'\\ \|f_u\|\le 1,\ \|f_v\|\le 1}} \begin{vmatrix} f_u(x) & f_v(x) \\ f_u(y) & f_v(y) \end{vmatrix}. \tag{1.4}$$

For $X = \ell^p$ and $x, y \in \ell^p$ the above formula, for convenience, can be rewritten as

$$\|x,y\|_p^G := \sup_{\substack{u,v\in \ell^q\\ \|u\|_q,\|v\|_q\le 1}} \begin{vmatrix} \langle x,u\rangle & \langle x,v\rangle \\ \langle y,u\rangle & \langle y,v\rangle \end{vmatrix}$$

where q is the conjugate of p such that $\frac{1}{q} = 1 - \frac{1}{p}$ $(1 < p < \infty)$ and ℓ^q is the dual of ℓ^p, which consists of bounded linear functionals f_u, f_v on ℓ^p where $f_u(x) = \langle x,u\rangle := \sum_{k=1}^{\infty} x_k u_k$ $(x \in \ell^p$ and $u \in \ell^q)$, $f_v(x) = \langle x,v\rangle := \sum_{k=1}^{\infty} x_k v_k$ $(x \in \ell^p$ and $v \in \ell^q)$ and $\|\cdot\|_q$ is usual norm on ℓ^q.

For $n = 2$ and $x, y \in \ell^p$ the equation (1.3) reduces to the following equation:

$$\|x,y\|_p^H := \left[\frac{1}{2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} abs\begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}^p\right]^{\frac{1}{p}}.$$

Wibawah-Kusuma and Gunawan [17] proved that Gunawan's and Gähler's n-norms on ℓ^p are strongly equivalent.

Lemma 1.1. (Theorem 2.3, [17]) For any $x_1, ..., x_n \in \ell^p$ we have

$$(n!)^{\frac{1}{p}-1} \|x_1, ..., x_n\|_p^H \leq \|x_1, ..., x_n\|_p^G \leq (n!)^{\frac{1}{p}} \|x_1, ..., x_n\|_p^H .$$

If we take $n = 2$, then the inequality given above for $x, y \in \ell^p$ reduces to

$$2^{\frac{1}{p}-1} \|x, y\|_p^H \leq \|x, y\|_p^G \leq 2^{\frac{1}{p}} \|x, y\|_p^H . \tag{1.5}$$

Lemma 1.2. (Lemma 2.1, [10]) For every $x, y \in \ell^p$ we have

$$\|x, y\|_p^H \leq 2^{1-\frac{1}{p}} \|x\|_p \|y\|_p .$$

In this work, we define a new 2-norm on ℓ^p and prove the equivalence of these three 2-norms on ℓ^p.

2. Main Results

Let $x, y \in \ell^p$ and ℓ^q be the dual of ℓ^p, which consists of bounded linear functionals where q is the conjugate of p such that $\frac{1}{q} = 1 - \frac{1}{p}$ $(1 < p < \infty)$ and f_u, f_v on ℓ^p where $f_u(x) = \langle x, u\rangle := \sum_{k=1}^{\infty} x_k u_k$ $(x \in \ell^p$ and $u \in \ell^q)$, $f_v(x) = \langle x, v\rangle := \sum_{k=1}^{\infty} x_k v_k$ $(x \in \ell^p$ and $v \in \ell^q)$. Throughout the paper, by $\|\cdot\|_p$ we mean the usual norm on ℓ^p. Now, we define a new formula on ℓ^p as follows:

$$\|x, y\|_p^{SI} := \sup_{\substack{u, v \in \ell^q \\ \|u\|_q , \|v\|_q \leq 1}} \frac{1}{2} \left(\|\langle y, u\rangle x - \langle x, u\rangle y\|_p + \|\langle y, v\rangle x - \langle x, v\rangle y\|_p \right) .$$

The following fact tells us that the function $\|.,.\|_p^{SI}$ is a 2-norm on ℓ^p.

Fact 2.1. The real-valued function $\|.,.\|_p^{SI}$ defines a 2-norm on ℓ^p.

Proof. We need to check that $\|.,.\|_p^{SI}$ satisfies the four properties of a 2-norm. The "if" part of (1), (2), (3) and (4) are obvious. To verify the "only if" part of (1), assume that $\|.,.\|_p^{SI} = 0$. Then from the definition of norm we have $\langle y, u\rangle x - \langle x, u\rangle y = 0$ and $\langle y, v\rangle x - \langle x, v\rangle y = 0$. Thus,

$$x = \frac{\langle x, u\rangle}{\langle y, u\rangle} y \quad (\langle y, u\rangle \neq 0)$$

and

$$x = \frac{\langle x, v\rangle}{\langle y, v\rangle} y \quad (\langle y, v\rangle \neq 0) .$$

Since $\langle .,.\rangle$ is bounded linear functional on ℓ^p, then $\frac{\langle y,u\rangle}{\langle x,u\rangle}$ and $\frac{\langle y,v\rangle}{\langle x,v\rangle}$ are real. For $x = \frac{\langle x,u\rangle}{\langle y,u\rangle} y \quad (\langle y, u\rangle \neq 0)$, if we insert the value of x in the equation $\langle y, v\rangle x - \langle x, v\rangle y = 0$, then clearly x satisfies the condition. In both cases, we conclude that x and y are linearly dependent. □

As a consequence of Fact 2.1 we have the following corollary.

Corollary 2.2. The space ℓ^p equipped with $\|.,.\|_p^{SI}$ is a 2-normed space.

Lemma 2.3. Let $1 \le p < \infty$. For $x, y \in \ell^p$, we have

$$\|x,y\|_p^{SI} \le 2^{\frac{1}{p}} \|x,y\|_p^H .$$

Proof. For $x, y \in \ell^p$ and $u \in \ell^q$, we have the following by utilizing Hölder's inequality

$$\begin{aligned}
\|\langle y,u\rangle x - \langle x,u\rangle y\|_p^p &= \sum_{i=1}^{\infty} |\langle y,u\rangle x_i - \langle x,u\rangle y_i|^p \\
&= \sum_{i=1}^{\infty} \left| \sum_{j=1}^{\infty} (x_i y_j - x_j y_i) u_j \right|^p \\
&\le \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} |x_i y_j - x_j y_i| \, |u_j| \right)^p \\
&\le \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |x_i y_j - x_j y_i|^p \left(\sum_{k=1}^{\infty} |u_k|^q \right)^{\frac{p}{q}} \\
&= 2 \|u\|_q^p \left(\|x,y\|_p^H \right)^p .
\end{aligned}$$

From the above equation, we have

$$\begin{aligned}
\|x,y\|_p^{SI} &= \sup_{\substack{u,v \in \ell^q \\ \|u\|_q, \|v\|_q \le 1}} \tfrac{1}{2} \left(\|\langle y,u\rangle x - \langle x,u\rangle y\|_p + \|\langle y,v\rangle x - \langle x,v\rangle y\|_p \right) \\
&\le \sup_{\substack{u,v \in \ell^q \\ \|u\|_q, \|v\|_q \le 1}} \tfrac{1}{2} \left(2^{\frac{1}{p}} \|u\|_q \|x,y\|_p^H + 2^{\frac{1}{p}} \|v\|_q \|x,y\|_p^H \right) \\
&\le 2^{\frac{1}{p}} \|x,y\|_p^H .
\end{aligned}$$

□

Now, we need the following lemma to show the equivalence between the 2-norms $\|.,.\|_p^{SI}$ and $\|.,.\|_p^H$.

Lemma 2.4. For $x, y \in \ell^p$ and $a, b, c, d \in \mathbb{R}$, we have

$$abs \begin{vmatrix} a & b \\ c & d \end{vmatrix} \|x,y\|_p^H = \|ax - by, -cx + dy\|_p^H .$$

Proof. Let $x, y \in \ell^p$ and $a, b, c, d \in \mathbb{R}$. Then we obtain

$$\begin{aligned}
\left(\|ax - by, -cx + dy\|_p^H \right)^p &= \frac{1}{2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |(ax_i - by_i)(-cx_j + dy_j) - (ax_j - by_j)(-cx_i + dy_i)|^p \\
&= \frac{1}{2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |adx_i y_j + bcx_j y_i - bcx_i y_j - adx_j y_i|^p \\
&= \frac{1}{2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |ad(x_i y_j - x_j y_i) - bc(x_i y_j - x_j y_i)|^p \\
&= |ad - bc|^p \left(\|x,y\|_p^H \right)^p .
\end{aligned}$$

Thus, we have $abs \begin{vmatrix} a & b \\ c & d \end{vmatrix} \|x,y\|_p^H = \|ax - by, -cx + dy\|_p^H$. □

Lemma 2.4 is important to obtain the equivalence between $\|.,.\|_p^{SI}$ and $\|.,.\|_p^{H}$.

Theorem 2.5. For $x, y \in \ell^p$ we have

$$2^{\frac{1}{p}-1}\|x,y\|_p^H \le \|x,y\|_p^{SI} \le 2^{\frac{1}{p}}\|x,y\|_p^H .$$

Proof. By the equation (1.5) in Lemma 1.1, we have

$$\left(\|x,y\|_p^H\right)^2 \le 2^{1-\frac{1}{p}}\|x,y\|_p^G \|x,y\|_p^H .$$

Then

$$\begin{aligned}
\|x,y\|_p^H &\le 2^{\frac{1}{2}-\frac{1}{2p}}\left(\|x,y\|_p^H\right)^{\frac{1}{2}}\left(\|x,y\|_p^G\right)^{\frac{1}{2}} \\
&= 2^{\frac{1}{2}-\frac{1}{2p}}\left(\sup_{\substack{u,v\in\ell^q\\ \|u\|_q,\ \|v\|_q\le 1}} \left|\begin{matrix}\langle x,u\rangle & \langle x,v\rangle\\ \langle y,u\rangle & \langle y,v\rangle\end{matrix}\right| \|x,y\|_p^H\right)^{\frac{1}{2}} . \\
&\le 2^{\frac{1}{2}-\frac{1}{2p}}\left(\sup_{\substack{u,v\in\ell^q\\ \|u\|_q,\ \|v\|_q\le 1}} abs\left|\begin{matrix}\langle y,u\rangle & \langle x,u\rangle\\ \langle y,v\rangle & \langle x,v\rangle\end{matrix}\right| \|x,y\|_p^H\right)^{\frac{1}{2}}
\end{aligned}$$

By Lemma 2.4, we have

$$\|x,y\|_p^H \le 2^{\frac{1}{2}-\frac{1}{2p}} \sup_{\substack{u,v\in\ell^q\\ \|u\|_q,\|v\|_q\le 1}} \|\langle y,u\rangle x-\langle x,u\rangle y,-\langle y,v\rangle x+(\langle x,v\rangle y\|_p^H)^{\frac{1}{2}} .$$

Meanwhile, Lemma 1.2 helps us to obtain the below inequality.

$$\begin{aligned}
\|x,y\|_p^H &\le 2^{1-\frac{1}{p}} \sup_{\substack{u,v\in\ell^q\\ \|u\|_q,\ \|v\|_q\le 1}} \left(\|\langle y,u\rangle x-\langle x,u\rangle y\|_p\|-\langle y,v\rangle x+\langle x,v\rangle y\|_p\right)^{\frac{1}{2}} \\
&\le 2^{1-\frac{1}{p}} \sup_{\substack{u,v\in\ell^q\\ \|u\|_q,\ \|v\|_q\le 1}} \left(\tfrac{1}{2}\|\langle y,u\rangle x-\langle x,u\rangle y\|_p+\tfrac{1}{2}\|\langle y,v\rangle x-\langle x,v\rangle y\|_p\right) \\
&= 2^{1-\frac{1}{p}}\|x,y\|_p^{SI} .
\end{aligned} \tag{2.1}$$

Hence, by Lemma 2.3 and equation (2.1) we obtain the following equivalence

$$2^{\frac{1}{p}-1}\|x,y\|_p^H \le \|x,y\|_p^{SI} \le 2^{\frac{1}{p}}\|x,y\|_p^H .$$

□

The equivalence relation between the 2-norms $\|x,y\|_p^G$ and $\|x,y\|_p^{SI}$ can be obtained with a simple process.

Theorem 2.6. For $x, y \in \ell^p$ we have

$$\frac{1}{2}\|x,y\|_p^G \le \|x,y\|_p^{SI} \le 2\|x,y\|_p^G$$

Proof. From the equation (1.5) and Theorem 2.5, we obtain

$$2^{\frac{1}{p}-1}\left\|x,y\right\|_p^H \le \left\|x,y\right\|_p^G \le 2^{\frac{1}{p}}\left\|x,y\right\|_p^H \le 2\left\|x,y\right\|_p^{SI} \le 2^{\frac{1}{p}+1}\left\|x,y\right\|_p^H \le 4\left\|x,y\right\|_p^G .$$

Hence, we have the result. □

Corollary 2.7. For $x, y \in \ell^p$ the three 2-norms $\|.,.\|_p^H$, $\|.,.\|_p^G$ and $\|.,.\|_p^{SI}$ on ℓ^p are equivalent.

Proof. It is easy to see from Theorem 2.5 and Theorem 2.6. □

We know that the space $\left(\ell^p, \|.,.\|_p^H\right)$ is complete (see, Theorem 2.6 in [7]). Since $\left(\ell^p, \|.,.\|_p^H\right)$ and $\left(\ell^p, \|.,.\|_p^{SI}\right)$ are equivalent two 2-norms, then we come to the main result which is given by the following corollary.

Corollary 2.8. The space $\left(\ell^p, \|.,.\|_p^{SI}\right)$ is complete. In other words, it is a Banach space.

Proof. Let $x(m)$ be a Cauchy sequence in ℓ^p with respect to $\|.,.\|_p^{SI}$. Then, by Theorem 2.5 $x(m)$ is Cauchy with respect to $\|.,.\|_p^H$. But we know that ℓ^p is complete with respect to $\|.,.\|_p^H$, and so $x(m)$ must converge to some $x \in \ell^p$ in $\|.,.\|_p^H$. By another application of Theorem 2.5, $x(m)$ also converges to x in $\|.,.\|_p^{SI}$. This shows that ℓ^p is complete with respect to the 2-norm $\|.,.\|_p^{SI}$. □

3. CONCLUDING REMARKS

In this work, a new 2-norm $\|.,.\|_p^{SI}$ is defined on ℓ^p. We have just seen that Gähler's and Gunawan's 2-norms on ℓ^p are (strongly) equivalent to this new 2-norm on ℓ^p. If one insists to involve the functionals on ℓ^p, one may use the fact that the dual space ℓ^q is also a 2-normed space, as well as ℓ^p $(1 \le p < \infty)$. Thus one may generalize this definition of 2-norm to n-norm on ℓ^p. The results obtained in this work can be extended to n-normed space.

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