

COMMENTS ON TWO COMPLETELY MONOTONIC FUNCTIONS INVOLVING THE q -TRIGAMMA FUNCTION

FENG QI, FANG-FANG LIU, XIAO-TING SHI

ABSTRACT. In the papers [F. Qi, *A completely monotonic function related to the q -trigamma function*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **76** (2014), no. 1, 107–114] and [J.-L. Zhao, *A completely monotonic function relating to the q -trigamma function*, J. Math. Inequal. **9** (2015), no. 1, 53–60; Available online at <http://dx.doi.org/10.7153/jmi-09-05>], Qi and Zhao proved the complete monotonicity of two functions involving the q -trigamma function. These two functions originate and generalize the same function involving the classical trigamma function. In current paper, the authors compare with and comments on these two functions and related results obtained in the above-mentioned two papers. Moreover, the authors correct some errors by repeating the proof supplied by Qi in the above-mentioned paper published in 2014.

1. NOTATION

It is well known [3, 6] that the Euler gamma function $\Gamma(z)$ may be defined by

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{n!n^z}{\prod_{k=0}^n (z+k)}, \quad z \neq 0, -1, -2, \dots$$

The logarithmic derivative of $\Gamma(z)$, denoted by $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$, is called the digamma function, and the derivatives $\psi^{(i)}(z)$ for $i \in \{0\} \cup \mathbb{N}$ are respectively called the polygamma functions, where \mathbb{N} stands for the set of all positive integers. In particular, the functions $\psi'(z)$ and $\psi''(z)$ are called the trigamma and tetragamma functions. For some behaving properties of $\Gamma(z)$ at $z = 0, -1, -2, \dots$, please refer to [12, 18].

The q -analogue $\Gamma_q(z)$ of the gamma function $\Gamma(z)$ may be defined for $\Re(z) > 0$ by, when $0 < q < 1$,

$$\Gamma_q(z) = (1-q)^{1-z} \prod_{i=0}^{\infty} \frac{1-q^{i+1}}{1-q^{i+z}},$$

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and by, when $q > 1$,

$$\Gamma_q(z) = (q-1)^{1-z} q^{\binom{z}{2}} \prod_{i=0}^{\infty} \frac{1 - q^{-(i+1)}}{1 - q^{-(i+z)}}.$$

When $\Re(z) > 0$, the q -digamma function $\psi_q(z)$, the q -analogue of the digamma function $\psi(z)$, may be defined by

$$\begin{aligned} \psi_q(z) &= \frac{\Gamma'_q(z)}{\Gamma_q(z)} = -\ln(1-q) + \ln q \sum_{k=0}^{\infty} \frac{q^{k+z}}{1 - q^{k+z}} \\ &= -\ln(1-q) + \ln q \sum_{k=1}^{\infty} \frac{q^{kz}}{1 - q^k} \end{aligned}$$

for $0 < q < 1$ and by

$$\psi_q(z) = -\ln(q-1) + \ln q \left(z - \frac{1}{2} - \sum_{n \geq 0} \frac{q^{-n-z}}{1 - q^{-n-z}} \right)$$

for $q > 1$. The functions $\psi_q^{(k)}(z)$, the q -analogues of the polygamma functions $\psi^{(k)}(z)$, for $k \in \mathbb{N}$ are called the q -polygamma functions. For detailed information about the above formulas, see [3, 7, 8, 11, 19] and closely related references therein. The above mentioned functions satisfy the following relations

$$\lim_{q \rightarrow 1^\pm} \Gamma_q(z) = \Gamma(z), \quad \lim_{q \rightarrow 1^\pm} \psi_q(z) = \psi(z), \quad \Gamma_q(z) = q^{\binom{z-1}{2}} \Gamma_{1/q}(z).$$

The proofs of the above limits can be found in [2, Appendix A], [3, pp. 493–496], [5, p. 17], and [9, Appendix B].

Recall from [10, Chapter XIII], [28, Chapter 1], and [29, Chapter IV] that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and

$$0 \leq (-1)^n f^{(n)}(x) < \infty$$

for $x \in I$ and $n \geq 0$. In [29, p. 161, Theorem 12b], it was stated that a necessary and sufficient condition that $f(x)$ should be completely monotonic for $0 < x < \infty$ is that

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t),$$

where $\alpha(t)$ is non-decreasing and the integral converges for $0 < x < \infty$. In other words, a function is completely monotonic on $(0, \infty)$ if and only if it is a Laplace transform.

2. COMMENTS

For $x > 0$, let

$$f(x) = \psi'(x) - \frac{1}{x} - \frac{1}{2x^2}. \quad (2.1)$$

In recent years, the complete monotonicity of the function (2.1) was proved, generalized, and applied in [1, 4, 7, 8, 11, 17, 19], [13, Theorem 1.1], [20, Theorem 1.3], [21, pp. 1977–1978], [22, Theorem 2]. For more information on this topic, please refer to related texts in the survey articles [15, 16, 24, 26, 27] and closely related references therein.

For $x > 0$ and $0 < q < 1$, let

$$f_q(x) = \psi'_q(x) - \frac{(1-q)q^x}{1-q^x} - \frac{1}{2} \left[\frac{(1-q)q^x}{1-q^x} \right]^2. \quad (2.2)$$

It is clear that

$$\lim_{q \rightarrow 1^-} f_q(x) = f(x).$$

So, we may regard $f_q(x)$ as the q -analogue of the function $f(x)$.

In the paper [14], the complete monotonicity of $f_q(x)$ for $0 < q < 1$ on $(0, \infty)$ was verified.

Theorem 2.1 ([14, Theorem 1.1]). *For $0 < q < 1$, the function $f_q(x)$ defined by (2.2) is completely monotonic on $(0, \infty)$.*

In [30], the author imitated the paper [14], considered an alternative q -analogue

$$F_q(x) = \psi'_q(x) - \frac{1-q}{1-q^x} - \frac{1}{2} \left(\frac{1-q}{1-q^x} \right)^2 + \frac{1}{2}(1-q)(3-q) \quad (2.3)$$

of the function $f(x)$ in (2.1), and confirmed that the function $F_q(x)$ for $q \in (0, 1)$ is completely monotonic on $(0, \infty)$.

Theorem 2.2 ([30, Theorem 1]). *For $0 < q < 1$, the function $F_q(x)$ defined by (2.3) is completely monotonic on $(0, \infty)$.*

To a great extent, the function $F_q(x)$ considered in [30] is a more natural q -analogue of the function $f(x)$ in (2.1), because the factor

$$\frac{1-q^z}{1-q},$$

denoted by $[z]_q$ in [6, 23], is, but the factor

$$\frac{1-q^z}{(1-q)q^z}$$

is not, the usually adopted q -analogue of $z \in \mathbb{C}$ for $q \neq 1$. For $q \neq 1$, the quantity $[z]_q$ may be called a q -number, a basic number, a q -analogue, a q -deformation, a q -extension, or a q -generalization of the complex number z . See [6, Eqs. (1.2.13) and (1.2.43)].

We also observe that the difference

$$f_q(x) - F_q(x) = \frac{(1-q)^2 q^x}{1-q^x}$$

is completely monotonic on $(0, \infty)$, which implies that the result obtained in [30] is surely better than the one in [14].

Basing on the above points, we do not simply think that the lately published paper [30] is a plagiarism of the formerly published paper [14]. However, after all, a high and heavy imitation is not good and appropriate behaviour in academic community.

3. CORRECTED PROOF OF THEOREM 2.1

Some errors appeared on page 112 in the proof of [14, Theorem 1.1]. We are now in a position to provide a corrected proof of [14, Theorem 1.1] as follows.

As did in the proof of [14, Theorem 1.1], a straightforward computation gives

$$f_q(x) - f_q(x+1) = \sum_{k=0}^{\infty} \left\{ \left[\frac{(1-q)k}{2} + 1 \right] (1-q)(q^{k+1} - 1) + (k+1)(\ln q)^2 \right\} q^{(k+1)x}.$$

Let

$$g_q(t) = \frac{1}{2}(1-q)[(1-q)(t-1) + 2](q^t - 1) + (\ln q)^2 t$$

for $0 < q < 1$ and $t \in [1, \infty)$. Then

$$\begin{aligned} g'_q(t) &= (\ln q)^2 + \frac{1}{2}(\ln q)(q-1)[(q-1)(t-1) - 2]q^t + \frac{1}{2}(1-q)^2(q^t - 1), \\ g''_q(t) &= \frac{1}{2}(\ln q)(q-1)q^t[(q-1)t \ln q + 2(q-1) - (q+1)\ln q] \\ &\triangleq \frac{1}{2}(\ln q)(q-1)q^t \varphi(t, q), \end{aligned}$$

where $\varphi(t, q)$ satisfies

$$\varphi(1, q) = 2(q-1-\ln q) \quad \text{and} \quad \frac{d\varphi(1, q)}{dq} = 2\left(1 - \frac{1}{q}\right) < 0.$$

Since $\varphi(1, q)$ is decreasing with respect to $q \in (0, 1)$ and $\varphi(1, 1) = 0$, so $\varphi(1, q) > 0$ for $q \in (0, 1)$. It is obvious that $\varphi(t, q)$ is increasing with respect to t , so $\varphi(t, q) > 0$ for $(t, q) \in [1, \infty) \times (0, 1)$. Hence, the second derivative $g''_q(t)$ is positive for $(t, q) \in [1, \infty) \times (0, 1)$ and $g'_q(t)$ is increasing with respect to $t \in [1, \infty)$. Making use of the easily verified double inequality

$$\frac{1}{2}\left(q - \frac{1}{q}\right) < \ln q < q - 1, \quad q \in (0, 1), \quad (3.1)$$

we have

$$\begin{aligned} g'_q(1) &= \frac{1}{2}[(q-1)^3 + 2(\log q)^2 - 2q(q-1)\ln q] \\ &> \frac{1}{2}[(q-1)^3 + 2(q-1)^2 - 2q(q-1)\ln q] \\ &= \frac{1}{2}(q-1)(q^2 - 2q\ln q - 1) \\ &> 0, \end{aligned}$$

we obtain $g'_q(t) > 0$ for $(t, q) \in (1, \infty) \times (0, 1)$. As a result, the function $g_q(t)$ for $0 < q < 1$ is increasing with respect to $t \in [1, \infty)$. By (3.1), it is easy to see that

$$g_q(1) = (\ln q)^2 - (q-1)^2 > 0$$

for $q \in (0, 1)$. Accordingly, the function $g_q(t)$ is positive for $(t, q) \in [1, \infty) \times (0, 1)$. Consequently,

$$[f_q(x) - f_q(x+1)]^{(i-1)} = (\ln q)^{i-1} \left\{ [(\ln q)^2 - (1-q)^2] q^x + \sum_{k=1}^{\infty} (k+1)^{i-1} [g_q(k+1) + (1-q)^2(1-q^{k+1})] q^{(k+1)x} \right\}$$

for $i \in \mathbb{N}$. This means that

$$(-1)^{i-1} [f_q(x) - f_q(x+1)]^{(i-1)} = (-1)^{i-1} (\ln q)^{i-1} \left\{ [(\ln q)^2 - (1-q)^2] q^x + \sum_{k=1}^{\infty} (k+1)^{i-1} [g_q(k+1) + (1-q)^2(1-q^{k+1})] q^{(k+1)x} \right\} > 0$$

which can be rearranged as

$$(-1)^{i-1} [f_q(x)]^{(i-1)} > (-1)^{i-1} [f_q(x+1)]^{(i-1)}.$$

By induction and [14, Lemma 2.4] which reads that

$$\lim_{x \rightarrow \infty} [f_q(x)]^{(i-1)} = 0$$

for $0 < q < 1$ and $i \in \mathbb{N}$, it follows that

$$\begin{aligned} (-1)^{i-1} [f_q(x)]^{(i-1)} &> (-1)^{i-1} [f_q(x+1)]^{(i-1)} > (-1)^{i-1} [f_q(x+2)]^{(i-1)} > \dots \\ &> (-1)^{i-1} [f_q(x+k)]^{(i-1)} \geq (-1)^{i-1} \lim_{k \rightarrow \infty} [f_q(x+k)]^{(i-1)} = 0 \end{aligned}$$

for $(i, k) \in \mathbb{N}^2$. Consequently, the function $f_q(x)$ for $0 < q < 1$ is completely monotonic on $(0, \infty)$. The proof of Theorem 2.1 is complete.

Remark 1. *This paper is a slightly modified version of the preprint [25].*

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FENG QI

INSTITUTE OF MATHEMATICS, HENAN POLYTECHNIC UNIVERSITY, JIAOZUO CITY, HENAN PROVINCE, 454010, CHINA; COLLEGE OF MATHEMATICS, INNER MONGOLIA UNIVERSITY FOR NATIONALITIES, TONGLIAO CITY, INNER MONGOLIA AUTONOMOUS REGION, 028043, CHINA; DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE, TIANJIN POLYTECHNIC UNIVERSITY, TIANJIN CITY, 300387, CHINA

E-mail address: qifeng618@gmail.com, qifeng618@hotmail.com

URL: <https://qifeng618.wordpress.com>

FANG-FANG LIU

DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE, TIANJIN POLYTECHNIC UNIVERSITY, TIANJIN CITY, 300387, CHINA

E-mail address: fang-liu@qq.com

XIAO-TING SHI

DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE, TIANJIN POLYTECHNIC UNIVERSITY, TIANJIN CITY, 300387, CHINA

E-mail address: xiao-ting.shi@hotmail.com, xiao-ting.shi@qq.com