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STUDIES ON THE TSALLIS ENTROPY DEFINED BY THE ALEXANDER OPERATOR IN A COMPLEX DOMAIN

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Abstract. In this article, we develop the definition of the Tsallis entropy in a complex domain. This development depends on the Alexander operator in a complex domain. We illustrate some new classes of analytic functions, which are studied in view of the geometry function theory. This type of entropy is called fractional entropy; therefore, we call them fractional entropic geometry classes. Moreover, we discuses the boundedness of the integral operator by applying the concept of Schwarz derivatives. Other geometric properties are demonstrated in the sequel. Our performance is carried by the Maxwell Lemma and Jack Lemma.

1. Introduction

Operators (differential and integral) are of serious significance to functional analysis, and they catch application in various other areas of pure and applied mathematics. For instance, in usual mechanics, the derivative is utilized ubiquitously, and in quantum mechanics, observations are denoted by hermitian operators. Important possessions that different operators may show contain linearity, compactness, boundedness and continuity.

An entropy of variables is defined for the first time by Tsallis and modified by many researchers. All of these modifications were in real cases. The benefit of interesting physical structures, attitudes to entropic methods that are auxiliary joint than the ordinary entropy. In 1988, Tsallis familiarized a new type of fractional entropy. The Tsallis Entropy has been exploited alongside with the Standard of maximum entropy to improve the Tsallis distribution. This entropy has been active in many fields such as thermodynamics, chaos, statistical mechanics and information theory. For continuous probability distributions, the entropy is formulated by (see [3],[4]):

$$\mathcal{T}_\gamma[\phi] = \frac{1}{\gamma - 1}\left(1 - \int (\phi(x))^\gamma \, dx\right), \quad \gamma \neq 1,$$

or in the functional form

$$\mathcal{T}_\gamma[\phi](x) = \frac{1}{\gamma - 1}\left(1 - \int_0^x (\phi(u))^\gamma \, du\right),$$

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where $\phi(x)$ is a probability density function.

Here, we extend the definition into the unit disk $U := \{z : |z| < 1\}$ by applying the analytic function $f(z), z \in U$ (type Schwarz function $|f(z)| < |z|$,) where it is normalized by $f(0) = 0$, $f'(0) = 1$. The class of all normalized functions is denoted by $\mathcal{A}$. In [1] (for recent work [6]), Alexander introduced a first order integral operator

$$A[f](z) = \int_0^z \frac{f(\xi)}{\xi} d\xi, \quad f \in \mathcal{A}.$$

Let $\phi(z) := \frac{f(z)}{z}$, $f \in \mathcal{A}$, $z \neq 0 \in U$ then we have the integral operator

$$A_\gamma[\phi](z) = \int_0^z (\phi(t))^\gamma dt \quad \gamma \neq 1,\, z \in U.$$

The advantage of using the Alexander operator is of subordinating-preserving integral operator. Moreover, if the function f is of bounded turning ($\Re(f'(z)) > 0$) then $A[f](z)$ is also of bounded turning ($\Re(A[f]'(z)) > 0$).

By using the above fractional Alexander operator, we formulate the complex Tsallis entropy by

$$\mathcal{T}_\gamma[\phi](z) = \frac{1 - A_\gamma[\phi](z)}{\gamma - 1}, \quad \gamma \neq 1,\, z \in U.$$

In this note, we suggest the fractional entropy, in order to study some geometric properties and information regarding the classes of analytic functions. Receiving information about the properties of a function from properties of its derivatives indicates a significant part in many areas of mathematical analysis. There are two patterns of these outcomes characterizations and bound to the function.

The function $f \in \mathcal{A}$ has enough information in each geometric classes; such as the starlikeness ($\mathcal{S}^*(\alpha)$)

$$\Re(zf'(z)/f(z)) > \alpha,\, \alpha \in [0, 1)$$

and convexity ($\mathcal{K}(\alpha)$);

$$\Re(1 + zf''(z)/f'(z)) > \alpha,\, \alpha \in [0, 1),$$

if the complex entropy satisfies

$$\Re\Big(\mathcal{T}_\gamma[\phi](z)\Big) > 0,$$

which is equivalent to

$$\Re\Big(1 - A_\gamma[\phi](z)\Big) > 0, \quad \gamma > 1.$$

Let

$$\rho(z) := 1 - A_\gamma[\phi](z), \quad z \in U.$$

The set of information for each geometric class can be realized by the conclusion

$$\mathcal{I}_\phi^\gamma = \Big\{f \in \mathcal{A} : 0 < \Re\Big(\mathcal{T}_\gamma[\phi](z)\Big) < \frac{\gamma}{\gamma - 1},\ \phi(z) = \frac{f(z)}{z},\ z \in U\backslash\{0\},\, \gamma \neq 1\Big\}.$$

Remark. 1.1 Studies regarding the operator $A_\gamma[\phi](z)$ can be viewed as concerning the operator

$$\mathcal{J}_\gamma[f](z) = \int_0^z (f'(t))^\gamma dt \quad \gamma \neq 1,\, z \in U.$$

The relation can be formulated by using $\circ$ as follows (see [2]):

$$A_\gamma[\phi](z) = \mathcal{J}_\gamma[f](z) \circ A[\phi](z).$$

It is well known that

$$A[\mathcal{S}^*] = \mathcal{J}[\mathcal{K}], \quad A_\gamma[\mathcal{S}^*] = \mathcal{J}_\gamma[\mathcal{K}].$$

Remark. 1.2 Under the norm $\|f\| = \sup_z (1-|z|^2)|\top_f(z)|$, $z \in U$ where $\top_f(z) := f''/f'$, and f is univalent function in U, we have

$$\|\mathcal{J}_\gamma[f](z)\| \leq |\gamma|\|f\| \leq 6|\gamma|$$

and

$$\|A_\gamma[\phi](z)\| \leq |\gamma|\|f\| \leq 4|\gamma|.$$

Our discussion is based on the Maxwell Lemma as well as Jack Lemma respectively

Lemma 1.1. [7] *If κ is real and ρ is analytic in the unit disk, then*

$$\Re\big(\rho(z) + \kappa z \rho'(z)/\rho(z)\big) > 0 \Longrightarrow \Re\big(\rho(z)\big) > 0.$$

Lemma 1.2. [5] *Let $h(z)$ be analytic in U with $h(0) = 0$. Then if $|h(z)|$ approaches its maximization when $|z| = r$ at a point $z_0 \in U$, then $z_0 h'(z_0) = \epsilon\, h(z_0)$, where $\epsilon \geq 1$ is a real number.*

Moreover, we need the subordination concept in the sequel, which is defined as follows: Assume that $\mu(z)$ and $\nu(z)$ are two analytic functions in U. Then $\mu(z)$ is said to be subordinate to $\nu(z)$ if there exists an analytic function $\psi(z)$ in U satisfying $\psi(0) = 0, |\psi(z)| < 1 (z \in U)$ and $\mu(z) = \nu(\psi(z))$. This subordination is noted by

$$\mu(z) \prec \nu(z), \quad z \in U.$$

2. The main finding

Our aim is to achieve the property of the set $\mathcal{I}^\gamma_\phi$.

Theorem 2.1. *Let $f \in \mathcal{A}$, $\gamma \geq 2$, and $1 < \alpha < 2$. If $\phi(z) = f(z)/z$ satisfies*

$$0 < \Re\Big(1 + \frac{zf''(z)}{f'(z)}\Big) < \frac{\alpha(2-\alpha)}{2(1+\alpha)} \tag{2.1}$$

then $\phi \in \mathcal{I}^\gamma_\phi$. Moreover, f is starlike in the open unit disk.

Proof. First, we show that $\Re\Big(\mathcal{T}_\gamma[\phi](z)\Big) < \frac{\gamma}{\gamma-1}$. Define a function g as follows:

$$g(z) = \mathcal{T}_\gamma[\phi](z) - \frac{1}{\gamma-1},\, \gamma \neq 1$$

such that

$$g'(z) = \Big(\mathcal{T}_\gamma[\phi](z)\Big)',$$

considering the relation

$$\frac{zf'(z)}{f(z)} = \frac{\alpha(1-g(z))}{\alpha - g(z)}, \quad g(z) \neq \alpha.$$

It is clear that g is analytic because f is analytic in the open disk U. In addition, $g(0)=0$, we need only to show that $|g(z)|<1$. A calculation yields

$$\begin{aligned}\Re\Big(1+\frac{zf''(z)}{f'(z)}\Big) &= \Re\Big(\frac{\alpha(1-g(z))}{\alpha-g(z)} - \frac{zg'(z)}{1-g(z)} + \frac{zg'(z)}{\alpha-g(z)}\Big)\\
&= \Re\Big(\frac{\alpha(1-[\mathcal{T}_\gamma[\phi](z)-\frac{1}{\gamma-1}])}{\alpha-[\mathcal{T}_\gamma[\phi](z)-\frac{1}{\gamma-1}]} - \frac{z\Big(\mathcal{T}_\gamma[\phi](z)\Big)'}{1-[\mathcal{T}_\gamma[\phi](z)-\frac{1}{\gamma-1}]} + \frac{z\Big(\mathcal{T}_\gamma[\phi](z)\Big)'}{\alpha-[\mathcal{T}_\gamma[\phi](z)-\frac{1}{\gamma-1}]}\Big)\\
&= \Re\Big(\frac{\alpha(1+\frac{1}{\gamma-1}-\mathcal{T}_\gamma[\phi](z))}{\alpha+\frac{1}{\gamma-1}-\mathcal{T}_\gamma[\phi](z)} - \frac{z\Big(\mathcal{T}_\gamma[\phi](z)\Big)'}{1+\frac{1}{\gamma-1}-\mathcal{T}_\gamma[\phi](z)} + \frac{z\Big(\mathcal{T}_\gamma[\phi](z)\Big)'}{\alpha+\frac{1}{\gamma-1}-\mathcal{T}_\gamma[\phi](z)}\Big)\\
&< \frac{\alpha(2-\alpha)}{2(1+\alpha)}.\end{aligned}$$

In view of Lemma 1.2, there exists a complex number $z_0 \in U$ such that $h(z_0) = e^{i\theta}$ and

$$z_0 h'(z_0) = \epsilon h(z_0) = \epsilon e^{i\theta},\ \epsilon \geq 1.$$

Therefore, we obtain

$$\begin{aligned}1+\frac{z_0 f''(z_0)}{f'(z_0)} &= \frac{\alpha(1+\frac{1}{\gamma-1}-\mathcal{T}_\gamma[\phi](z_0))}{\alpha+\frac{1}{\gamma-1}-\mathcal{T}_\gamma[\phi](z_0)} - \frac{z_0\Big(\mathcal{T}_\gamma[\phi](z_0)\Big)'}{1+\frac{1}{\gamma-1}-\mathcal{T}_\gamma[\phi](z_0)} + \frac{z_0\Big(\mathcal{T}_\gamma[\phi](z_0)\Big)'}{\alpha+\frac{1}{\alpha-1}-\mathcal{T}_\gamma[\phi](z_0)}\\
&= \frac{\alpha(1-e^{i\theta})}{\alpha-e^{i\theta}} - \frac{\epsilon e^{i\theta}}{1-e^{i\theta}} + \frac{\epsilon e^{i\theta}}{\alpha-e^{i\theta}}\\
&= \frac{\alpha+(\epsilon-\alpha)e^{i\theta}}{\alpha-e^{i\theta}} - \frac{\epsilon e^{i\theta}}{1-e^{i\theta}}.\end{aligned}$$

Since

$$\Re\big(\frac{1}{1-e^{i\theta}}\big) = \frac{1}{2}, \quad \Re\big(\frac{1}{\alpha-e^{i\theta}}\big) = \frac{1}{2\alpha} + \frac{\alpha^2-1}{2\alpha(1+\alpha^2-2\alpha\cos(\theta))}$$

then, we attain

$$\Re\Big(1+\frac{z_0 f''(z_0)}{f'(z_0)}\Big) = \frac{1+\alpha}{2} + \frac{(\alpha^2-1)(1-\alpha+\epsilon)}{2(1+\alpha^2-2\alpha\cos\theta)}.$$

For $\alpha \in (1,2)$, and $\epsilon = 1$, we get

$$\begin{aligned}\Re\Big(1+\frac{z_0 f''(z_0)}{f'(z_0)}\Big) &\geq \frac{2-\alpha}{2(1+\alpha)} + \frac{(\alpha-1)(2-\alpha)}{2(1+\alpha)}\\
&= \frac{\alpha(2-\alpha)}{2(1+\alpha)},\end{aligned}$$

which contradicts the assumption of the theorem. Therefore, $|g(z)| < 1$, consequently $\Re\Big(\mathcal{T}_\gamma[\phi](z)\Big) < \frac{\gamma}{\gamma-1}$. Obviously, we have

$$\begin{aligned} g(z) &= \mathcal{T}_\gamma[\phi](z) - \frac{1}{\gamma-1} \\ &= \frac{\alpha(\frac{zf'(z)}{f(z)}-1)}{\frac{zf'(z)}{f(z)}-\alpha}, \quad z \in U \\ &= \frac{\alpha(1-\frac{zf'(z)}{f(z)})}{\alpha-\frac{zf'(z)}{f(z)}}. \end{aligned} \tag{2.2}$$

Hence, we obtain

$$\frac{zf'(z)}{f(z)} \prec \frac{\alpha(1-z)}{\alpha-z}$$

This conclude that f is starlike in U.

Now, we proceed to show that $\Re\Big(\mathcal{T}_\gamma[\varphi](z)\Big) > 0$. Since φ is starlike, then we define a function $\rho(z)$ such that

$$\frac{zf'(z)}{f(z)} = \rho(z).$$

If, we let

$$\Phi(z) := \frac{zf'(z)}{f(z)} = \rho(z)$$

then we receive

$$\begin{aligned} \frac{z\Phi'(z)}{\Phi(z)} + \Phi(z) &= 1 + \frac{zf''(z)}{f'(z)} \\ &= \rho(z) + \frac{z\rho'(z)}{\rho(z)}. \end{aligned}$$

Hence, we have

$$\Re\Big(1 + \frac{zf''(z)}{f'(z)}\Big) = \Re\Big(\rho(z) + \frac{z\rho'(z)}{\rho(z)}\Big) > 0.$$

By applying Lemma 1.1, we have $\Re(\rho(z)) > 0$ and this implies that $\Re\Big(\mathcal{T}_\gamma[\phi](z)\Big) > 0$. Moreover, the above conclusion shows that $\phi(z) = f(z)/z \in \mathcal{I}_\phi^\gamma$. This completes the proof. □

A direct application of Theorem 2.1 and Remark 1.2, we have the following results:

Corollary 2.2. *Let the assumptions of Theorem 2.1 hold. Then $A_\gamma[\phi](z)$ is convex in the open unit disk and hence $\mathcal{T}_2[\phi](z)$.*

Corollary 2.3. *Let the assumptions of Theorem 2.1 hold. Then*

$$\|\mathcal{T}_\gamma[\phi](z)\| \le \frac{5\gamma+1}{\gamma-1}, \quad \gamma \ge 2.$$

Corollary 2.4. *Let the assumptions of Theorem 2.1 hold with $\gamma = 2$. Then $\mathcal{T}_\gamma[\phi](z)$ is locally univalent with the Hornich operation*

$$f \oplus g = \int_0^z f'(t)g'(t)dt, \quad \gamma \star f = \int_0^z f'(t)^\gamma dt.$$

3. Applications

We illustrate some examples of functions in the set $\mathcal{I}_\phi^\alpha$ by using Theorem 2.1.

Example 3.1 Consider the function

$$f(z) = \Big(\frac{1+\alpha}{2}\Big)(1-z)^{-\frac{1-\alpha}{1+\alpha}-1} - \Big(\frac{1+\alpha}{2}\Big), \quad z \in U.$$

Obviously, $f(0) = 0$ and $f'(0) = 1$, where

$$f'(z) = (1-z)^{-\frac{1-\alpha}{1+\alpha}-2}, \quad z \in U.$$

Now, we have

$$\begin{aligned} 0 < 1 + \Re\Big(\frac{zf''(z)}{f'(z)}\Big) &= 1 + (2 + \frac{1-\alpha}{1+\alpha})\Re\Big(\frac{z}{1-z}\Big), \quad z \in U \\ &= \frac{\alpha-1}{2(1+\alpha)}, \quad z \to^- 1 \\ &< \frac{\alpha(2-\alpha)}{2(1+\alpha)}, \quad \alpha \in (1,2). \end{aligned}$$

Hence, in view of Theorem 2.1, $\phi(z) = f(z)/z \in \mathcal{I}_\phi^\gamma$ and $f(z)$ is starlike in the open unit disk.

Example 3.2 Consider the function

$$f(z) = \Big(\frac{2(\alpha+1)}{\alpha^2+2}\Big)(1-z)^{-\frac{\alpha(\alpha-2)}{2(1+\alpha)}-1} - \Big(\frac{2(\alpha+1)}{\alpha^2+2}\Big), \quad z \in U.$$

Obviously, $f(0) = 0$ and $f'(0) = 1$, where

$$f'(z) = (1-z)^{-\frac{\alpha(\alpha-2)}{2(1+\alpha)}-2}, \quad z \in U.$$

A calculation implies that

$$\begin{aligned} 0 < 1 + \Re\Big(\frac{zf''(z)}{f'(z)}\Big) &= 1 + \big(2 + \frac{\alpha(\alpha-2)}{2(1+\alpha)}\big)\Re\Big(\frac{z}{1-z}\Big), \quad z \in U \\ &= \frac{\alpha(2-\alpha)}{4(1+\alpha)}, \quad z \to^- 1 \\ &< \frac{\alpha(2-\alpha)}{2(1+\alpha)}, \quad \alpha \in (1,2). \end{aligned}$$

Thus, in view of Theorem 2.1, $\phi \in \mathcal{I}_\phi^\gamma$ and f is starlike in the open unit disk.

Example 3.3 Consider the function

$$f(z) = \Big(\frac{\alpha+1}{1-\alpha}\Big)(1-z)^{-\frac{\alpha-2}{1+\alpha}-1} - \Big(\frac{\alpha+1}{1-\alpha}\Big), \quad z \in U.$$

Obviously, $f(0) = 0$ and $f'(0) = 1$, where

$$f'(z) = (1-z)^{-\frac{\alpha-2}{1+\alpha}-2}, \quad z \in U.$$

A computation gives

$$\begin{aligned} 0 < 1 + \Re\Big(\frac{zf''(z)}{f'(z)}\Big) &= 1 + \big(2 + \frac{\alpha-2}{1+\alpha}\big)\Re\Big(\frac{z}{1-z}\Big), \quad z \in U \\ &= \frac{2-\alpha}{2(1+\alpha)}, \quad z \to^{-} 1 \\ &< \frac{\alpha(2-\alpha)}{2(1+\alpha)}, \quad \alpha \in (1,2). \end{aligned}$$

Thus, in view of Theorem 2.1, $\phi \in \mathcal{I}_{\phi}^{\gamma}$ and f is starlike in the open unit disk.

Example 3.4 Consider the function

$$f(z) = \Big(\frac{2(1+\alpha)}{1-\alpha}\Big)(1-z)^{-\frac{\alpha-2}{2(1+\alpha)}-1} - \Big(\frac{2(1+\alpha)}{1-\alpha}\Big), \quad z \in U.$$

Obviously, $f(0) = 0$ and $f'(0) = 1$, where

$$f'(z) = (1-z)^{-\frac{\alpha-2}{2(1+\alpha)}-2}, \quad z \in U.$$

A calculation implies that

$$\begin{aligned} 0 < 1 + \Re\Big(\frac{zf''(z)}{f'(z)}\Big) &= 1 + \big(2 + \frac{\alpha-2}{2(1+\alpha)}\big)\Re\Big(\frac{z}{1-z}\Big), \quad z \in U \\ &= \frac{2-\alpha}{4(1+\alpha)}, \quad z \to^{-} 1 \\ &< \frac{\lambda(2-\alpha)}{2(1+\alpha)}, \quad \alpha \in (1,2). \end{aligned}$$

Thus, in view of Theorem 2.1, $\phi \in \mathcal{I}_{\phi}^{\gamma}$ and it is starlike in the open unit disk.

4. Conclusion

An extended definition of Tsallis entropy in a complex domain is imposed based on some classes of analytic functions given by the Alexander operator in a complex domain. This definition is developed the fractional entropy. Moreover, we defined a new class of analytic function, may call the class of geometric information of analytic functions. We introduced a sufficient condition of the existing. Also, under the same condition, the function was starlike (Theorem 2.1) and the integral of the first order was convex (Corollary 2.2). We discussed the boundedness of the operator by applying the concept of Schwarz derivatives (see Corollary 2.3). The conditions of locally univalent of the operator is determined in Corollary 2.4. Some examples are shown the type of the normalized functions.

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