

## ON HADAMARD TYPE INEQUALITIES FOR $m$ -CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

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**ABSTRACT.** In this paper, we prove the Hadamard type inequalities for  $m$ -convex functions via fractional integrals and related inequalities. These results have some relationships with the Hadamard inequalities for fractional integrals and related inequalities.

### 1. INTRODUCTION

The Hadamard inequality states that: If  $f : I \rightarrow \mathbb{R}$  is a convex function on the interval  $I$  of real numbers and  $a, b \in I$  with  $a < b$ , then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

The Hadamard inequality have got attention of many mathematicians and many generalizations and refinements have been found so far for example see, [2, 3, 4, 7, 10, 11, 12, 13, 14, 15, 18, 19, 20, 22, 23].

In [24] Toader define the concept of  $m$ -convexity, an intermediate between usual convexity and star shape functions.

**Definition 1.1.** A function  $f : [0, b] \rightarrow \mathbb{R}$ ,  $b > 0$ , is said to be  $m$ -convex, where  $m \in [0, 1]$ , if we have

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

for all  $x, y \in [0, b]$  and  $t \in [0, 1]$ .

If we take  $m = 1$ , then we recapture the concept of convex functions defined on  $[0, b]$  and if we take  $m = 0$ , then we get the concept of starshaped functions on  $[0, b]$ . We recall that  $f : [0, b] \rightarrow \mathbb{R}$  is called *starshaped* if

$$f(tx) \leq tf(x) \text{ for all } t \in [0, b] \text{ and } x \in [0, b].$$

Denote by  $K_m(b)$  the set of the  $m$ -convex functions on  $[0, b]$  for which  $f(0) < 0$ , then one has

$$K_1(b) \subset K_m(b) \subset K_0(b),$$

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whenever  $m \in (0, 1)$ . Note that in the class  $K_1(b)$  are only convex functions  $f : [0, b] \rightarrow \mathbb{R}$  for which  $f(0) \leq 0$  (see, [10]).

**Example 1.2.** [16] The function  $f : [0, \infty) \rightarrow \mathbb{R}$ , given by

$$f(x) = \frac{1}{12} (4x^3 - 15x^2 + 18x - 5)$$

is  $\frac{16}{17}$ -convex function but it is not convex function.

Let  $f \in L_1[a, b]$ . Then the Riemann-Liouville fractional integrals  $J_{a+}^\alpha f$  and  $J_{b-}^\alpha f$  of order  $\alpha > 0$  with  $a \geq 0$  are defined as:

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b$$

where  $\Gamma(\alpha)$  is the Gamma function defined as:

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt,$$

also

$$\Gamma(\alpha + 1) = \alpha \Gamma(\alpha).$$

For more information one can consult [1, 5, 6, 8, 9].

Now a days well known inequalities have been considered for fractional integrals and a huge amount of research publications can be found in this regard for example one can see references and references there in.

In this paper we are motivated to give a version of the Hadamard inequality and some related inequalities for  $m$ -convex functions via fractional integrals. Also we connect our results some already known inequalities. In Section 2 we prove the Hadamard type inequality for  $m$ -convex functions via fractional integrals and deduce some related results. In Section 3 we prove a version of the Hadamard inequality for  $m$ -convex functions on co-ordinates via fractional integrals defined on coordinates. Also as particular cases some known results are mentioned.

## 2. HADAMARD TYPE INEQUALITIES FOR $m$ -CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

In this section we give Hadamard type inequalities for  $m$ -convex functions via fractional integrals and related fractional inequalities.

**Theorem 2.1.** Let  $f : [0, \infty) \rightarrow \mathbb{R}$  be a positive integrable function. If  $f$  is  $m$ -convex function on  $[0, \infty)$ , then for  $0 \leq a \leq mb$  following inequalities hold

$$(1) \quad f\left(\frac{a+mb}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha} \left[ J_{a+}^\alpha f(mb) + m^{\alpha+1} J_{b-}^\alpha f\left(\frac{a}{m}\right) \right] \\ \leq \frac{\alpha}{2(\alpha+1)} \left[ f(a) - m^2 f\left(\frac{a}{m^2}\right) \right] + \frac{m}{2} \left[ f(b) + m f\left(\frac{a}{m^2}\right) \right]$$

with  $\alpha > 0$ .

*Proof.* Since  $f$  is  $m$ -convex function, we have for  $x, y \in [a, mb]$  with  $\lambda = \frac{1}{2}$

$$(2) \quad f\left(\frac{x+my}{2}\right) \leq \frac{f(x)+mf(y)}{2}.$$

For  $x = ta + m(1-t)b$ ,  $y = tb + \frac{1}{m}(1-t)a$ , we have

$$(3) \quad 2f\left(\frac{a+mb}{2}\right) \leq f(ta + m(1-t)b) + mf\left(tb + \frac{1}{m}(1-t)a\right).$$

Multiplying both sides of (3) by  $t^{\alpha-1}$ , then integrating with respect to  $t$  over  $[0, 1]$ , we get

$$\begin{aligned} (4) \quad & \frac{2}{\alpha} f\left(\frac{a+mb}{2}\right) \leq \int_0^1 t^{\alpha-1} f(ta + m(1-t)b) dt + m \int_0^1 t^{\alpha-1} f(tb + \frac{1}{m}(1-t)a) dt \\ &= \int_{mb}^a \left(\frac{mb-u}{mb-a}\right)^{\alpha-1} f(u) \frac{du}{a-mb} + m^2 \int_{\frac{a}{m}}^b \left(\frac{v-\frac{a}{m}}{b-\frac{a}{m}}\right)^{\alpha-1} f(v) \frac{dv}{mb-a} \\ &= \Gamma(\alpha) \frac{1}{(mb-a)^\alpha} \left[ J_{a+}^\alpha f(mb) + m^{\alpha+1} J_{b-}^\alpha f\left(\frac{a}{m}\right) \right]. \end{aligned}$$

From which we get the first inequality in (1). For the prove of second inequality, in (2) note that if  $f$  is  $m$ -convex, then for  $\lambda \in [0, 1]$ , it yields

$$(5) \quad f(ta + m(1-t)b) \leq tf(a) + m(1-t)f(b)$$

and

$$(6) \quad mf\left(tb + \frac{(1-t)}{m}a\right) \leq mt f(b) + m^2(1-t)f\left(\frac{a}{m^2}\right).$$

By adding above two inequalities, we have

$$\begin{aligned} (7) \quad & f(ta + m(1-t)b) + mf\left(tb + \frac{(1-t)}{m}a\right) \\ & \leq tf(a) + m(1-t)f(b) + mt f(b) + m^2(1-t)f\left(\frac{a}{m^2}\right). \end{aligned}$$

Multiplying both sides of (7) by  $\frac{\alpha}{2}t^{\alpha-1}$ , then integrating with respect to  $t$  over  $[0, 1]$ , we get

$$\begin{aligned} & \frac{\alpha}{2} \int_0^1 t^{\alpha-1} f(ta + m(1-t)b) dt + \frac{\alpha}{2} m \int_0^1 t^{\alpha-1} f\left(tb + \frac{(1-t)}{m}a\right) dt \\ & \leq \frac{\alpha}{2} \int_0^1 t^\alpha f(a) dt + m \frac{\alpha}{2} \int_0^1 t^{\alpha-1} (1-t)f(b) dt + \frac{\alpha}{2} m \int_0^1 t^\alpha f(b) dt \\ & \quad + \frac{\alpha}{2} m^2 \int_0^1 t^{\alpha-1} (1-t)f\left(\frac{a}{m^2}\right) dt. \end{aligned}$$

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha} \left[ J_{a+}^\alpha f(mb) + m^{\alpha+1} J_{b-}^\alpha f\left(\frac{a}{m}\right) \right] \\ & \leq \frac{\alpha}{2(\alpha+1)} f(a) - m^2 \frac{\alpha}{2(\alpha+1)} f\left(\frac{a}{m^2}\right) + \frac{m}{2} f(b) + \frac{m^2}{2} f\left(\frac{a}{m^2}\right). \end{aligned}$$

From which we get second inequality in (1).  $\square$

**Corollary 2.2.** *If in Theorem 2.1, we take  $m = 1$ , then inequality (1) becomes*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \leq \frac{f(a) + f(b)}{2}.$$

In [20] above result is proved.

**Remark 2.3.** If we take  $\alpha = 1$  along with  $m = 1$  in Theorem 2.1 then we get the Hadamard inequality.

For next result we need the following lemma.

**Lemma 2.4.** *Let  $f : [a, mb] \rightarrow \mathbb{R}$  be a differentiable mapping on  $(a, mb)$  with  $a < mb$ . If  $f' \in L[a, mb]$ , then the following equality for fractional integrals holds:*

$$(8) \quad \begin{aligned} \frac{f(a) + f(mb)}{2} - \frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha} [J_{a^+}^\alpha f(mb) + J_{mb^-}^\alpha f(a)] \\ = \frac{mb-a}{2} \int_0^1 [(1-t)^\alpha - t^\alpha] f'(ta + m(1-t)b) dt, \end{aligned}$$

with  $\alpha > 0$ .

*Proof.* Since

$$(9) \quad \begin{aligned} & \frac{mb-a}{2} \int_0^1 [(1-t)^\alpha - t^\alpha] f'(ta + m(1-t)b) dt \\ &= \frac{mb-a}{2} \int_0^1 (1-t)^\alpha f'(ta + m(1-t)b) dt - \frac{mb-a}{2} \int_0^1 t^\alpha f'(ta + m(1-t)b) dt. \end{aligned}$$

First term of right hand side is calculated as

$$\begin{aligned} & \frac{mb-a}{2} \int_0^1 (1-t)^\alpha f'(ta + m(1-t)b) dt \\ &= \frac{f(mb)}{2} + \frac{\alpha}{2} \int_{mb}^a \left(\frac{a-x}{a-mb}\right)^{\alpha-1} \frac{f(x)}{a-mb} dx \\ &= \frac{f(mb)}{2} - \frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha} J_{mb^-}^\alpha f(a), \end{aligned}$$

while second term of right side is calculated as

$$\begin{aligned} & -\frac{mb-a}{2} \int_0^1 t^\alpha f'(ta + m(1-t)b) dt \\ &= \frac{f(a)}{2} - \frac{\alpha}{2} \int_{mb}^a \left(\frac{mb-x}{mb-a}\right)^{\alpha-1} \frac{f(x)}{a-mb} dx \\ &= \frac{f(a)}{2} - \frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha} J_{a^+}^\alpha f(mb). \end{aligned}$$

Now using these value in (9), we get required result.  $\square$

**Remark 2.5.** If we take  $m = 1$  in Lemma 2.4, then equality (8) becomes

$$\begin{aligned} \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] \\ = \frac{b-a}{2} \int_0^1 [(1-t)^\alpha - t^\alpha] f'(ta + (1-t)b) dt, \end{aligned}$$

which is proved in [20]. If we take  $\alpha = 1$  along with  $m = 1$  in Lemma 2.4, then equality (8) gives an equality in [12, Lemma 2.1].

Using above lemma we give the following Hadamard-type inequality for  $m$ -convex functions.

**Theorem 2.6.** Let  $f : [a, mb] \rightarrow \mathbb{R}$  be a differentiable mapping on  $(a, mb)$  with  $0 \leq a < mb$ . If  $|f'|$  is  $m$ -convex on  $[a, mb]$ , then the following inequality for fractional integrals holds

$$(10) \quad \left| \frac{f(a) + f(mb)}{2} - \frac{\Gamma(\alpha + 1)}{2(mb-a)^\alpha} [J_{a+}^\alpha f(mb) + J_{mb-}^\alpha f(a)] \right| \leq \frac{mb-a}{2} \left( 1 - \frac{1}{2^\alpha} \right) [f'(a) + mf'(b)],$$

with  $\alpha > 0$ .

*Proof.* Using Lemma 2.4 and  $m$ -convexity of  $|f'|$ , we find

$$\begin{aligned} & \left| \frac{f(a) + f(mb)}{2} - \frac{\Gamma(\alpha + 1)}{2(mb-a)^\alpha} [J_{a+}^\alpha f(mb) + J_{mb-}^\alpha f(a)] \right| \\ & \leq \frac{mb-a}{2} \int_0^1 |(1-t)^\alpha - t^\alpha| |f'(ta + m(1-t)b)| dt \\ & \leq \frac{mb-a}{2} \int_0^1 |(1-t)^\alpha - t^\alpha| \left[ t |f'(a)| + m(1-t) |f'(b)| \right] dt \\ & = \frac{mb-a}{2} \left( \int_0^{\frac{1}{2}} |(1-t)^\alpha - t^\alpha| \left[ t |f'(a)| + m(1-t) |f'(b)| \right] dt \right. \\ & \quad \left. + \int_{\frac{1}{2}}^1 |t^\alpha - (1-t)^\alpha| \left[ t |f'(a)| + m(1-t) |f'(b)| \right] dt \right) \\ (11) \quad & \left| \frac{f(a) + f(mb)}{2} - \frac{\Gamma(\alpha + 1)}{2(mb-a)^\alpha} [J_{a+}^\alpha f(mb) + J_{mb-}^\alpha f(a)] \right| \\ & \leq \frac{mb-a}{2} (K_1 + K_2). \end{aligned}$$

Now calculating  $K_1$  and  $K_2$ , we have

$$\begin{aligned} K_1 &= |f'(a)| \left[ \int_0^{\frac{1}{2}} t(1-t)^\alpha dt - \int_0^{\frac{1}{2}} t^{\alpha+1} dt \right] \\ &\quad + m |f'(b)| \left[ \int_0^{\frac{1}{2}} (1-t)^{\alpha+1} dt - \int_0^{\frac{1}{2}} (1-t)t^\alpha dt \right] \end{aligned}$$

$$= \left| f'(a) \right| \left[ \frac{1}{(\alpha+1)(\alpha+2)} - \frac{(\frac{1}{2})^{\alpha+1}}{(\alpha+1)} \right] + m \left| f'(b) \right| \left[ \frac{1}{(\alpha+2)} - \frac{(\frac{1}{2})^{\alpha+1}}{(\alpha+1)} \right]$$

and

$$\begin{aligned} K_2 &= \left| f'(a) \right| \left[ \int_{\frac{1}{2}}^1 t^{\alpha+1} dt - \int_{\frac{1}{2}}^1 t(1-t)^{\alpha} dt \right] \\ &\quad + m \left| f'(b) \right| \left[ \int_{\frac{1}{2}}^1 (1-t)t^{\alpha} dt - \int_{\frac{1}{2}}^1 (1-t)^{\alpha+1} dt \right] \\ &= \left| f'(a) \right| \left[ \frac{1}{(\alpha+2)} - \frac{(\frac{1}{2})^{\alpha+1}}{(\alpha+1)} \right] + m \left| f'(b) \right| \left[ \frac{1}{(\alpha+1)(\alpha+2)} - \frac{(\frac{1}{2})^{\alpha+1}}{(\alpha+1)} \right]. \end{aligned}$$

Using values of  $K_1$  and  $K_2$  in (11), we have

$$\begin{aligned} &\left| \frac{f(a) + f(mb)}{2} - \frac{\Gamma(\alpha+1)}{2(mb-a)^{\alpha}} [J_{a+}^{\alpha} f(mb) + J_{mb-}^{\alpha} f(a)] \right| \\ &\leq \frac{mb-a}{2} \left( \left| f'(a) \right| \left[ \frac{1}{(\alpha+1)(\alpha+2)} - \frac{(\frac{1}{2})^{\alpha+1}}{(\alpha+1)} + \frac{1}{(\alpha+2)} - \frac{(\frac{1}{2})^{\alpha+1}}{(\alpha+1)} \right] \right. \\ &\quad \left. + m \left| f'(b) \right| \left[ \frac{1}{(\alpha+1)(\alpha+2)} - \frac{(\frac{1}{2})^{\alpha+1}}{(\alpha+1)} + \frac{1}{(\alpha+1)} - \frac{(\frac{1}{2})^{\alpha+1}}{(\alpha+1)} \right] \right) \\ &= \frac{mb-a}{2} \left( \left| f'(a) \right| \left[ \frac{1}{(\alpha+1)} - \frac{(\frac{1}{2})^{\alpha}}{(\alpha+1)} \right] + m \left| f'(b) \right| \left[ \frac{1}{(\alpha+1)} - \frac{(\frac{1}{2})^{\alpha}}{(\alpha+1)} \right] \right) \\ &= \frac{mb-a}{2} \left( 1 - \frac{1}{2^{\alpha}} \right) [\left| f'(a) \right| + m \left| f'(b) \right|]. \end{aligned}$$

The proof is completed.  $\square$

**Corollary 2.7.** *If in Theorem 2.6, we take  $m = 1$ , then inequality (10) becomes*

$$\begin{aligned} &\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} [J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)] \right| \\ &\leq \frac{b-a}{2} \left( 1 - \frac{1}{2^{\alpha}} \right) [f'(a) + f'(b)], \end{aligned}$$

which is proved in [20].

**Remark 2.8.** If we take  $\alpha = 1$  along with  $m = 1$  in Theorem 2.6, then inequality (10) gives an inequality in [12, Theorem 2.2].

### 3. HADAMARD-TYPE INEQUALITIES FOR COORDINATED $m$ -CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

In this section we use co-ordinated  $m$ -convex functions to give Hadamard type inequalities for fractional integrals and related fractional inequalities for two co-ordinates. First we give preliminaries for this section. Throughout we consider  $\Delta = [a, b] \times [c, d]$ .

**Definition 3.1.** [13] A function  $f : \Delta \rightarrow \mathbb{R}$  is said to be co-ordinated m-convex on  $\Delta$  if the following inequality holds:

(12)

$$\begin{aligned} & f(tx + m(1-t)y, su + m(1-s)w) \\ & \leq tsf(x, u) + ms(1-t)f(y, u) + mt(1-s)f(x, w) + m^2(1-t)(1-s)f(y, w), \end{aligned}$$

for all  $(x, y), (u, w) \in \Delta$ .

In [21] Riemann-Liouville integrals on two co-ordinates are defined as:

**Definition 3.2.** Consider  $f \in L_1(\Delta)$ , then the Riemann-Liouville integrals  $J_{a+,c+}^{\alpha,\beta}, J_{a+,d-}^{\alpha,\beta}, J_{b-,c+}^{\alpha,\beta}, J_{b-,d-}^{\alpha,\beta}$ , of order  $\alpha, \beta > 0$  with  $a, c \geq 0$  are defined as:

$$J_{a+,c+}^{\alpha,\beta} f(x, y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_c^y (x-t)^{\alpha-1}(y-s)^{\beta-1} f(t, s) ds dt, x > a, y > c,$$

$$J_{a+,d-}^{\alpha,\beta} f(x, y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_y^d (x-t)^{\alpha-1}(s-y)^{\beta-1} f(t, s) ds dt, x > a, y < d,$$

$$J_{b-,c+}^{\alpha,\beta} f(x, y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_c^y (t-x)^{\alpha-1}(y-s)^{\beta-1} f(t, s) ds dt, x < b, y > c,$$

and

$$J_{b-,d-}^{\alpha,\beta} f(x, y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_y^d (t-x)^{\alpha-1}(s-y)^{\beta-1} f(t, s) ds dt, x < b, y < d$$

respectively. Also

$$J_{a+,c+}^{0,0} f(x, y) = J_{a+,d-}^{0,0} f(x, y) = J_{b-,c+}^{0,0} f(x, y) = J_{b-,d-}^{0,0} f(x, y) = f(x, y).$$

There in [21] also defined:

$$\begin{aligned} J_{a+}^\alpha f\left(x, \frac{c+d}{2}\right) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f\left(t, \frac{c+d}{2}\right) dt, x > a, \\ J_{b-}^\alpha f\left(x, \frac{c+d}{2}\right) &= \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f\left(t, \frac{c+d}{2}\right) dt, x < b, \\ J_{c+}^\beta f\left(\frac{a+b}{2}, y\right) &= \frac{1}{\Gamma(\beta)} \int_c^y (y-s)^{\beta-1} f\left(\frac{a+b}{2}, s\right) ds, y > c, \\ J_{d-}^\beta f\left(\frac{a+b}{2}, y\right) &= \frac{1}{\Gamma(\beta)} \int_y^d (s-y)^{\beta-1} f\left(\frac{a+b}{2}, s\right) ds, y < d. \end{aligned}$$

In the following we give Hadamard type inequalities for co-ordinated  $m$ -convex functions on co-ordinates.

**Theorem 3.3.** Let  $f : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$  be co-ordinated  $m$ -convex functions with  $0 \leq a < mb$ ,  $0 \leq c < md$  and  $f \in L_1([0, \infty) \times [0, \infty))$ . Then for  $0 \leq a < mb$ ,

$0 \leq c < md$  the following inequalities hold

$$\begin{aligned}
(13) \quad & f\left(\frac{a+mb}{2}, \frac{c+md}{2}\right) \leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(mb-a)^\alpha(md-c)^\beta} \left[ J_{a+,c+}^{\alpha,\beta} f(mb, md) \right. \\
& + m J_{a+,d-}^{\alpha,\beta} f\left(mb, \frac{c}{m}\right) + m J_{b-,c+}^{\alpha,\beta} f\left(\frac{a}{m}, md\right) + m^2 J_{b-,d-}^{\alpha,\beta} f\left(\frac{a}{m}, \frac{c}{m}\right) \left. \right] \\
& \leq \frac{m^2}{4} \left[ mf\left(b, \frac{c}{m^2}\right) + m^2 f\left(\frac{a}{m^2}, \frac{c}{m^2}\right) + mf\left(\frac{a}{m^2}, d\right) + f(b, d) \right] \\
& + \frac{\alpha\beta}{4(\alpha+1)(\beta+1)} \left[ f(a, c) - m^2 f\left(a, \frac{c}{m^2}\right) + m^4 f\left(\frac{a}{m^2}, \frac{c}{m^2}\right) - m^2 f\left(\frac{a}{m^2}, c\right) \right] \\
& + \frac{\alpha}{4(\alpha+1)} \left[ m^2 f\left(a, \frac{c}{m^2}\right) - m^3 f\left(\frac{a}{m^2}, d\right) - m^4 f\left(\frac{a}{m^2}, \frac{c}{m^2}\right) + mf(a, d) \right] \\
& + \frac{\beta}{4(\beta+1)} \left[ mf(b, c) - m^3 f\left(b, \frac{c}{m^2}\right) - m^4 f\left(\frac{a}{m^2}, \frac{c}{m^2}\right) + m^2 f\left(\frac{a}{m^2}, c\right) \right].
\end{aligned}$$

*Proof.* From inequality (12) with  $x = t_1a + m(1-t_1)b$ ,  $y = \frac{(1-t_1)a}{m} + t_1b$ ,  $u = s_1c + m(1-s_1)d$ ,  $w = \frac{(1-s_1)c}{m} + s_1d$ ,  $t = s = \frac{1}{2}$ , we get

$$\begin{aligned}
(14) \quad & f\left(\frac{a+mb}{2}, \frac{c+md}{2}\right) \leq \frac{1}{4} [f(t_1a + m(1-t_1)b, s_1c + m(1-s_1)d) \\
& + mf\left(t_1a + m(1-t_1)b, \frac{1}{m}(1-s_1)c + s_1d\right) + mf\left(\frac{1}{m}(1-t_1)a + t_1b, s_1c + m(1-s_1)d\right) \\
& + m^2 f\left(\frac{1}{m}(1-t_1)a + t_1b, \frac{1}{m}(1-s_1)c + s_1d\right)].
\end{aligned}$$

Multiplying both sides of inequality (14) with  $t_1^{\alpha-1}s_1^{\beta-1}$  and integrating the resulting inequality over  $[0, 1] \times [0, 1]$ , we get

$$\begin{aligned}
(15) \quad & \frac{4}{\alpha\beta} f\left(\frac{a+mb}{2}, \frac{c+md}{2}\right) \\
& \leq \frac{1}{4} \left[ \int_0^1 \int_0^1 t_1^{\alpha-1} s_1^{\beta-1} f(t_1a + m(1-t_1)b, s_1c + m(1-s_1)d) ds_1 dt_1 \right. \\
& + m \int_0^1 \int_0^1 t_1^{\alpha-1} s_1^{\beta-1} f\left(t_1a + m(1-t_1)b, \frac{1}{m}(1-s_1)c + s_1d\right) ds_1 dt_1 \\
& + m \int_0^1 \int_0^1 t_1^{\alpha-1} s_1^{\beta-1} f\left(\frac{1}{m}(1-t_1)a + t_1b, s_1c + m(1-s_1)d\right) ds_1 dt_1 \\
& \left. + m^2 \int_0^1 \int_0^1 t_1^{\alpha-1} s_1^{\beta-1} f\left(\frac{1}{m}(1-t_1)a + t_1b, \frac{1}{m}(1-s_1)c + s_1d\right) ds_1 dt_1 \right].
\end{aligned}$$

Using the change of variables we have

$$\begin{aligned} & \frac{4}{\alpha\beta} f\left(\frac{a+mb}{2}, \frac{c+md}{2}\right) \\ & \leq \frac{1}{(mb-a)^\alpha(md-c)^\beta} \left[ \int_a^{mb} \int_c^{md} (mb-x)^{\alpha-1}(md-y)^{\beta-1} f(x,y) dy dx \right. \\ & + m \int_a^{mb} \int_{\frac{c}{m}}^d (mb-x)^{\alpha-1} \left(y - \frac{c}{m}\right)^{\beta-1} f(x,y) dy dx + m \int_{\frac{a}{m}}^b \int_c^{md} \left(x - \frac{a}{m}\right)^{\alpha-1} \right. \\ & \left. (md-y)^{\beta-1} f(x,y) dy dx + m^2 \int_{\frac{a}{m}}^b \int_{\frac{c}{m}}^d \left(x - \frac{a}{m}\right)^{\alpha-1} \left(y - \frac{c}{m}\right)^{\beta-1} f(x,y) dy dx \right]. \end{aligned}$$

From which one can have first inequality of (13).

On the other hand from (12) for  $x = a, y = b, u = c, w = d$ , we have

$$\begin{aligned} & f(ta + m(1-t)b, sc + m(1-s)d) \\ & \leq tsf(a,c) + ms(1-t)f(b,c) + mt(1-s)f(a,d) + m^2(1-t)(1-s)f(b,d) \\ & f\left(ta + m(1-t)b, m\frac{(1-s)}{m^2}c + sd\right) \\ & \leq mt(1-s)f\left(a, \frac{c}{m^2}\right) + m^2(1-t)(1-s)f\left(b, \frac{c}{m^2}\right) + tsf(a,d) + ms(1-t)f(b,d) \\ & f\left(m\frac{(1-t)}{m^2}a + tb, sc + m(1-s)d\right) \\ & \leq ms(1-t)f\left(\frac{a}{m^2}, c\right) + tsf(b,c) + m^2(1-t)(1-s)f\left(\frac{a}{m^2}, d\right) + mt(1-s)f(b,d) \\ & f\left(m\frac{(1-t)}{m^2}a + tb, m\frac{(1-s)c}{m^2} + sd\right) \\ & \leq m^2(1-t)(1-s)f\left(\frac{a}{m^2}, \frac{c}{m^2}\right) + mt(1-s)f\left(b, \frac{c}{m^2}\right) + m(1-t)sf\left(\frac{a}{m^2}, d\right) \\ & + tsf(b,d). \end{aligned}$$

Adding the above four inequalities we get

(16)

$$\begin{aligned} & f(ta + m(1-t)b, sc + m(1-s)d) + mf\left(ta + m(1-t)b, m\frac{(1-s)}{m^2}c + sd\right) \\ & + mf\left(m\frac{(1-t)}{m^2}a + tb, sc + m(1-s)d\right) + m^2f\left(m\frac{(1-t)}{m^2}a + tb, m\frac{(1-s)}{m^2}c + sd\right) \\ & \leq tsf(a,c) + m^2t(1-s)f\left(a, \frac{c}{m^2}\right) + m^2s(1-t)f\left(\frac{a}{m^2}, c\right) + m^3(1-t)f\left(\frac{a}{m^2}, d\right) \\ & + m^4(1-t)(1-s)f\left(\frac{a}{m^2}, \frac{c}{m^2}\right) + msf(b,c) + mtf(a,d) + m^3(1-s)f\left(b, \frac{c}{m^2}\right) \\ & + m^2f(b,d). \end{aligned}$$

Multiplying both sides of inequality (16) with  $t^{\alpha-1}s^{\beta-1}$  and integrating the resulting inequality over  $[0, 1] \times [0, 1]$ , we get

$$\begin{aligned}
(17) \quad & \int_0^1 \int_0^1 t^{\alpha-1}s^{\beta-1} f(ta + m(1-t)b, sc + m(1-s)d) ds dt \\
& + m \int_0^1 \int_0^1 t^{\alpha-1}s^{\beta-1} f \left( ta + m(1-t)b, m \frac{(1-s)c}{m^2} + sd \right) ds dt \\
& + m \int_0^1 \int_0^1 t^{\alpha-1}s^{\beta-1} f \left( m \frac{(1-t)a}{m^2} + tb, sc + m(1-s)d \right) ds dt \\
& + m^2 \int_0^1 \int_0^1 t^{\alpha-1}s^{\beta-1} f \left( m \frac{(1-t)a}{m^2} + tb, m \frac{(1-s)c}{m^2} + sd \right) ds dt \\
& \leq \int_0^1 \int_0^1 t^\alpha s^\beta \left[ f(a, c) - m^2 f \left( a, \frac{c}{m^2} \right) + m^4 f \left( \frac{a}{m^2}, \frac{c}{m^2} \right) - m^2 f \left( \frac{a}{m^2}, c \right) \right] ds dt \\
& + \int_0^1 \int_0^1 t^{\alpha-1}s^{\beta-1} \left[ m^3 f \left( b, \frac{c}{m^2} \right) + m^3 f \left( \frac{a}{m^2}, d \right) + m^4 f \left( \frac{a}{m^2}, \frac{c}{m^2} \right) + m^2 f(b, d) \right] ds dt \\
& + \int_0^1 \int_0^1 t^\alpha s^{\beta-1} \left[ m^2 f \left( a, \frac{c}{m^2} \right) - m^3 f \left( \frac{a}{m^2}, d \right) - m^4 f \left( \frac{a}{m^2}, \frac{c}{m^2} \right) + m f(a, d) \right] ds dt \\
& + \int_0^1 \int_0^1 t^{\alpha-1}s^\beta \left[ m f(b, c) - m^3 f \left( b, \frac{c}{m^2} \right) - m^4 f \left( \frac{a}{m^2}, \frac{c}{m^2} \right) + m^2 f \left( \frac{a}{m^2}, c \right) \right] ds dt.
\end{aligned}$$

Using change of variables we have

$$\begin{aligned}
(18) \quad & \frac{\Gamma(\alpha)\Gamma(\beta)}{(mb-a)^\alpha(m-d)^\beta} \left[ J_{a+,c+}^{\alpha,\beta} f(mb, md) + m J_{a+,d-}^{\alpha,\beta} f \left( mb, \frac{c}{m} \right) \right. \\
& \left. + m J_{b-,c+}^{\alpha,\beta} f \left( \frac{a}{m}, md \right) + m^2 J_{b-,d-}^{\alpha,\beta} f \left( \frac{a}{m}, \frac{c}{m} \right) \right] \\
& \leq \frac{1}{(\alpha+1)(\beta+1)} \left[ f(a, c) - m^2 f \left( a, \frac{c}{m^2} \right) + m^4 f \left( \frac{a}{m^2}, \frac{c}{m^2} \right) - m^2 f \left( \frac{a}{m^2}, c \right) \right] \\
& + \frac{1}{\alpha\beta} \left[ m^3 f \left( b, \frac{c}{m^2} \right) + m^4 f \left( \frac{a}{m^2}, \frac{c}{m^2} \right) + m^3 f \left( \frac{a}{m^2}, d \right) + m^2 f(b, d) \right] \\
& + \frac{1}{(\alpha+1)\beta} \left[ m^2 f \left( a, \frac{c}{m} \right) - m^3 f \left( \frac{a}{m^2}, d \right) - m^4 f \left( \frac{a}{m^2}, \frac{c}{m^2} \right) + m f(a, d) \right] \\
& + \frac{1}{\alpha(\beta+1)} \left[ m f(b, c) - m^3 f \left( b, \frac{c}{m^2} \right) - m^4 f \left( \frac{a}{m^2}, \frac{c}{m^2} \right) + m^2 f \left( \frac{a}{m^2}, c \right) \right].
\end{aligned}$$

From which one can have second inequality of (13).  $\square$

**Corollary 3.4.** *If in Theorem 3.3 we take  $m = 1$ , then inequality (13) becomes*

$$\begin{aligned}
& f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) \\
& \leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta} \left[ J_{a+,c+}^{\alpha,\beta} f(b, d) + J_{a+,d-}^{\alpha,\beta} f(b, c) + J_{b-,c+}^{\alpha,\beta} f(a, d) + J_{b-,d-}^{\alpha,\beta} f(a, c) \right] \\
& \leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}.
\end{aligned}$$

which is proved in [21].

**Remark 3.5.** If we put  $\alpha = \beta = 1$  along with  $m = 1$  in Theorem 3.3 then we get Hadamard inequalities in two co-ordinates.

**Theorem 3.6.** Let  $f : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$  be co-ordinated  $m$ -convex and  $f \in L_1([0, \infty) \times [0, \infty))$ . Then for  $0 \leq a \leq mb$  following inequalities hold:

$$\begin{aligned}
& (19) \\
& f\left(\frac{a+mb}{2}, \frac{c+md}{2}\right) \\
& \leq \frac{\Gamma(\alpha+1)}{4(mb-a)^\alpha} \left[ J_{a+}^\alpha f\left(mb, \frac{c+md}{2}\right) + m^{\alpha+1} J_{b-}^\alpha f\left(\frac{a}{m}, \frac{c+md}{2}\right) \right] \\
& + \frac{\Gamma(\beta+1)}{4(md-c)^\beta} \left[ J_{c+}^\beta f\left(\frac{a+mb}{2}, md\right) + m^{\beta+1} J_{d-}^\beta f\left(\frac{a+mb}{2}, \frac{c}{m}\right) \right] \\
& \leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(mb-a)^\alpha(md-c)^\beta} \left[ J_{a+,c+}^{\alpha,\beta} f(mb, md) + m^{\beta+1} J_{a+,d-}^{\alpha,\beta} f\left(mb, \frac{c}{m}\right) \right. \\
& \quad \left. + \left(\frac{m^{\alpha+1} + m^{\beta+1}}{2}\right) J_{b-,c+}^{\alpha,\beta} f\left(\frac{a}{m}, md\right) + m^{\alpha+\beta+2} J_{b-,d-}^{\alpha,\beta} f\left(\frac{a}{m}, \frac{c}{m}\right) \right] \\
& \leq \frac{m\alpha}{8(\alpha+1)} \left[ f(a, d) + mf\left(a, \frac{c}{m^2}\right) - m^2 f\left(\frac{a}{m^2}, d\right) - m^3 f\left(\frac{a}{m^2}, \frac{c}{m^2}\right) \right] \\
& + \frac{m\beta}{8(\beta+1)} \left[ f(b, c) + mf\left(\frac{a}{m^2}, c\right) - m^2 f\left(b, \frac{c}{m^2}\right) - m^3 f\left(\frac{a}{m^2}, \frac{c}{m^2}\right) \right] \\
& + \frac{m^2}{4} \left[ f(b, d) + mf\left(\frac{a}{m^2}, d\right) + mf\left(b, \frac{c}{m^2}\right) + m^2 f\left(\frac{a}{m^2}, \frac{c}{m^2}\right) \right] \\
& \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{8(\beta+1)(mb-a)^\alpha} \left[ J_{a+}^\alpha f(mb, c) - m^2 J_{a+}^\alpha f\left(mb, \frac{c}{m^2}\right) + m^{\alpha+1} J_{b-}^\alpha f\left(\frac{a}{m}, c\right) \right. \\
& \quad \left. - m^{\alpha+3} J_{b-}^\alpha f\left(\frac{a}{m}, \frac{c}{m^2}\right) \right] + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{8(\alpha+1)(md-c)^\beta} \left[ J_{c+}^\beta f(a, md) - m^2 J_{c+}^\beta f\left(\frac{a}{m^2}, md\right) \right. \\
& \quad \left. + m^{\beta+1} J_{d-}^\beta f\left(a, \frac{c}{m}\right) - m^{\beta+3} J_{d-}^\beta f\left(\frac{a}{m^2}, \frac{c}{m}\right) \right].
\end{aligned}$$

*Proof.* Since  $f : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$  is co-ordinated  $m$ -convex, so the mapping  $g_x : [0, \infty) \rightarrow \mathbb{R}$ ,  $g_x(t) = f(x, t)$ , is  $m$ -convex on  $[0, \infty)$  for all  $x \in [0, \infty)$ . By using inequality (1), we can write

$$\begin{aligned}
g_x\left(\frac{c+md}{2}\right) & \leq \frac{\Gamma(\beta+1)}{2(md-c)^\beta} \left[ J_{c+}^\beta g_x(md) + m^{\beta+1} J_{d-}^\beta g_x\left(\frac{c}{m}\right) \right] \\
& \leq \frac{\beta}{2(\beta+1)} \left[ g_x(c) - m^2 g_x\left(\frac{c}{m^2}\right) \right] + \frac{m}{2} \left[ g_x(d) + mg_x\left(\frac{c}{m^2}\right) \right],
\end{aligned}$$

which implies,

$$\begin{aligned}
& (20) \\
& f\left(x, \frac{c+md}{2}\right) \leq \frac{\beta}{2(md-c)^\beta} \left[ \int_c^{md} (md-y)^{\beta-1} f(x, y) dy + m^{\beta+1} \int_{\frac{c}{m}}^d \left(y - \frac{c}{m}\right)^{\beta-1} f(x, y) dy \right] \\
& \leq \frac{\beta}{2(\beta+1)} \left[ f(x, c) - m^2 f\left(x, \frac{c}{m^2}\right) \right] + \frac{m}{2} \left[ f(x, d) + mf\left(x, \frac{c}{m^2}\right) \right].
\end{aligned}$$

Multiplying both sides of inequality (20) with  $\frac{\alpha(mb-x)^{\alpha-1}}{2(mb-a)^\alpha}$  and integrating the resulted inequality over  $[a, mb]$ , we get

(21)

$$\begin{aligned} & \frac{\alpha}{2(mb-a)^\alpha} \int_a^{mb} (mb-x)^{\alpha-1} f\left(x, \frac{c+md}{2}\right) dx \\ & \leq \frac{\alpha\beta}{4(md-c)^\beta(mb-a)^\alpha} \left[ \int_a^{mb} \int_c^{md} (mb-x)^{\alpha-1} (md-y)^{\beta-1} f(x, y) dy dx \right. \\ & \quad \left. + m^{\beta+1} \int_a^{mb} \int_{\frac{c}{m}}^d (mb-x)^{\alpha-1} \left(y - \frac{c}{m}\right)^{\beta-1} f(x, y) dy dx \right] \\ & \leq \frac{\alpha\beta}{4(\beta+1)(mb-a)^\alpha} \left[ \int_a^{mb} (mb-x)^{\alpha-1} f(x, c) dx - m^2 \int_a^{mb} (mb-x)^{\alpha-1} f\left(x, \frac{c}{m^2}\right) dx \right] \\ & \quad + \frac{m\alpha}{4(mb-a)^\alpha} \left[ \int_a^{mb} (mb-x)^{\alpha-1} f(x, d) dx + m \int_a^{mb} (mb-x)^{\alpha-1} f\left(x, \frac{c}{m^2}\right) dx \right]. \end{aligned}$$

Multiplying both sides of inequality (20) with  $\frac{m^{\alpha+1}\alpha(x-\frac{a}{m})^{\alpha-1}}{2(mb-a)^\alpha}$  and integrating the resulted inequality over  $[\frac{a}{m}, b]$ , we get,

(22)

$$\begin{aligned} & \frac{m^{\alpha+1}\alpha}{2(mb-a)^\alpha} \int_{\frac{a}{m}}^b \left(x - \frac{a}{m}\right)^{\alpha-1} f\left(x, \frac{c+md}{2}\right) dx \\ & \leq \frac{m^{\alpha+1}\alpha\beta}{4(md-c)^\beta(mb-a)^\alpha} \left[ \int_{\frac{a}{m}}^b \int_c^{md} \left(x - \frac{a}{m}\right)^{\alpha-1} (md-y)^{\beta-1} f(x, y) dy dx \right. \\ & \quad \left. + m^{\beta+1} \int_{\frac{a}{m}}^b \int_{\frac{c}{m}}^d \left(x - \frac{a}{m}\right)^{\alpha-1} \left(y - \frac{c}{m}\right)^{\beta-1} f(x, y) dy dx \right] \\ & \leq \frac{m^{\alpha+2}\alpha}{4(mb-a)^\alpha} \left[ \int_{\frac{a}{m}}^b \left(x - \frac{a}{m}\right)^{\alpha-1} f(x, d) dx + m \int_{\frac{a}{m}}^b \left(x - \frac{a}{m}\right)^{\alpha-1} f\left(x, \frac{c}{m^2}\right) dx \right] \\ & \quad + \frac{m^{\alpha+1}\alpha\beta}{4(\beta+1)(mb-a)^\alpha} \left[ \int_{\frac{a}{m}}^b \left(x - \frac{a}{m}\right)^{\alpha-1} f(x, c) dx - m^2 \int_{\frac{a}{m}}^b \left(x - \frac{a}{m}\right)^{\alpha-1} f\left(x, \frac{c}{m^2}\right) dx \right]. \end{aligned}$$

Similarly for the mapping  $g_y : [0, \infty) \rightarrow \mathbb{R}$ ,  $g_y(t) = f(t, y)$ , we have

(23)

$$\begin{aligned}
& \frac{\beta}{2(md-c)^\beta} \int_c^{md} (md-y)^{\beta-1} f\left(\frac{a+mb}{2}, y\right) dy \\
& \leq \frac{\alpha\beta}{4(md-c)^\beta(mb-a)^\alpha} \left[ \int_a^{mb} \int_c^{md} (mb-x)^{\alpha-1} (md-y)^{\beta-1} f(x, y) dy dx \right. \\
& \quad \left. + m^{\alpha+1} \int_{\frac{a}{m}}^b \int_c^{md} \left(x - \frac{a}{m}\right)^{\alpha-1} (md-y)^{\beta-1} f(x, y) dy dx \right] \\
& \leq \frac{m\beta}{4(md-c)^\beta} \left[ \int_c^{md} (md-y)^{\beta-1} f(b, y) dy + m \int_c^{md} (md-y)^{\beta-1} f\left(\frac{a}{m^2}, y\right) dy \right] \\
& \quad + \frac{\alpha\beta}{4(\alpha+1)(md-c)^\beta} \left[ \int_c^{md} (md-y)^{\beta-1} f(a, y) dy - m^2 \int_c^{md} (md-y)^{\beta-1} f\left(\frac{a}{m^2}, y\right) dy \right]
\end{aligned}$$

and

(24)

$$\begin{aligned}
& \frac{m^{\beta+1}\beta}{2(md-c)^\beta} \int_{\frac{c}{m}}^d \left(y - \frac{c}{m}\right)^{\beta-1} f\left(\frac{a+mb}{2}, y\right) dy \\
& \leq \frac{m^{\beta+1}\alpha\beta}{4(md-c)^\beta(mb-a)^\alpha} \left[ \int_a^{mb} \int_{\frac{c}{m}}^d (mb-x)^{\alpha-1} \left(y - \frac{c}{m}\right)^{\beta-1} f(x, y) dy dx \right. \\
& \quad \left. + m^{\beta+1} \int_{\frac{a}{m}}^b \int_{\frac{c}{m}}^d \left(x - \frac{a}{m}\right)^{\alpha-1} \left(y - \frac{c}{m}\right)^{\frac{\beta}{k}-1} f(x, y) dy dx \right] \\
& \leq \frac{m^{\beta+2}\beta}{4(md-c)^\beta} \left[ \int_{\frac{c}{m}}^d \left(y - \frac{c}{m}\right)^{\beta-1} f(b, y) dy + m \int_{\frac{c}{m}}^d \left(y - \frac{c}{m}\right)^{\beta-1} f\left(\frac{a}{m^2}, y\right) dy \right] \\
& \quad + \frac{m^{\beta+1}\alpha\beta}{4(\alpha+1)(md-c)^\beta} \left[ \int_{\frac{c}{m}}^d \left(y - \frac{c}{m}\right)^{\beta-1} f(a, y) dy - m^2 \int_{\frac{c}{m}}^d \left(y - \frac{c}{m}\right)^{\beta-1} f\left(\frac{a}{m^2}, y\right) dy \right].
\end{aligned}$$

Adding (21), (22), (23), (24), we get,

$$\begin{aligned}
(25) \quad & \frac{\Gamma(\alpha+1)}{4(mb-a)^\alpha} \left[ J_{a+}^\alpha f \left( mb, \frac{c+md}{2} \right) + m^{\alpha+1} J_{b-}^\alpha f \left( \frac{a}{m}, \frac{c+md}{2} \right) \right] \\
& + \frac{\Gamma(\beta+1)}{4(md-c)^\beta} \left[ J_{c+}^\beta f \left( \frac{a+mb}{2}, md \right) + m^{\beta+1} J_{d-}^\beta f \left( \frac{a+mb}{2}, \frac{c}{m} \right) \right] \\
& \leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(mb-a)^\alpha(md-c)^\beta} \left[ J_{a+,c+}^{\alpha,\beta} f(mb, md) + m^{\beta+1} J_{a+,d-}^{\alpha,\beta} f \left( mb, \frac{c}{m} \right) \right. \\
& \quad \left. + m^{\alpha+1} J_{b-,c+}^{\alpha,\beta} f \left( \frac{a}{m}, md \right) + m^{\alpha+\beta+2} J_{b-,d-}^{\alpha,\beta} f \left( \frac{a}{m}, \frac{c}{m} \right) \right] \\
& \leq \frac{m\Gamma(\alpha+1)}{8(mb-a)^\alpha} \left[ J_{a+}^\alpha f(mb, d) + m J_{a+}^\alpha f \left( mb, \frac{c}{m^2} \right) + m^{\alpha+1} J_{b-}^\alpha f \left( \frac{a}{m}, d \right) \right. \\
& \quad \left. + m^{\alpha+2} J_{b-}^\alpha f \left( \frac{a}{m}, \frac{c}{m^2} \right) \right] + \frac{m\Gamma(\beta+1)}{8(md-c)^\beta} \left[ J_{c+}^\beta f(b, md) + m J_{c+}^\beta f \left( \frac{a}{m^2}, md \right) \right. \\
& \quad \left. + m^{\beta+1} J_{d-}^\beta f \left( b, \frac{c}{m} \right) + m^{\beta+2} J_{d-}^\beta f \left( \frac{a}{m^2}, \frac{c}{m} \right) \right] + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{8(\beta+1)(mb-a)^\alpha} \left[ J_{a+}^\alpha f(mb, c) \right. \\
& \quad \left. - m^2 J_{a+}^\alpha f \left( mb, \frac{c}{m^2} \right) + m^{\alpha+1} J_{b-}^\alpha f \left( \frac{a}{m}, c \right) - m^{\alpha+3} J_{b-}^\alpha f \left( \frac{a}{m}, \frac{c}{m^2} \right) \right] \\
& \quad + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{8(\alpha+1)(md-c)^\beta} \left[ J_{c+}^\beta f(a, md) - m^2 J_{c+}^\beta f \left( \frac{a}{m^2}, md \right) + m^{\beta+1} J_{d-}^\beta f \left( a, \frac{c}{m} \right) \right. \\
& \quad \left. - m^{\beta+3} J_{d-}^\beta f \left( \frac{a}{m^2}, \frac{c}{m} \right) \right]
\end{aligned}$$

Now from the first inequality of (1) via  $g_x$  and  $g_y$  we get

$$\begin{aligned}
f \left( \frac{a+mb}{2}, \frac{c+md}{2} \right) & \leq \frac{\alpha}{2(mb-a)^\alpha} \left[ \int_a^{mb} (mb-x)^{\alpha-1} f \left( x, \frac{c+md}{2} \right) dx \right. \\
& \quad \left. + m^{\alpha+1} \int_{\frac{a}{m}}^b \left( x - \frac{a}{m} \right)^{\alpha-1} f \left( x, \frac{c+md}{2} \right) dx \right]
\end{aligned}$$

and

$$\begin{aligned}
f \left( \frac{a+mb}{2}, \frac{c+md}{2} \right) & \leq \frac{\beta}{2(md-c)^\beta} \left[ \int_c^{md} (md-y)^{\beta-1} f \left( \frac{a+mb}{2}, y \right) dy \right. \\
& \quad \left. + m^{\beta+1} \int_{\frac{c}{m}}^{md} \left( y - \frac{c}{m} \right)^{\beta-1} f \left( \frac{a+mb}{2}, y \right) dy \right].
\end{aligned}$$

Adding above two inequalities we get,

$$\begin{aligned}
(26) \quad & f \left( \frac{a+mb}{2}, \frac{c+md}{2} \right) \leq \frac{\Gamma(\alpha+1)}{4(mb-a)^\alpha} \left[ J_{a+}^\alpha f \left( mb, \frac{c+md}{2} \right) + m^{\alpha+1} J_{b-}^\alpha f \left( \frac{a}{m}, \frac{c+md}{2} \right) \right] \\
& + \frac{\Gamma(\beta+1)}{4(md-c)^\beta} \left[ J_{c+}^\beta f \left( \frac{a+mb}{2}, md \right) + m^{\beta+1} J_{d-}^\beta f \left( \frac{a+mb}{2}, \frac{c}{m} \right) \right].
\end{aligned}$$

Similarly using the second inequality in (1) one can get

$$\begin{aligned}
& \frac{m^2\alpha}{2(mb-a)^\alpha} \left[ \int_a^{mb} (mb-x)^{\alpha-1} f\left(x, \frac{c}{m^2}\right) dx + m^{\alpha+1} \int_{\frac{a}{m}}^b \left(x - \frac{a}{m}\right)^{\alpha-1} f\left(x, \frac{c}{m^2}\right) dx \right] \\
& \leq \frac{m\alpha}{2(\alpha+1)} \left[ f\left(a, \frac{c}{m^2}\right) - m^2 f\left(\frac{a}{m^2}, \frac{c}{m^2}\right) \right] + \frac{m^2}{2} \left[ f\left(b, \frac{c}{m^2}\right) + mf\left(\frac{a}{m^2}, \frac{c}{m^2}\right) \right] \\
& \quad \frac{m\alpha}{2(mb-a)^\alpha} \left[ \int_a^{mb} (mb-x)^{\alpha-1} f(x, d) dx + m^{\alpha+1} \int_{\frac{a}{m}}^b \left(x - \frac{a}{m}\right)^{\alpha-1} f(x, d) dx \right] \\
& \leq \frac{m\alpha}{2(\alpha+1)} \left[ f(a, d) - m^2 f\left(\frac{a}{m^2}, d\right) \right] + \frac{m^2}{2} \left[ f(b, d) + mf\left(\frac{a}{m^2}, d\right) \right] \\
& \quad \frac{m^2\beta}{2(md-c)^\beta} \left[ \int_c^{md} (md-y)^{\beta-1} f\left(\frac{a}{m^2}, y\right) dy + m^{\beta+1} \int_{\frac{c}{m}}^d \left(y - \frac{c}{m}\right)^{\beta-1} f\left(\frac{a}{m^2}, y\right) dy \right] \\
& \leq \frac{m\beta}{2(\beta+1)} \left[ f\left(\frac{a}{m^2}, c\right) - m^2 f\left(\frac{a}{m^2}, \frac{c}{m^2}\right) \right] + \frac{m^2}{2} \left[ f\left(\frac{a}{m^2}, d\right) + mf\left(\frac{a}{m^2}, \frac{c}{m^2}\right) \right]
\end{aligned}$$

and

$$\begin{aligned}
& \frac{m\beta}{2(md-c)^\beta} \left[ \int_c^{md} (md-y)^{\beta-1} f(b, y) dy + m^{\beta+1} \int_{\frac{c}{m}}^d \left(y - \frac{c}{m}\right)^{\beta-1} f(b, y) dy \right] \\
& \leq \frac{m\beta}{2(\beta+1)} \left[ f(b, c) - m^2 f\left(b, \frac{c}{m^2}\right) \right] + \frac{m^2}{2} \left[ f(b, d) + mf\left(b, \frac{c}{m^2}\right) \right].
\end{aligned}$$

By adding the above four inequalities we get

$$\begin{aligned}
(27) \quad & \frac{m\Gamma(\alpha+1)}{8(mb-a)^\alpha} \left[ J_{a+}^\alpha f(mb, d) + mJ_{a+}^\alpha f\left(mb, \frac{c}{m^2}\right) + m^{\alpha+1} J_{b-}^\alpha f\left(\frac{a}{m}, d\right) \right. \\
& \quad \left. + m^{\alpha+2} J_{b-}^\alpha f\left(\frac{a}{m}, \frac{c}{m^2}\right) \right] + \frac{m\Gamma(\beta+1)}{8(md-c)^\beta} \left[ J_{c+}^\beta f(b, md) + mJ_{c+}^\beta f\left(\frac{a}{m^2}, md\right) \right. \\
& \quad \left. + m^{\beta+1} J_{d-}^\beta f\left(b, \frac{c}{m}\right) + m^{\beta+2} J_{d-}^\beta f\left(\frac{a}{m^2}, \frac{c}{m^2}\right) \right] \\
& \leq \frac{m\alpha}{8(\alpha+1)} \left[ f(a, d) + mf\left(a, \frac{c}{m^2}\right) - m^2 f\left(\frac{a}{m^2}, d\right) - m^3 f\left(\frac{a}{m^2}, \frac{c}{m^2}\right) \right] \\
& \quad + \frac{m\beta}{8(\beta+1)} \left[ f(b, c) + mf\left(\frac{a}{m^2}, c\right) - m^2 f\left(b, \frac{c}{m^2}\right) - m^3 f\left(\frac{a}{m^2}, \frac{c}{m^2}\right) \right] \\
& \quad + \frac{m^2}{4} \left[ f(b, d) + mf\left(\frac{a}{m^2}, d\right) + mf\left(b, \frac{c}{m^2}\right) + m^2 f\left(\frac{a}{m^2}, \frac{c}{m^2}\right) \right].
\end{aligned}$$

Adding the following term on both sides of (27)

$$\begin{aligned}
& \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{8(\beta+1)(mb-a)^\alpha} \left[ J_{a+}^\alpha f(mb, c) - m^2 J_{a+}^\alpha f\left(mb, \frac{c}{m^2}\right) + m^{\alpha+1} J_{b-}^\alpha f\left(\frac{a}{m}, c\right) \right. \\
& \quad \left. - m^{\alpha+3} J_{b-}^\alpha f\left(\frac{a}{m}, \frac{c}{m^2}\right) \right] + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{8(\alpha+1)(md-c)^\beta} \left[ J_{c+}^\beta f(a, md) - m^2 J_{c+}^\beta f\left(\frac{a}{m^2}, md\right) \right. \\
& \quad \left. + m^{\beta+1} J_{d-}^\beta f\left(a, \frac{c}{m}\right) - m^{\beta+3} J_{d-}^\beta f\left(\frac{a}{m^2}, \frac{c}{m}\right) \right].
\end{aligned}$$

After adding the above term we get

$$\begin{aligned}
(28) \quad & \frac{m\Gamma(\alpha+1)}{8(mb-a)^\alpha} \left[ J_{a+}^\alpha f(mb, d) + mJ_{a+}^\alpha f\left(mb, \frac{c}{m^2}\right) + m^{\alpha+1} J_{b-}^\alpha f\left(\frac{a}{m}, d\right) \right. \\
& + m^{\alpha+2} J_{b-}^\alpha f\left(\frac{a}{m}, \frac{c}{m^2}\right) \left. \right] + \frac{m\Gamma(\beta+1)}{8(md-c)^\beta} \left[ J_{c+}^\beta f(b, md) + mJ_{c+}^\beta f\left(\frac{a}{m^2}, md\right) \right. \\
& + m^{\beta+1} J_{d-}^\beta f\left(b, \frac{c}{m}\right) + m^{\beta+2} J_{d-}^\beta f\left(\frac{a}{m^2}, \frac{c}{m^2}\right) \left. \right] \\
& + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{8(\beta+1)(mb-a)^\alpha} \left[ J_{a+}^\alpha f(mb, c) - m^2 J_{a+}^\alpha f\left(mb, \frac{c}{m^2}\right) + m^{\alpha+1} J_{b-}^\alpha f\left(\frac{a}{m}, c\right) \right. \\
& - m^{\alpha+3} J_{b-}^\alpha f\left(\frac{a}{m}, \frac{c}{m^2}\right) \left. \right] + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{8(\alpha+1)(md-c)^\beta} \left[ J_{c+}^\beta f(a, md) - m^2 J_{c+}^\beta f\left(\frac{a}{m^2}, md\right) \right. \\
& + m^{\beta+1} J_{d-}^\beta f\left(a, \frac{c}{m}\right) - m^{\beta+3} J_{d-}^\beta f\left(\frac{a}{m^2}, \frac{c}{m}\right) \left. \right] \\
& \leq \frac{m\alpha}{8(\alpha+1)} \left[ f(a, d) + mf\left(a, \frac{c}{m^2}\right) - m^2 f\left(\frac{a}{m^2}, d\right) - m^3 f\left(\frac{a}{m^2}, \frac{c}{m^2}\right) \right] \\
& + \frac{m\beta}{8(\beta+1)} \left[ f(b, c) + mf\left(\frac{a}{m^2}, c\right) - m^2 f\left(b, \frac{c}{m^2}\right) - m^3 f\left(\frac{a}{m^2}, \frac{c}{m^2}\right) \right] \\
& + \frac{m^2}{4} \left[ f(b, d) + mf\left(\frac{a}{m^2}, d\right) + mf\left(b, \frac{c}{m^2}\right) + m^2 f\left(\frac{a}{m^2}, \frac{c}{m^2}\right) \right] \\
& + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{8(\beta+1)(mb-a)^\alpha} \left[ J_{a+}^\alpha f(mb, c) - m^2 J_{a+}^\alpha f\left(mb, \frac{c}{m^2}\right) + m^{\alpha+1} J_{b-}^\alpha f\left(\frac{a}{m}, c\right) \right. \\
& - m^{\alpha+3} J_{b-}^\alpha f\left(\frac{a}{m}, \frac{c}{m^2}\right) \left. \right] + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{8(\alpha+1)(md-c)^\beta} \left[ J_{c+}^\beta f(a, md) - m^2 J_{c+}^\beta f\left(\frac{a}{m^2}, md\right) \right. \\
& + m^{\beta+1} J_{d-}^\beta f\left(a, \frac{c}{m}\right) - m^{\beta+3} J_{d-}^\beta f\left(\frac{a}{m^2}, \frac{c}{m}\right) \left. \right]
\end{aligned}$$

by combining (25), (26), (28) we get (19).  $\square$

**Corollary 3.7.** *In Theorem 3.6 if we consider the inequality that exists between the second and third inequalities and take  $m = 1$  then we get the following inequalities*

$$\begin{aligned}
f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) & \leq \frac{\Gamma(\alpha+1)}{4(b-a)^\alpha} \left[ J_{a+}^\alpha f\left(b, \frac{c+d}{2}\right) + J_{b-}^\alpha f\left(a, \frac{c+d}{2}\right) \right] \\
& + \frac{\Gamma(\beta+1)}{4(d-c)^\beta} \left[ J_{c+}^\beta f\left(\frac{a+b}{2}, d\right) + J_{d-}^\beta f\left(\frac{a+b}{2}, c\right) \right] \\
& \leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta} \left[ J_{a+,c+}^{\alpha,\beta} f(b, d) + J_{a+,d-}^{\alpha,\beta} f(b, c) \right. \\
& \quad \left. + J_{b-,c+}^{\alpha,\beta} f(a, d) + J_{b-,d-}^{\alpha,\beta} f(a, c) \right] \\
& \leq \frac{\Gamma(\alpha+1)}{8(b-a)^\alpha} \left[ J_{a+}^\alpha f(b, c) + J_{a+}^\alpha f(b, d) + J_{b-}^\alpha f(a, c) + J_{b-}^\alpha f(a, d) \right] \\
& + \frac{\Gamma(\beta+1)}{8(d-c)^\beta} \left[ J_{c+}^\beta f(a, d) + J_{c+}^\beta f(b, d) + J_{d-}^\beta f(a, c) + J_{d-}^\beta f(b, c) \right] \\
& \leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}.
\end{aligned}$$

In this result we note a misprint that there should be  $\frac{1}{8}$  instead of  $\frac{1}{4}$  in the first term of last inequality and this result is proved in [21].

#### 4. CONCLUSION

In Section 2, we generalized many results given by Sarikaya et al. in [20], specially in Theorem 2.1, a version of the Hadamard inequality for  $m$ -convex functions via fractional integral is presented. In Section 3, the results are extended for  $m$ -convex function on coordinates, Theorem 3.3 is a generalization of [21, Theorem 3], while Theorem 3.6 is generalization of [21, Theorem 4]. This work may be further extended for  $m$ -convex functions on coordinates defined for suitable domain in  $\mathbb{R}^n$ .

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