

NEW GENERAL INTEGRAL INEQUALITIES FOR LIPSCHITZIAN FUNCTIONS VIA RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS AND APPLICATIONS

İMDAT İŞCAN, MEHMET KUNT*, NAZLI YAZICI GÖZÜTOK, TUNCAY KÖROĞLU

ABSTRACT. In this paper, we obtain some new estimates on generalization of Hermite-Hadamard, Ostrowski and Simpson type inequalities for Lipschitzian functions via Riemann-Liouville fractional integrals. Also, we give some applications to special means of two positive real numbers.

1. INTRODUCTION

In recent years, many authors have studied errors estimations for Hermite-Hadamard, Ostrowski and Simpson type inequalities, see [1, 2, 3, 4, 7, 14].

But this inequalities rarely studied for M -Lipschitzian functions. In our observation, some works for this subject as follows:

In [5], Dragomir et. al. give some inequalities of Hadamard's type for M -Lipschitzian functions and give some applications which are connected some special means of two positive numbers. In [21], Yang and Tseng establish several inequalities of Hadamard's type for Lipschitzian mappings. In [17, 18], Wang study several inequalities of Hadamard's type for Lipschitzian mappings and give some applications. In [16], Tseng et. al. establish some Hermite-type and Bullen-type inequalities for Lipschitzian functions and give several applications for special means. In [6], Hwang et. al. establish some Hadamard-type inequalities for Lipschitzian functions in one and two variables and give several applications for special means. In [11], İşcan study Hadamard, Ostrowski and Simpson type inequalities for Lipschitzian functions via Hadamard fractional integrals and give some applications to special means of positive real numbers.

Because of the wide application of Hermite-Hadamard type inequalities and fractional integrals, many researchers extend their studies to Hermite-Hadamard type inequalities involving fractional integrals not limited to integer integrals. Recently, more and more Hermite-Hadamard inequalities involving fractional integrals have been obtained for different classes of functions; see [8, 9, 15, 19, 20].

2000 *Mathematics Subject Classification.* 26A51, 26A33, 26D10.

Key words and phrases. Hermite-Hadamard inequality, Ostrowski inequality, Simpson inequality, Riemann-Liouville fractional integral.

©2016 Universiteti i Prishtinës, Prishtinë, Kosovë.

Submitted February 15, 2016. Published October 3, 2016.

*Corresponding author.

In this work, we study Hadamard, Ostrowski and Simpson type inequalities for Lipschitzian functions via Riemann-Liouville fractional integrals and give some applications to special means of positive real numbers.

2. PRELIMINARIES AND GENERAL CONDITIONS

Let a real function f be defined on some nonempty interval I of real line \mathbb{R} . The function f is said to be convex on I if inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

The following inequalities are well known in the literature as Hermite-Hadamard, Ostrowski and Simpson inequalities respectively.

Theorem 1. *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following double inequality holds:*

$$(2.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}.$$

Theorem 2. *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a mapping differentiable in I° , the interior of I , and let $a, b \in I^\circ$ with $a < b$. If $|f'(x)| \leq M$, $x \in [a, b]$; then the following inequality holds:*

$$(2.2) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{M}{b-a} \left[\frac{(x-a)^2 + (b-x)^2}{2} \right]$$

for all $x \in [a, b]$.

Theorem 3. *Let $f : [a, b] \rightarrow \mathbb{R}$ be four times continuously differentiable mapping on (a, b) and $\|f^{(4)}\|_\infty = \sup_{x \in (a, b)} |f^{(4)}(x)| < \infty$. Then the following inequality holds:*

$$(2.3) \quad \left| \frac{1}{3} \left[\frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{\|f^{(4)}\|_\infty}{2880} (b-a)^4.$$

The following definition of M -Lipschitzian function is well known in the literature.

Definition 1. *A function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ called an M -Lipschitzian function on the interval I of real numbers with $M \geq 0$ if*

$$(2.4) \quad |f(x) - f(y)| \leq M|x - y|$$

for all $x, y \in I$.

The following definitions and mathematical preliminaries of fractional calculus theory are used further in this paper.

Definition 2. [13]. *Let $f \in L[a, b]$. The Riemann-Liouville fractional integrals $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$ and $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

In [10], İşcan give the definition of harmonically convex functions and establish the following Hermite-Hadamard type inequality for harmonically convex functions.

Definition 3. Let $I \subset \mathbb{R} \setminus \{0\}$ be a real interval. A function $f : I \rightarrow \mathbb{R}$ is said to be harmonically convex, if

$$(2.5) \quad f\left(\frac{xy}{tx + (1-t)y}\right) \leq tf(y) + (1-t)f(x)$$

for all $x, y \in I$ and $t \in [0, 1]$. If the inequality in (2.5) is reversed, then f is said to be harmonically concave.

Theorem 4. [10]. Let $f : I \subset \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a, b]$ then the following inequalities holds:

$$(2.6) \quad f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b \frac{f(t)}{t^2} dt \leq \frac{f(a) + f(b)}{2}.$$

In [12], Kunt et al. establish Hermite-Hadamard's inequalities for harmonically convex functions in Riemann-Liouville fractional integral forms as follows:

Theorem 5. Let $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a function such that $f \in L[a, b]$, where $a, b \in I$ with $a < b$. If f is a harmonically convex function on $[a, b]$, then the following inequalities for fractional integrals holds:

$$(2.7) \quad f\left(\frac{2ab}{a+b}\right) \leq \frac{\Gamma(\alpha+1)}{2^{1-\alpha}} \left(\frac{ab}{b-a}\right)^\alpha \left\{ \begin{array}{l} J_{\frac{a+b}{2ab}+}^\alpha (f \circ g)(1/a) \\ + J_{\frac{a+b}{2ab}-}^\alpha (f \circ g)(1/b) \end{array} \right\} \leq \frac{f(a) + f(b)}{2}$$

with $\alpha > 0$ and $g(x) = 1/x$, $x \in [\frac{1}{b}, \frac{1}{a}]$.

3. GENERAL RESULTS

Let $I \subseteq (0, \infty)$ be a real interval and $f : I \rightarrow \mathbb{R}$ be a M -Lipschitzian function on I ; throughout this section, we will take

$$(3.1) \quad \begin{aligned} I_f(x, \lambda, \alpha, a, b) &= (1-\lambda) \left[\left(\frac{1}{a} - \frac{1}{x}\right)^\alpha + \left(\frac{1}{x} - \frac{1}{b}\right)^\alpha \right] f(x) \\ &+ \lambda \left[f(a) \left(\frac{1}{a} - \frac{1}{x}\right)^\alpha + f(b) \left(\frac{1}{x} - \frac{1}{b}\right)^\alpha \right] \\ &- \Gamma(\alpha+1) \left[J_{\frac{1}{x}+}^\alpha (f \circ g)(1/a) + J_{\frac{1}{x}-}^\alpha (f \circ g)(1/b) \right] \end{aligned}$$

$$(3.2) \quad \begin{aligned} S_f(x, y, \alpha, a, b) &= f(x) + f(y) - \left(\frac{2ab}{b-a}\right)^\alpha \Gamma(\alpha+1) \\ &\times \left[J_{\frac{a+b}{2ab}+}^\alpha (f \circ g)(1/a) + J_{\frac{a+b}{2ab}-}^\alpha (f \circ g)(1/b) \right] \end{aligned}$$

where $a, b \in I$ with $a < b$, $x, y \in [a, b]$, $g(t) = 1/t$, $\lambda \in [0, 1]$, $\alpha > 0$ and Γ is Euler Gamma function.

Theorem 6. Let $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a M -Lipschitzian function on I and $a, b \in I$ with $a < b$. Then for all $x \in [a, b]$, $\lambda \in [0, 1]$ and $\alpha > 0$ we have the following inequality for Riemann-Liouville fractional integrals

$$(3.3) \quad |I_f(x, \lambda, \alpha, a, b)| \leq M \left\{ [(1-\lambda)x - \lambda a] \left(\frac{1}{a} - \frac{1}{x} \right)^\alpha \right. \\ \left. + \alpha(2\lambda - 1) \int_{\frac{1}{x}}^{\frac{1}{a}} \left(\frac{1}{a} - t \right)^{\alpha-1} \frac{1}{t} dt + [\lambda b - (1-\lambda)x] \right. \\ \left. \times \left(\frac{1}{x} - \frac{1}{b} \right)^\alpha + \alpha(1-2\lambda) \int_{\frac{1}{b}}^{\frac{1}{x}} \left(t - \frac{1}{b} \right)^{\alpha-1} \frac{1}{t} dt \right\}.$$

Proof. Since f is a M -Lipschitzian function, then we have the following inequality

$$\begin{aligned} |I_f(x, \lambda, \alpha, a, b)| &= \left| (1-\lambda) \left[\left(\frac{1}{a} - \frac{1}{x} \right)^\alpha + \left(\frac{1}{x} - \frac{1}{b} \right)^\alpha \right] f(x) \right. \\ &\quad \left. + \lambda \left[f(a) \left(\frac{1}{a} - \frac{1}{x} \right)^\alpha + f(b) \left(\frac{1}{x} - \frac{1}{b} \right)^\alpha \right] \right. \\ &\quad \left. - \Gamma(\alpha+1) \left[J_{\frac{1}{x}+}^\alpha (f \circ g)(1/a) + J_{\frac{1}{x}-}^\alpha (f \circ g)(1/b) \right] \right| \\ &\leq (1-\lambda) \left| \left(\frac{1}{a} - \frac{1}{x} \right)^\alpha f(x) - \alpha \int_{\frac{1}{x}}^{\frac{1}{a}} \left(\frac{1}{a} - t \right)^{\alpha-1} f\left(\frac{1}{t}\right) dt \right. \\ &\quad \left. + \left(\frac{1}{x} - \frac{1}{b} \right)^\alpha f(x) - \alpha \int_{\frac{1}{b}}^{\frac{1}{x}} \left(t - \frac{1}{b} \right)^{\alpha-1} f\left(\frac{1}{t}\right) dt \right| \\ &\quad + \lambda \left| \left(\frac{1}{a} - \frac{1}{x} \right)^\alpha f(a) - \alpha \int_{\frac{1}{x}}^{\frac{1}{a}} \left(\frac{1}{a} - t \right)^{\alpha-1} f\left(\frac{1}{t}\right) dt \right. \\ &\quad \left. + \left(\frac{1}{x} - \frac{1}{b} \right)^\alpha f(b) - \alpha \int_{\frac{1}{b}}^{\frac{1}{x}} \left(t - \frac{1}{b} \right)^{\alpha-1} f\left(\frac{1}{t}\right) dt \right| \\ &\leq \alpha(1-\lambda) \left\{ \int_{\frac{1}{x}}^{\frac{1}{a}} \left(\frac{1}{a} - t \right)^{\alpha-1} \left| f(x) - f\left(\frac{1}{t}\right) \right| dt \right. \\ &\quad \left. + \int_{\frac{1}{b}}^{\frac{1}{x}} \left(t - \frac{1}{b} \right)^{\alpha-1} \left| f(x) - f\left(\frac{1}{t}\right) \right| dt \right\} \\ &\quad + \alpha\lambda \left\{ \int_{\frac{1}{x}}^{\frac{1}{a}} \left(\frac{1}{a} - t \right)^{\alpha-1} \left| f(a) - f\left(\frac{1}{t}\right) \right| dt \right. \\ &\quad \left. + \int_{\frac{1}{b}}^{\frac{1}{x}} \left(t - \frac{1}{b} \right)^{\alpha-1} \left| f(b) - f\left(\frac{1}{t}\right) \right| dt \right\} \\ &\leq \alpha(1-\lambda) M \left\{ \int_{\frac{1}{x}}^{\frac{1}{a}} \left(\frac{1}{a} - t \right)^{\alpha-1} \left(x - \frac{1}{t} \right) dt \right. \\ &\quad \left. + \int_{\frac{1}{b}}^{\frac{1}{x}} \left(t - \frac{1}{b} \right)^{\alpha-1} \left(\frac{1}{t} - x \right) dt \right\} \end{aligned}$$

$$\begin{aligned}
& + \alpha \lambda M \left\{ \int_{\frac{1}{x}}^{\frac{1}{a}} \left(\frac{1}{a} - t \right)^{\alpha-1} \left(\frac{1}{t} - a \right) dt \right. \\
& \left. + \int_{\frac{1}{b}}^{\frac{1}{x}} \left(t - \frac{1}{b} \right)^{\alpha-1} \left(b - \frac{1}{t} \right) dt \right\}.
\end{aligned}$$

By a simple computation from this inequality we have the inequality (3.3). This completes the proof. \square

Corollary 7. *In Theorem 6, if one takes $\lambda = 0$, then one has*

$$\begin{aligned}
(3.4) \quad & \left| \frac{\left(\frac{1}{a} - \frac{1}{x} \right)^\alpha + \left(\frac{1}{x} - \frac{1}{b} \right)^\alpha}{2^{1-\alpha}} \left(\frac{ab}{b-a} \right)^\alpha f(x) - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}} \left(\frac{ab}{b-a} \right)^\alpha \right. \\
& \times \left[J_{\frac{1}{x}+}^\alpha (f \circ g)(1/a) + J_{\frac{1}{x}-}^\alpha (f \circ g)(1/b) \right] \Big| \\
& \leq \frac{M}{2^{1-\alpha}} \left(\frac{ab}{b-a} \right)^\alpha \left\{ x \left[\left(\frac{1}{a} - \frac{1}{x} \right)^\alpha - \left(\frac{1}{x} - \frac{1}{b} \right)^\alpha \right] \right. \\
& \left. + \alpha \left[\int_{\frac{1}{b}}^{\frac{1}{x}} \left(t - \frac{1}{b} \right)^{\alpha-1} \frac{1}{t} dt - \int_{\frac{1}{x}}^{\frac{1}{a}} \left(\frac{1}{a} - t \right)^{\alpha-1} \frac{1}{t} dt \right] \right\}.
\end{aligned}$$

In (3.4),

(1) *If one takes $\alpha = 1$, then one has*

$$(3.5) \quad \left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(t)}{t^2} dt \right| \leq M \frac{ab}{b-a} \left\{ \frac{x(a+b)}{ab} - 2 + \ln \frac{ab}{x^2} \right\},$$

(2) *If one takes $x = \frac{2ab}{a+b}$, then one has*

$$\begin{aligned}
(3.6) \quad & \left| f\left(\frac{2ab}{a+b} \right) - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}} \left(\frac{ab}{b-a} \right)^\alpha \left[J_{\frac{2ab}{2ab}+}^\alpha (f \circ g)(1/a) \right. \right. \\
& \left. \left. + J_{\frac{2ab}{2ab}-}^\alpha (f \circ g)(1/b) \right] \right| \\
& \leq \frac{M}{2^{1-\alpha}} \left(\frac{ab}{b-a} \right)^\alpha \alpha \left\{ \int_{\frac{1}{b}}^{\frac{a+b}{2ab}} \left(t - \frac{1}{b} \right)^{\alpha-1} \frac{1}{t} dt - \int_{\frac{a+b}{2ab}}^{\frac{1}{a}} \left(\frac{1}{a} - t \right)^{\alpha-1} \frac{1}{t} dt \right\},
\end{aligned}$$

(3) *If one takes $\alpha = 1$ and $x = \frac{2ab}{a+b}$, then one has*

$$(3.7) \quad \left| f\left(\frac{2ab}{a+b} \right) - \frac{ab}{b-a} \int_a^b \frac{f(t)}{t^2} dt \right| \leq M \frac{ab}{b-a} \ln \frac{(a+b)^2}{4ab}.$$

Corollary 8. *In Theorem 6, if one takes $\lambda = 1$, then one has*

$$\begin{aligned}
(3.8) \quad & \left| \frac{f(a) \left(\frac{1}{a} - \frac{1}{x} \right)^\alpha + f(b) \left(\frac{1}{x} - \frac{1}{b} \right)^\alpha}{2^{1-\alpha}} \left(\frac{ab}{b-a} \right)^\alpha - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}} \left(\frac{ab}{b-a} \right)^\alpha \right. \\
& \times \left[J_{\frac{1}{x}+}^\alpha (f \circ g)(1/a) + J_{\frac{1}{x}-}^\alpha (f \circ g)(1/b) \right] \Big| \\
& \leq \frac{M}{2^{1-\alpha}} \left(\frac{ab}{b-a} \right)^\alpha \left\{ \left[b \left(\frac{1}{x} - \frac{1}{b} \right)^\alpha - a \left(\frac{1}{a} - \frac{1}{x} \right)^\alpha \right] \right. \\
& \left. + \alpha \left[\int_{\frac{1}{x}}^{\frac{1}{a}} \left(\frac{1}{a} - t \right)^{\alpha-1} \frac{1}{t} dt - \int_{\frac{1}{b}}^{\frac{1}{x}} \left(t - \frac{1}{b} \right)^{\alpha-1} \frac{1}{t} dt \right] \right\}.
\end{aligned}$$

In (3.8),

(1) If one takes $x = \frac{2ab}{a+b}$, then one has

$$(3.9) \quad \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2^{1-\alpha}} \left(\frac{ab}{b-a} \right)^\alpha \right. \\ \times \left[J_{\frac{a+b}{2ab}+}^\alpha (f \circ g)(1/a) + J_{\frac{a+b}{2ab}-}^\alpha (f \circ g)(1/b) \right] \Big| \\ \leq \frac{M}{2^{1-\alpha}} \left(\frac{ab}{b-a} \right)^\alpha \left\{ (b-a) \left(\frac{b-a}{2ab} \right)^\alpha \right. \\ \left. + \alpha \left[\int_{\frac{a+b}{2ab}}^{\frac{1}{a}} \left(\frac{1}{a} - t \right)^{\alpha-1} \frac{1}{t} dt - \int_{\frac{1}{b}}^{\frac{a+b}{2ab}} \left(t - \frac{1}{b} \right)^{\alpha-1} \frac{1}{t} dt \right] \right\},$$

(2) If one takes $\alpha = 1$ and $x = \frac{2ab}{a+b}$, then one has

$$(3.10) \quad \left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(t)}{t^2} dt \right| \leq M \frac{ab}{b-a} \left\{ \frac{(b-a)^2}{2ab} + \ln \frac{4ab}{(a+b)^2} \right\}.$$

Remark. In (3.6) and (3.7) we get new inequalities about the left hand side of Hermite-Hadamard's inequalities of (2.7) and (2.6), in (3.9) and (3.10) we get new inequalities about the right hand side of Hermite-Hadamard's inequalities of (2.7) and (2.6) respectively for M -Lipschitzian functions.

Corollary 9. In Theorem 6,

(1) If one takes $\lambda = \frac{1}{3}$, $x = \frac{2ab}{a+b}$, then one has

$$(3.11) \quad \left| \frac{1}{3} \left[\frac{f(a) + f(b)}{2} + 2f \left(\frac{2ab}{a+b} \right) \right] - \frac{\Gamma(\alpha + 1)}{2^{1-\alpha}} \left(\frac{ab}{b-a} \right)^\alpha \right. \\ \times \left[J_{\frac{a+b}{2ab}+}^\alpha (f \circ g)(1/a) + J_{\frac{a+b}{2ab}-}^\alpha (f \circ g)(1/b) \right] \Big| \\ \leq \frac{M}{2^{1-\alpha}} \left(\frac{ab}{b-a} \right)^\alpha \left\{ \frac{b-a}{3} \left(\frac{b-a}{2ab} \right)^\alpha \right. \\ \left. + \frac{\alpha}{3} \left(\int_{\frac{1}{b}}^{\frac{a+b}{2ab}} \left(t - \frac{1}{b} \right)^{\alpha-1} \frac{1}{t} dt - \int_{\frac{a+b}{2ab}}^{\frac{1}{a}} \left(\frac{1}{a} - t \right)^{\alpha-1} \frac{1}{t} dt \right) \right\}.$$

Specially if one takes $\alpha = 1$ in (3.11), then one has

$$(3.12) \quad \left| \frac{1}{3} \left[\frac{f(a) + f(b)}{2} + 2f \left(\frac{2ab}{a+b} \right) \right] - \frac{ab}{b-a} \int_a^b \frac{f(t)}{t^2} dt \right| \\ \leq M \frac{ab}{b-a} \left\{ \frac{(b-a)^2}{6ab} + \frac{1}{3} \ln \frac{(a+b)^2}{4ab} \right\},$$

(2) If one takes $\lambda = \frac{1}{2}$, $x = \frac{2ab}{a+b}$, then one has

$$(3.13) \quad \left| \frac{1}{2} \left[\frac{f(a) + f(b)}{2} + f \left(\frac{2ab}{a+b} \right) \right] - \frac{\Gamma(\alpha + 1)}{2^{1-\alpha}} \left(\frac{ab}{b-a} \right)^\alpha \right. \\ \times \left[J_{\frac{a+b}{2ab}+}^\alpha (f \circ g)(1/a) + J_{\frac{a+b}{2ab}-}^\alpha (f \circ g)(1/b) \right] \Big| \leq M \frac{b-a}{4}.$$

Specially if one takes $\alpha = 1$ in (3.13), then one has

$$(3.14) \quad \left| \frac{1}{2} \left[\frac{f(a) + f(b)}{2} + f \left(\frac{2ab}{a+b} \right) \right] - \frac{ab}{b-a} \int_a^b \frac{f(t)}{t^2} dt \right| \leq M \frac{b-a}{4}.$$

Corollary 10. *In Theorem 6, if one takes $\alpha = 1$, then one has*

$$(3.15) \quad \left| (1-\lambda) f(x) + \lambda \left[\frac{f(a) \left(\frac{1}{a} - \frac{1}{x} \right) + f(b) \left(\frac{1}{x} - \frac{1}{b} \right)}{\frac{b-a}{ab}} \right] - \frac{ab}{b-a} \int_a^b \frac{f(t)}{t^2} dt \right|$$

$$\leq M \frac{ab}{b-a} \left\{ [(1-\lambda)x - \lambda a] \left(\frac{1}{a} - \frac{1}{x} \right) + [\lambda b - (1-\lambda)x] \left(\frac{1}{x} - \frac{1}{b} \right) \right.$$

$$\left. + (2\lambda - 1) \ln \frac{x^2}{ab} \right\}.$$

Specially, if one takes $x = \frac{2ab}{a+b}$ in (3.15), then one has

$$(3.16) \quad \left| (1-\lambda) f\left(\frac{2ab}{a+b}\right) + \lambda \left[\frac{f(a) + f(b)}{2} \right] - \frac{ab}{b-a} \int_a^b \frac{f(t)}{t^2} dt \right|$$

$$\leq M \frac{ab}{b-a} \left\{ \lambda \frac{(b-a)^2}{2ab} + (2\lambda - 1) \ln \frac{4ab}{(a+b)^2} \right\}.$$

Remark. *If we take $\lambda = 0$, $\lambda = 1$, $\lambda = \frac{1}{3}$ and $\lambda = \frac{1}{2}$ in inequality (3.16) we obtain inequalities (3.7), (3.10), (3.12) and (3.14), respectively.*

Let $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a M -Lipschitzian function on I and $a, b \in I$ with $a < b$. In the next Theorem $a \leq x \leq y \leq b$ and define $U_\alpha(x, y)$, $\alpha > 0$ as follows:

(1) If $a \leq \frac{2ab}{a+b} \leq x \leq y \leq b$, then

$$(3.17) \quad U_\alpha(x, y) = \frac{x}{\alpha} \left(\frac{b-a}{2ab} \right)^\alpha - \frac{y}{\alpha} \left[2 \left(\frac{b-y}{by} \right)^\alpha - \left(\frac{b-a}{2ab} \right)^\alpha \right]$$

$$- \int_{\frac{a+b}{2ab}}^{\frac{1}{a}} \left(\frac{1}{a} - t \right)^{\alpha-1} \frac{dt}{t} + \int_{\frac{1}{b}}^{\frac{1}{y}} \left(t - \frac{1}{b} \right)^{\alpha-1} \frac{dt}{t} - \int_{\frac{1}{y}}^{\frac{a+b}{2ab}} \left(t - \frac{1}{b} \right)^{\alpha-1} \frac{dt}{t},$$

(2) If $a \leq x \leq \frac{2ab}{a+b} \leq y \leq b$, then

$$(3.18) \quad U_\alpha(x, y) = \frac{x}{\alpha} \left[2 \left(\frac{x-a}{ax} \right)^\alpha - \left(\frac{b-a}{2ab} \right)^\alpha \right] - \frac{y}{\alpha} \left[2 \left(\frac{b-y}{by} \right)^\alpha - \left(\frac{b-a}{2ab} \right)^\alpha \right]$$

$$+ \int_{\frac{a+b}{2ab}}^{\frac{1}{x}} \left(\frac{1}{a} - t \right)^{\alpha-1} \frac{dt}{t} - \int_{\frac{1}{x}}^{\frac{1}{a}} \left(\frac{1}{a} - t \right)^{\alpha-1} \frac{dt}{t}$$

$$+ \int_{\frac{1}{b}}^{\frac{1}{y}} \left(t - \frac{1}{b} \right)^{\alpha-1} \frac{dt}{t} - \int_{\frac{a+b}{2ab}}^{\frac{1}{y}} \left(t - \frac{1}{b} \right)^{\alpha-1} \frac{dt}{t},$$

(3) If $a \leq x \leq y \leq \frac{2ab}{a+b} \leq b$, then

$$(3.19) \quad U_\alpha(x, y) = \frac{x}{\alpha} \left[2 \left(\frac{x-a}{ax} \right)^\alpha - \left(\frac{b-a}{2ab} \right)^\alpha \right] - \frac{y}{\alpha} \left(\frac{b-a}{2ab} \right)^\alpha$$

$$+ \int_{\frac{a+b}{2ab}}^{\frac{1}{x}} \left(\frac{1}{a} - t \right)^{\alpha-1} \frac{dt}{t} - \int_{\frac{1}{x}}^{\frac{1}{a}} \left(\frac{1}{a} - t \right)^{\alpha-1} \frac{dt}{t} + \int_{\frac{1}{b}}^{\frac{a+b}{2ab}} \left(t - \frac{1}{b} \right)^{\alpha-1} \frac{dt}{t}.$$

Theorem 11. *Let $x, y, \alpha, U_\alpha(x, y)$ and function f be defined as above. Then we have following inequality for Riemann-Louville fractional integrals*

$$(3.20) \quad |S_f(x, y, \alpha, a, b)| \leq M \alpha \left(\frac{2ab}{b-a} \right)^\alpha U_\alpha(x, y).$$

Proof. Since f is a M -Lipschitzian function, then we have the following inequality

$$(3.21) \quad |S_f(x, y, \alpha, a, b)| = \alpha \left(\frac{2ab}{b-a} \right)^\alpha \left| \frac{\left(\frac{b-a}{2ab} \right)^\alpha}{\alpha} f(x) + \frac{\left(\frac{b-a}{2ab} \right)^\alpha}{\alpha} f(y) \right. \\ \left. - \Gamma(\alpha) \left[J_{\frac{a+b}{2ab}}^\alpha (f \circ g)(1/a) + J_{\frac{a+b}{2ab}}^{\alpha\alpha} (f \circ g)(1/b) \right] \right| \\ = \alpha \left(\frac{2ab}{b-a} \right)^\alpha \left| \int_{\frac{a+b}{2ab}}^{\frac{1}{a}} \left(\frac{1}{a} - t \right)^{\alpha-1} f(x) dt \right. \\ \left. + \int_{\frac{1}{b}}^{\frac{a+b}{2ab}} \left(t - \frac{1}{b} \right)^{\alpha-1} f(y) dt - \int_{\frac{a+b}{2ab}}^{\frac{1}{a}} \left(\frac{1}{a} - t \right)^{\alpha-1} f\left(\frac{1}{t}\right) dt \right. \\ \left. - \int_{\frac{1}{b}}^{\frac{a+b}{2ab}} \left(t - \frac{1}{b} \right)^{\alpha-1} f\left(\frac{1}{t}\right) dt \right| \\ \leq \alpha \left(\frac{2ab}{b-a} \right)^\alpha \left\{ \int_{\frac{a+b}{2ab}}^{\frac{1}{a}} \left(\frac{1}{a} - t \right)^{\alpha-1} \left| f(x) - f\left(\frac{1}{t}\right) \right| dt \right. \\ \left. + \int_{\frac{1}{b}}^{\frac{a+b}{2ab}} \left(t - \frac{1}{b} \right)^{\alpha-1} \left| f(y) - f\left(\frac{1}{t}\right) \right| dt \right\} \\ \leq M \alpha \left(\frac{2ab}{b-a} \right)^\alpha \left\{ \int_{\frac{a+b}{2ab}}^{\frac{1}{a}} \left(\frac{1}{a} - t \right)^{\alpha-1} \left| x - \frac{1}{t} \right| dt \right. \\ \left. + \int_{\frac{1}{b}}^{\frac{a+b}{2ab}} \left(t - \frac{1}{b} \right)^{\alpha-1} \left| y - \frac{1}{t} \right| dt \right\}.$$

Using (3.17), (3.18) and (3.19), by a simple calculations, we have

$$(3.22) \quad \int_{\frac{a+b}{2ab}}^{\frac{1}{a}} \left(\frac{1}{a} - t \right)^{\alpha-1} \left| x - \frac{1}{t} \right| dt + \int_{\frac{1}{b}}^{\frac{a+b}{2ab}} \left(t - \frac{1}{b} \right)^{\alpha-1} \left| y - \frac{1}{t} \right| dt = U_\alpha(x, y).$$

Now using (3.21) and (3.22) we obtain (3.20). This completes the proof. \square

With using assumptions of Theorem 11 we have the following corollary and remarks.

Corollary 12. *In Theorem 11, if one takes $\alpha = 1$, then the inequality (3.20) reduces the following inequality*

$$(3.23) \quad \left| f(x) + f(y) - \frac{2ab}{b-a} \int_a^b \frac{f(t)}{t^2} dt \right| \leq M \frac{2ab}{b-a} U_1(x, y).$$

Remark. *In Theorem 11, if one takes $x = y = \frac{2ab}{a+b}$, then the inequality (3.20) reduces the inequality (3.6).*

Remark. *In Theorem 11, if one takes $x = a$ and $y = b$, then the inequality (3.20) reduces the inequality (3.9).*

4. APPLICATIONS TO SPECIAL MEANS

Let us recall the following special means of positive numbers a, b with $a < b$.

(1) The arithmetic mean:

$$(4.1) \quad A = A(a, b) := \frac{a+b}{2},$$

(2) The geometric mean:

$$(4.2) \quad G = G(a, b) := \sqrt{ab},$$

(3) The harmonic mean:

$$(4.3) \quad H = H(a, b) := \frac{2ab}{a+b},$$

(4) The logarithmic mean:

$$(4.4) \quad L = L(a, b) := \frac{b-a}{\ln b - \ln a},$$

(5) The identric mean:

$$(4.5) \quad I = I(a, b) := \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}}.$$

We need the following lemma to prove the results of this section.

Lemma 13. (see [16]). Let $f : [a, b] \rightarrow \mathbb{R}$ be differentiable with $\|f'\|_\infty < \infty$. Then f is a M -Lipschitzian function on $[a, b]$ where $M = \|f'\|_\infty$.

Proposition 14. For $b > a > 0$, $\lambda \in [0, 1]$ and $n \geq 1$, we have

$$(4.6) \quad \left| (1-\lambda) H^n(a, b) + \lambda A(a^n, b^n) - ab \frac{L(a^{n-1}, b^{n-1})}{L(a, b)} \right| \\ \leq nb^{n-1} \left\{ \lambda \frac{b-a}{2} + (2\lambda-1) \frac{ab}{b-a} \ln \frac{H(a, b)}{A(a, b)} \right\}.$$

Proof. The proof follows by inequality (3.16) applied for the Lipschitzian function $f(x) = x^n$ on $[a, b]$. \square

Remark. Let $\lambda = 0$ and $\lambda = 1$ in inequality (4.6). Then, using inequality (2.6), one has the following inequalities respectively,

$$(4.7) \quad 0 \leq ab \frac{L(a^{n-1}, b^{n-1})}{L(a, b)} - H^n(a, b) \leq -nb^{n-1} \frac{ab}{b-a} \ln \frac{H(a, b)}{A(a, b)},$$

$$(4.8) \quad 0 \leq A(a^n, b^n) - ab \frac{L(a^{n-1}, b^{n-1})}{L(a, b)} \leq nb^{n-1} \left\{ \frac{b-a}{2} + \frac{ab}{b-a} \ln \frac{H(a, b)}{A(a, b)} \right\}.$$

Proposition 15. For $b > a > 0$ and $\lambda \in [0, 1]$, one has

$$(4.9) \quad \left| (1-\lambda) H^2(a, b) e^{H(a, b)} + \lambda A(a^2 e^a, b^2 e^b) - ab L(e^a, e^b) \right| \\ \leq (2b + b^2) e^b \left\{ \lambda \frac{b-a}{2} + (2\lambda-1) \frac{ab}{b-a} \ln \frac{H(a, b)}{A(a, b)} \right\}.$$

Proof. The proof follows by inequality (3.16) applied for the Lipschitzian function $f(x) = x^2 e^x$ on $[a, b]$. \square

Remark. Let $\lambda = 0$ and $\lambda = 1$ in inequality (4.9). Then, using inequality (2.6), one has the following inequalities respectively,

$$(4.10) \quad 0 \leq abL(e^a, e^b) - H^2(a, b) e^{H(a, b)} \leq -(2b + b^2) e^b \frac{ab}{b-a} \ln \frac{H(a, b)}{A(a, b)},$$

$$(4.11) \quad 0 \leq A(a^2 e^a, b^2 e^b) - abL(e^a, e^b) \leq (2b + b^2) e^b \left\{ \frac{b-a}{2} + \frac{ab}{b-a} \ln \frac{H(a, b)}{A(a, b)} \right\}.$$

Proposition 16. For $b > a > 0$ and $\lambda \in [0, 1]$, one has

$$(4.12) \quad \left| (1-\lambda) H(a, b) + \lambda A(a, b) - abL^{-1}(a, b) \right| \\ \leq \lambda \frac{b-a}{2} + (2\lambda-1) \frac{ab}{b-a} \ln \frac{H(a, b)}{A(a, b)}.$$

Proof. The proof follows by inequality (3.16) applied for the Lipschitzian function $f(x) = x$ on $[a, b]$. \square

Remark. Let $\lambda = 0$ and $\lambda = 1$ in inequality (4.12). Then, using inequality (2.6), one has the following inequalities respectively,

$$(4.13) \quad 0 \leq abL^{-1}(a, b) - H(a, b) \leq -\frac{ab}{b-a} \ln \frac{H(a, b)}{A(a, b)},$$

$$(4.14) \quad 0 \leq A(a, b) - abL^{-1}(a, b) \leq \frac{b-a}{2} + \frac{ab}{b-a} \ln \frac{H(a, b)}{A(a, b)}.$$

Proposition 17. For $b > a > 0$ and $\lambda \in [0, 1]$, one has

$$(4.15) \quad \left| (1-\lambda) H(a, b) \ln H(a, b) + \lambda A(a \ln a, b \ln b) - \frac{ab \ln ab}{2} L^{-1}(a, b) \right| \\ \leq \ln eb \frac{ab}{b-a} \left\{ \lambda \frac{(b-a)^2}{2ab} + (2\lambda-1) \ln \frac{H(a, b)}{A(a, b)} \right\}.$$

Proof. The proof follows by inequality (3.16) applied for the Lipschitzian function $f(x) = x \ln x$ on $[a, b]$. \square

Remark. Let $\lambda = 0$ and $\lambda = 1$ in inequality (4.15). Then, using inequality (2.6), one has the following inequalities respectively,

$$(4.16) \quad 0 \leq \frac{ab \ln ab}{2} L^{-1}(a, b) - H(a, b) \ln H(a, b) \leq -\ln eb \frac{ab}{b-a} \ln \frac{H(a, b)}{A(a, b)},$$

$$(4.17) \quad 0 \leq A(a \ln a, b \ln b) - \frac{ab \ln ab}{2} L^{-1}(a, b) \leq \ln eb \left\{ \frac{b-a}{2} + \frac{ab}{b-a} \ln \frac{H(a, b)}{A(a, b)} \right\}.$$

Proposition 18. For $b > a > 0$ and $\lambda \in [0, 1]$, we have

$$(4.18) \quad \left| (1-\lambda) H^2(a, b) \ln H(a, b) + \lambda A(a^2 \ln a, b^2 \ln b) - ab \ln I(a, b) \right| \\ \leq (b + 2b \ln b) \frac{ab}{b-a} \left\{ \lambda \frac{(b-a)^2}{2ab} + (2\lambda-1) \ln \frac{H(a, b)}{A(a, b)} \right\}.$$

Proof. The proof follows by inequality (3.16) applied for the Lipschitzian function $f(x) = x^2 \ln x$ on $[a, b]$. \square

Remark. Let $\lambda = 0$ and $\lambda = 1$ in inequality (4.18). Then, using inequality (2.6), one has the following inequalities respectively,

$$(4.19) \quad 0 \leq \frac{ab \ln ab}{2} L^{-1}(a, b) - H^2(a, b) \ln H(a, b) \leq -(b + 2b \ln b) \frac{ab}{b-a} \ln \frac{H(a, b)}{A(a, b)},$$

$$(4.20) \quad 0 \leq A(a^2 \ln a, b^2 \ln b) - ab \ln I(a, b) \\ \leq (b + 2b \ln b) \left\{ \frac{b-a}{2} + \frac{ab}{b-a} \ln \frac{H(a, b)}{A(a, b)} \right\}.$$

REFERENCES

- [1] M. Alomari, M. Darus, S. S. Dragomir, P. Cerone, *Ostrowski type inequalities for functions whose derivatives are s -convex in the second sense*, Applied Mathematics Letters **23** 9 (2010) 1071–1076.
- [2] M. Avcı, H. Kavurmacı, M. E. Özdemir, *New inequalities of Hermite-Hadamard type via s -convex functions in the second sense with applications*, Applied Mathematics and Computation **217** 12 (2011) 5171–5176.
- [3] Y. Chu, G. Wang, X. Zhang, *Schur convexity and Hadamard's inequality* Mathematical Inequalities and Applications **13** 4 (2010) 725–731.
- [4] Y.-M. Chu, X.-M. Zhang, X.-H. Zhang, *The Hermite-Hadamard type inequality of GA-convex functions and its application*, Journal of Inequalities and Applications **11** 507560 (2010).
- [5] S. S. Dragomir, Y. J. Cho, S. S. Kim, *Inequalities of Hadamard's type for Lipschitzian mappings and their applications*, Journal of Mathematical Analysis and Applications **245** 2 (2000) 489–501.
- [6] S.-R. Hwang, K.-C. Hsu, K.-L. Tseng, *Hadamard-type inequalities for Lipschitzian functions in one and two variables with applications*, Journal of Mathematical Analysis and Applications **405** 2 (2013), 546–554.
- [7] İ. İşcan, *New estimates on generalization of some integral inequalities for s -convex functions and their applications*, International Journal of Pure and Applied Mathematics **86** 4 (2013), 727–746.
- [8] İ. İşcan, *Generalization of different type integral inequalities for s -convex functions via fractional integrals*, Applicable Analysis, **93** 9 (2014) 1846–1862.
- [9] İ. İşcan, *On generalization of different type integral inequalities for s -convex functions via fractional integrals*, Mathematical Sciences and Applications E-Notes **2** 1 (2014) 55–67.
- [10] İ. İşcan, *Hermite-Hadamard type inequalities for harmonically convex functions*, Hacettepe Journal of Mathematics and Statistics **43** 6 (2014) 935–942.
- [11] İ. İşcan, *New general integral inequalities for Lipschitzian functions via Hadamard fractional integrals*, International Journal of Analysis **353924** (2014).
- [12] M. Kunt, İ. İşcan, N. Y. Gözütok, U. Gözütok, *On new inequalities of Hermite-Hadamard-Fejer type for harmonically convex functions via fractional integrals*, SpringerPlus **5** 635 (2016) 1–19.
- [13] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, 2006.
- [14] E. Set, M. E. Özdemir, M. Z. Sarıkaya, *On new inequalities of Simpson's type for quasi convex functions with applications*, Tamkang Journal of Mathematics **43** 3 (2012) 357–364.
- [15] M. Z. Sarıkaya, E. Set, H. Yıldız, N. Başak, *Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities*, Mathematical and Computer Modelling **37** (2013) 2403–2407.
- [16] K.-L. Tseng, S.-R. Hwang, K.-C. Hsu, *Hadamard-type and Bullen-type inequalities for Lipschitzian functions and their applications*, Computers and Mathematics with Applications **64** 4 (2012) 651–660.
- [17] L.-C. Wang, *New inequalities of Hadamard's type for Lipschitzian mappings*, Journal of Inequalities in Pure and Applied Mathematics **6** 2(37) (2005).
- [18] L.-C. Wang, *On new inequalities of Hadamard-type for Lipschitzian mappings and their applications*, Journal of Inequalities in Pure and Applied Mathematics, **8** 1(30) (2007) 1–11.

- [19] J. Wang, X. Li, M. Fečkan, Y. Zhou, *Hermite-Hadamard-type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity*, *Applicable Analysis* **92** **11** (2012) 2241–2253.
- [20] J. Wang, C. Zhu, Y. Zhou, *New generalized Hermite-Hadamard type inequalities and applications to special means*, *Journal of Inequalities and Applications* **325** (2013) 1-15.
- [21] G.-S. Yang, K.-L. Tseng, *Inequalities of Hadamard's type for Lipschitzian mappings*, *Journal of Mathematical Analysis and Applications*, **260** **1** (2001) 230–238.

İMDAT İŞCAN

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES AND ARTS, 28200, GİRESUN UNIVERSITY, GİRESUN, TURKEY

E-mail address: `imdat.iscan@giresun.edu.tr; imdati@yahoo.com`

MEHMET KUNT, NAZLI YAZICI GÖZÜTOK, TUNCAY KÖROĞLU

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES, KARADENİZ TECHNICAL UNIVERSITY, 61080, TRABZON, TURKEY

E-mail address, M. Kunt: `mkunt@ktu.edu.tr`

E-mail address, N. Y. Gözütok: `nazliyazici@ktu.edu.tr`

E-mail address, T. Köroğlu: `tkor@ktu.edu.tr`